

The Orbit Structure of 2-reducible Algebraic Monoids

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Abstract

In this paper we study the orbit structure of semisimple algebraic monoids with exactly two nonzero minimal $G \times G$ -orbits. The case of one minimal orbit was solved earlier by the authors. The key notion for reductive monoids is the type map λ , which is the monoid notion of the Dynkin diagram. It is the ultimate combinatorial invariant of a reductive monoid. To calculate λ , we associate with each 2-reducible monoid M , certain invariants (I_+, I_-) and (Δ_+, Δ_-) . These invariants are not entirely independent, but can be regarded as the minimal information needed to determine the much sought after type map of M . We obtain a combinatorial recipe for the cross-section lattice and the type map in terms of (I_+, I_-) and (Δ_+, Δ_-) .

We end the discussion with several carefully chosen examples. The first one illustrates how the Cartan matrix can be involved in calculating (Δ_+, Δ_-) from (I_+, I_-) and the positive vertices of the polytope of M .

1 Introduction

Throughout this paper we assume without exception that K is an algebraically closed field. Let M be a reductive algebraic monoid defined over K . We assume that M has a zero element $0 \in M$. One of the basic problems here is to determine the finite lattice of $G \times G$ -orbits of M in terms of the minimal orbits of $M \setminus \{0\}$ and combinatorial information obtained from the underlying root system. The authors have worked this out in [9] for the \mathcal{J} -irreducible case. A reductive monoid M with 0 is *\mathcal{J} -irreducible* if $M \setminus \{0\}$ has exactly one minimal $G \times G$ -orbit. In this case, the lattice of orbits can be described in terms of the *type* of that minimal orbit. See Theorem 4.16 of [9]. One can interpret this set of $G \times G$ -orbits as a fundamental domain for the action of the Weyl group on the face lattice of a certain rational polytope. One consequence is the complete combinatorial classification of such polytopes, when there is only one W -orbit of vertices.

Vinberg obtained a similar description of the $G \times G$ -orbits of his universal, flat deformation monoid $Env(G)$ of the semisimple group G . See [15] for Vinberg's discussion, along with some decisive results about monoid deformations of a semisimple group.

In this paper we consider the orbit structure of 2-reducible, semisimple, algebraic monoids. A reductive monoid M is *semisimple* if $0 \in M$ and $\dim(Z(G)) = 1$. M is *2-reducible* if $M \setminus \{0\}$

has exactly two minimal $G \times G$ -orbits. Given a 2-reducible, semisimple monoid M , we obtain certain invariants (I_+, I_-) and (Δ_+, Δ_-) . From these, we calculate the *cross section lattice* Λ , and the *type map*

$$\lambda : \Lambda \rightarrow 2^\Delta.$$

See Definitions 2.1 and 2.2 below for the precise definitions of these key notions. The cross section lattice is a certain set of representatives for the $G \times G$ -orbits of M , and the type map tells us how these orbits fit together to make the monoid structure feasible. Once we know the type map it is then possible to calculate the set R of $B \times B$ -orbits, along with the ‘‘Bruhat’’ order relation on R . From there, certain Iwahori-Hecke algebra constructions are made possible [13, 8].

These invariants (I_-, I_+) and (Δ_+, Δ_-) are not entirely independent. It appears that the final answer depends on the polyhedral/combinatorial ‘‘shape’’ of the inverse of the Cartan matrix; and not just the shape of the Dynkin diagram.

In the parlance of spherical varieties, we are finding the *colored faces* of the *colored cone* of $M \setminus \{0\}$, in terms of the invariants (I_-, I_+) and (Δ_+, Δ_-) . (I_-, I_+) is the pair of *colors* of the two minimal $G \times G$ -orbits of $M \setminus \{0\}$. (Δ_+, Δ_-) is the minimal extra data needed to nail down the rest of the colored faces of $M \setminus \{0\}$. See [12] or §4.3 of [11], for much of what is known or written down about reductive monoids considered as spherical varieties.

The reader will need to be familiar with the basic theory of reductive algebraic monoids. Putcha’s monograph [6] provides an introduction to the fundamental techniques and concepts, with an emphasis on the semigroup theory. Solomon’s survey [14] provides an engaging introduction to reductive monoids from the point of view of root systems and algebraic combinatorics. As already mentioned, these orbit problems can be interpreted in terms of reflection groups acting on the face lattices of rational polytopes.

The reader should consult [6] for any unproved statements about algebraic monoids.

2 Reductive Monoids and Type Maps

In this section we summarize the relevant background information about reductive monoids and type maps that will be used in the paper.

Let G be a reductive group with Borel subgroup $B \subseteq G$ and maximal torus $T \subseteq B$. G is generated by T , along with a collection of one-dimensional, unipotent subgroups U_α . Each U_α is normalized by T . $\Phi = \{\alpha\} \subseteq X(T)$ is the set of *roots*. Let $\Delta \subseteq \Phi$ be the set of *simple roots* relative to B (see §27.3 of [1]). Δ is determined by a choice of Borel subgroup containing T . B is generated by T along with the U_α coming from *positive* roots $\alpha \in \Phi^+$.

The *Weyl group* W of T is $W = N(T)/T$ and it is generated by a certain set of *simple reflections* $\{s_\alpha \mid \alpha \in \Delta\}$. Furthermore, for each α , $s_\alpha U_\alpha s_\alpha = U_{-\alpha}$.

Let M be a reductive monoid with unit group G . For $X \subseteq M$, let $E(X) = \{e \in X \mid e^2 = e\}$ be the set of *idempotents* of X . Let $e \in E(M)$ be an idempotent. Let

$$P(e) = \{g \in G \mid ge = ege\}.$$

By Theorem 7.1 of [6], $P(e)$ is a parabolic subgroup of G .

Definition 2.1. Let M be reductive with unit group G . The *cross section lattice* Λ of M , relative to T and B is

$$\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\}$$

We let

$$\Lambda' = \Lambda \setminus \{0\}.$$

By Chapter 9 of [6],

$$M = \bigsqcup_{e \in \Lambda} GeG$$

Since Λ is a lattice, it has two operations, the *meet* \wedge , and the *join* \vee . The meet $e \wedge f$, of two idempotents, is just their product ef in M . The join $e \vee f$ of e and f in Λ is the smallest idempotent h in Λ such that $he = e$ and $hf = f$.

Definition 2.2. Let M be reductive with unit group G . Let T be a maximal torus of the Borel subgroup B of G , and let Δ be the set of simple roots of T relative to T and B . Let $\Lambda \subseteq E(\overline{T})$ be a cross section lattice of M . Let $W = N_G(T)/T$ be the Weyl group.

a) The *type map*

$$\lambda : \Lambda \rightarrow 2^\Delta$$

is defined so that $P_{\lambda(e)} = P(e)$, where P_I denotes the *parabolic subgroup of type* $I \subseteq \Delta$ containing B . Let $W(e) = W_{\lambda(e)}$, the associated parabolic subgroup of W .

b) Let $\lambda_*(e) = \bigcap_{f \leq e} \lambda(f)$ and $W_*(e) = \bigcap_{f \leq e} W(f)$.

c) Let $\lambda^*(e) = \bigcap_{f \geq e} \lambda(f)$ and $W^*(e) = \bigcap_{f \geq e} W(f)$.

Remark 2.3. It follows from these definitions that

- i) for $e \in \Lambda$, $\lambda(e) = \lambda^*(e) \sqcup \lambda_*(e)$;
- ii) for $e, f \in \Lambda$, $\lambda(e) \cap \lambda(f) \subseteq \lambda(e \vee f) \cap \lambda(e \wedge f)$;
- iii) for $e \in \Lambda$, $W(e) \cong W^*(e) \times W_*(e)$;
- iv) if $e \geq f$ then $\lambda_*(e) \subseteq \lambda_*(f)$ and $\lambda^*(f) \subseteq \lambda^*(e)$. Furthermore, λ^* restricted to eMe is the λ^* of eMe , and λ_* restricted to M_e is the λ_* of M_e .

Here, eMe is the reductive monoid with unit group $eC_G(e)$, and $M_e = \overline{G_e}$, where $G_e = \{g \in G \mid ge = eg = e\}^0$. M_e is also a reductive monoid.

The type map is the most important single combinatorial invariant in the structure theory of reductive monoids. It is in some sense, the monoid analogue of the Coxeter-Dynkin graph. It also shows us which $G \times G$ -orbits are involved in the monoid, as well as how the monoid structure is built up from these orbits [7].

Definition 2.4. Let M, Λ and λ be as above. Let $\Lambda_1 \subseteq \Lambda$ be the subset of nonzero minimal elements.

a) The *core* C of Λ is

$$C = \{e \in \Lambda \mid e = e_1 \vee \cdots \vee e_k, \text{ for some } e_i \in \Lambda_1\}.$$

b) Define $\theta : \Lambda' \rightarrow C$ by

$$\theta(e) = \vee \{f \in \Lambda_1 \mid f \leq e\}$$

so that, if $e_1 \leq e_2$, then $\theta(e_1) \leq \theta(e_2)$.

c) Write $\Lambda' = \bigsqcup_{h \in C} \Lambda_h$, where $\Lambda_h = \theta^{-1}(h)$.

Recall that a reductive monoid M with zero is *semisimple* if $\dim(Z(G)) = 1$. For any semisimple monoid M , there is a special relationship between Λ' and $S \subseteq X(T)$. If $\alpha \in \Delta$ then by Theorem 10.20 of [6] there exists $e_\alpha \in \Lambda'$ such that $P(e_\alpha) = P_{\Delta \setminus \{\alpha\}}$. Moreover, e_α is unique. See Lemma 4.7 below.

Definition 2.5. Let $C \subseteq \Lambda'$ be the core.

a) Define $\pi : \Delta \rightarrow C$ by $\pi(\alpha) = \theta(e_\alpha)$.

b) Write $\Delta = \sqcup_{h \in C} \Delta_h$, where $\Delta_h = \pi^{-1}(h)$.

Proposition 2.6. a) If $e \in \Lambda_h$ then

$$\lambda_*(e) = \{\alpha \in \lambda_*(h) \mid s_\alpha s_\beta = s_\beta s_\alpha \text{ for all } \beta \in \lambda^*(e)\}.$$

b) If $e \in \Lambda_h$ and $f \in \Lambda_k$ then

$$e \leq f \text{ if and only if } h \leq k \text{ and } \lambda^*(e) \subseteq \lambda^*(f).$$

Proof. Consider a). Since $e \geq h$, $\lambda_*(e) \subseteq \lambda_*(h)$. Let $\alpha \in \lambda^*(e)$. Since $W(e) = W_*(e) \times W^*(e)$, $s_\alpha s_\beta = s_\beta s_\alpha$ for all $\beta \in \lambda^*(e)$. So it remains to prove the reverse inclusion. Now $E(e\overline{T})$ is the face lattice of a polytope (see § 3). So e is the join of the nonzero minimal idempotents of $E(e\overline{T})$. Hence,

$$e = \vee \{xe'x^{-1} \mid e' \in \Lambda_1, e \geq e', x \in W^*(e)\}.$$

Let $\alpha \in \lambda_*(h)$ be such that $s_\alpha s_\beta = s_\beta s_\alpha$ for all $\beta \in \lambda^*(e)$. Then $s_\alpha x = x s_\alpha$ for all $x \in W^*(e)$. Let $e' \in \Lambda_1, e \geq e'$. Since $e \in \Lambda_h, h \geq e'$. Let $x \in W^*(e)$. However, $\alpha \in \lambda^*(h)$. So,

$$s_\alpha x e' x^{-1} = x s_\alpha e' x^{-1} = x e' x^{-1} = x e' s_\alpha x^{-1} = x e' x^{-1} s_\alpha.$$

By the above join formula for e , and Proposition 7.5 of [6], $s_\alpha x = x s_\alpha$. Thus, $s_\alpha \in W(e)$. Now s_α commutes with all the nonzero minimal idempotents in $E(e\bar{T})$, and thus, $e s_\alpha$ has the same property. Thus, $e s_\alpha$ commutes with all idempotents of $e\bar{T}$. Since $eW(e)$ acts faithfully on $E(e\bar{T})$, it follows from Chapter 10 of [6] that $e s_\alpha = e$. Hence, $s_\alpha \in W_*(e)$ and $\alpha \in \lambda_*(e)$.

For b), let $h \leq k$ and $\lambda^*(e) \subseteq \lambda^*(f)$. Let $e' \in \Lambda_1, e \geq e'$. Then $e' \leq h \leq k \leq f$. Let $x \in W^*(e) \subseteq W^*(f)$. Then,

$$f x e' x^{-1} = x f e' x^{-1} = x e' x^{-1}.$$

Hence, $x e' x^{-1} \leq f$. So, by the above join formula for e , $e \leq f$. The converse is clear. \square

3 The Cross Section Lattice and the Weyl Chamber

Let M be a normal, irreducible, reductive algebraic monoid with zero element. For the results of this section, it is not necessary to impose any other restrictions on M .

As always, we denote by G the unit group of M . G is a connected, reductive group which is open and dense in M in the Zariski topology. We let $T \subseteq G$ be a maximal torus of G and $\bar{T} \subseteq M$ the Zariski closure of T in M . $X(T)$ is the group of *characters* of T and $X(\bar{T}) \subseteq X(T)$ the monoid of characters of \bar{T} .

\bar{T} is called a *D-monoid* since its unit group is an algebraic torus, which is often referred to as a *D-group*. We can write

$$\bar{T} = \sqcup_{e \in E(\bar{T})} eT$$

where $E(\bar{T}) = \{e \in \bar{T} \mid e^2 = e\}$ is the set of *idempotents* of \bar{T} .

$E(\bar{T})$ is a finite semilattice under multiplication and consequently it is partially ordered by declaring $e \leq f$ if $ef = fe = e$. We denote by $E_1(\bar{T})$ the set of minimal elements of $E(\bar{T})$ for this ordering. For $e \in E(\bar{T})$ we let

$$E_1(\bar{e}T) = \{f \in E_1(\bar{T}) \mid ef = f\}.$$

We need to show how the cross section lattice Λ can be described in terms relating $X(\bar{T})$ with the set of dominant weights

$$X(T)_+ = \{\chi \in X(T) \mid \Delta_\alpha(\chi) \geq 0 \text{ for all } \alpha \in \Delta\}$$

where $\Delta_\alpha : X(T) \rightarrow \mathbb{Z}$ is defined by the equation $\chi - s_\alpha(\chi) = \Delta_\alpha(\chi)\alpha$.

Let $e \in E(\bar{T})$ and consider

$$\mu_e = \left\{ \chi \in X(e\overline{T}) \subseteq X(\overline{T}) \mid \begin{array}{l} \chi \neq 0 \text{ and} \\ \chi_{e\overline{T} \setminus eT} = 0 \end{array} \right\}$$

where $X(e\overline{T}) \subseteq X(\overline{T})$ via $\overline{T} \rightarrow e\overline{T}$, $z \rightarrow ez$. One can easily check that

$$X(\overline{T}) \setminus \{0\} = \bigsqcup_{e \in E(\overline{T})} \mu_e .$$

Lemma 3.1. *Let $\alpha \in \Delta$ and $e \in E(\overline{T})$. The following are equivalent:*

- a) $U_\alpha e = eU_\alpha e$.
- b) *Either $U_\alpha e = eU_\alpha$, or else $U_\alpha f = f$ for all $f \in E_1(\overline{eT})$.*

Proof. In case $U_\alpha e \neq e$, and $eU_\alpha \neq e$, one obtains that $U_\alpha e = eU_\alpha e$. Otherwise, if $U_\alpha f = f$ for all $f \in E_1(\overline{eT})$ yet $U_\alpha e \neq eU_\alpha e$ (i.e. $U_\alpha e \neq eU_\alpha$ and $U_\alpha e \neq e$) then $eU_\alpha = e \neq U_\alpha e$ is the only other possibility. So $\sigma_\alpha e \sigma_\alpha \neq e$ and thus, $\sigma_\alpha f \sigma_\alpha \neq f$ for some $f \in E_1(\overline{eT})$. But then $fU_\alpha = feU_\alpha = fe = f$. Hence, $U_\alpha f \neq f$, since $f\sigma_\alpha \neq \sigma_\alpha f$. Contradiction. \square

The key to getting geometric control of the cross section lattice is to study the relationship between the U_α 's and the idempotents of \overline{T} . For this we need a result from [10]. Lemma 3.6 a) of [10] states the following.

Let $f \in E_1(\overline{T})$, and let $\chi \in \mu_f$. Then the following are equivalent:

- a) $U_\alpha f = \{f\}$,
- b) $\Delta_\alpha(\chi) \geq 0$.

Lemma 3.2. a) *The following are equivalent:*

- i) $U_\alpha e = eU_\alpha$ (equivalently, $s_\alpha e = es_\alpha$);
- ii) *there exists $\chi \in \mu_e$ such that $\Delta_\alpha(\chi) = 0$.*

b) *Assuming $U_\alpha e \neq eU_\alpha$, the following are equivalent:*

- i) $U_\alpha f = f$ for all $f \in E_1(\overline{eT})$;
- ii) $\Delta_\alpha(\chi) > 0$ for all $\chi \in \mu_e$.

Proof. For a), first note that, for $\chi \in \mu_e$, $\Delta_\alpha(\chi) = 0$ if and only if $s_\alpha(\chi) = \chi$. But $s_\alpha(\mu_e) = \mu_{e'}$ where $e' = s_\alpha e s_\alpha$. So $s_\alpha(\mu_e) \cap \mu_e \neq \emptyset$ iff $s_\alpha e s_\alpha = e$. So if $\chi \in \mu_e$ and $s_\alpha(\chi) = \chi$, then $\chi \in s_\alpha(\mu_e) \cap \mu_e$. Conversely, if $\chi_1 \in s_\alpha(\mu_e) \cap \mu_e \neq \emptyset$, then $\chi_1 s_\alpha(\chi_1) \in \mu_e$ and $\Delta_\alpha(\chi) = 0$.

For b), assume first that $U_\alpha f = f$ for all $f \in E_1(\overline{eT})$. Since $f \in E_1(\overline{T})$, \overline{fT} is isomorphic with K , being a normal, one-dimensional reductive monoid with zero. Thus $K[\overline{fT}] = K[\chi_f]$ for some unique $\chi_f \in X(\overline{T})$. Then from Lemma 3.6 of [10], $\Delta_\alpha(\chi_f) \geq 0$. Now

$$K[\overline{eT}] = K[\chi \mid \chi^n \in \langle \chi_{f_1}, \dots, \chi_{f_2} \rangle \text{ for some } n > 0] \quad (*)$$

where $\{f_i\}_{i=1}^s = E_1(e\bar{T})$. Now if $\Delta_\alpha(\chi_f) = 0$ for all $f \in E_1(e\bar{T})$ then $s_\alpha(\chi_f) = \chi_f$ for all $f \in E_1(e\bar{T})$. So by (*), $s_\alpha(K[e\bar{T}]) = K[e\bar{T}]$ and so $s_\alpha e = es_\alpha$. Thus, $U_\alpha e = eU_\alpha$, a contradiction. So $\Delta_\alpha(\chi_f) > 0$ for some $f \in E_1(e\bar{T})$. Hence, $\Delta_\alpha(\chi) > 0$ for all $\chi \in \mu_e$. Conversely, suppose $\Delta_\alpha(\chi_f) > 0$ for some $f \in E_1(e\bar{T})$. Consider

$$\chi = \chi_f^N \chi_{f_2} \cdot \dots \cdot \chi_{f_s} \in \mu_e$$

where $N > 0$ and $E_1(e\bar{T}) = \{f, f_2, \dots, f_s\}$. Then $\Delta_\alpha(\chi) < 0$ if $N \gg 0$. Contradiction. \square

Theorem 3.3. *The following are equivalent for $e \in E(\bar{T}) \setminus \{0\}$:*

- a) $e \in \Lambda \setminus \{0\}$
- b) *There exists $\chi \in \mu_e$ such that $\Delta_\alpha(\chi) \geq 0$ for all $\alpha \in \Delta$.*

Proof. Now $e \in \Lambda' = \Lambda \setminus \{0\}$ iff for all $\alpha \in \Delta$ either $U_\alpha e = eU_\alpha$ or else $U_\alpha e \neq eU_\alpha$ and $U_\alpha f = f$ for all $f \in E_1(e\bar{T})$. By Lemma 3.2 this is equivalent to:

For each $\alpha \in \Delta$, either

$$\Delta_\alpha(\chi) = 0 \text{ for some } \chi \in \mu_e$$

or else

$$\Delta_\alpha(\chi) > 0 \text{ for all } \chi \in \mu_e .$$

Thus, $e \in \Lambda \setminus \{0\}$ if and only if for all $\alpha \in \Delta$ either

- i) $\Delta_\alpha(\chi) = 0$ for some $\chi \in \mu_e$, or else
- ii) $\Delta_\alpha(\chi) > 0$ for all $\chi \in \mu_e$.

So b) implies a).

Conversely, if $e \in \Lambda \setminus \{0\}$ then $\Delta = \Delta_1 \sqcup \Delta_2$, where

$$\begin{aligned} \Delta_1 &= \{\alpha \in \Delta \mid s_\alpha e = es_\alpha\} \quad \text{and} \\ \Delta_2 &= \{\alpha \in \Delta \mid s_\alpha e \neq es_\alpha\} . \end{aligned}$$

Let $\chi_0 \in \mu_e$ and define

$$\chi = \prod_{w \in W_{\Delta_1}} w(\chi_0) \in \mu_e .$$

Then $\Delta_\alpha(\chi) = 0$ for all $\alpha \in \Delta$. But $\Delta_\alpha(\chi) > 0$ for all $\alpha \in \Delta_2$. \square

Theorem 3.3 has a very appealing geometric interpretation.

One can identify $E(\bar{T}) \setminus \{0\}$ with the face lattice \mathcal{F} of the rational polyhedral cone $X(\bar{T}) \otimes \mathbb{Q}^+ \subseteq X(T) \otimes \mathbb{Q}$. Furthermore, $X(\bar{T}) \otimes \mathbb{Q}^+$ is W -invariant. We can think of $\mu_e \otimes \mathbb{Q}^+$ as the topological interior of $X(e\bar{T}) \otimes \mathbb{Q}^+ \in \mathcal{F}$. Theorem 3.3 says that

$$\Lambda = \left\{ e \in E(\bar{T}) \mid \begin{array}{l} \text{the interior of } X(e\bar{T}) \otimes \mathbb{Q}^+ \\ \text{meets } X(\bar{T})_+ \otimes \mathbb{Q}^+ \end{array} \right\} .$$

Clearly, $|C\ell_W(e) \cap \Lambda| = 1$ for all $e \in E(\overline{T})$.

Recall that a reductive monoid M is *semisimple* if the center of G is one-dimensional and M has a zero element. In this case the zero element of M is in the closure \overline{Z} of Z the one-dimensional connected center of M . As Z is contained in any maximal torus T of G , we have in particular that $\overline{Z} \subseteq \overline{T}$. Thus we obtain the induced (dual) map on the corresponding character monoids:

$$\gamma : X(\overline{T}) \rightarrow X(\overline{Z}) \cong \mathbb{N}.$$

This γ determines, on the associated rational polyhedral cones, a homomorphism

$$\zeta : X(\overline{T}) \otimes \mathbb{Q}^+ \rightarrow X(\overline{Z}) \otimes \mathbb{Q}^+ \cong \mathbb{Q}^+,$$

by setting $\zeta = \gamma \otimes 1$. For M semisimple we make the following definition.

Definition 3.4. Let

$$\mathcal{P} = \zeta^{-1}(1).$$

\mathcal{P} is the *polytope of M* .

From the above results, \mathcal{P} is W -invariant, and the face lattice \mathcal{F} of \mathcal{P} is canonically identified with $E(\overline{T})$. Furthermore, we can identify Λ as a subset of \mathcal{F} using Theorem 3.3.

Example 3.5. Let $M = M_n(K)$, the semisimple monoid of $n \times n$ matrices over K . In this case $Z = \{\alpha I_n \mid \alpha \in K^*\}$, where I_n is the identity $n \times n$ matrix. If T is the D -group of invertible diagonal matrices then \overline{T} is the set of diagonal matrices and $\zeta : X(\overline{T}) \otimes \mathbb{Q}^+ \rightarrow X(\overline{Z}) \otimes \mathbb{Q}^+$ is easily identified with the map

$$\rho : (\mathbb{Q}^+)^n \rightarrow \mathbb{Q}^+$$

defined by $\rho(s_1, \dots, s_n) = \sum_i s_i$. The polytope here is

$$\mathcal{P} = \{(s_1, \dots, s_n) \in (\mathbb{Q}^+)^n \mid \sum_i s_i = 1\}.$$

The face lattice of \mathcal{P} is easily identified with $E(\overline{T}) \setminus \{0\}$. Notice that characters are written additively in this setup.

4 The Type Map of a 2-reducible Monoid

Let M be a 2-reducible, semisimple monoid. Our terminology is well chosen because of the following.

Proposition 4.1. *Let M be a semisimple monoid. The following are equivalent:*

- a) M is 2-reducible.

- b) i) *There is a rational representation $\rho : M \rightarrow \text{End}(V \oplus W)$ so that ρ is finite as a morphism, and V and W are irreducible M -summands.*
- ii) *M is not \mathcal{J} -irreducible (see Remark 4.6 below)*

Proof. If M is 2-reducible, let $\Lambda_1 = \{e, f\}$. There exist irreducible representations $\rho_1 : M \rightarrow \text{End}(V)$ and $\rho_2 : M \rightarrow \text{End}(W)$ such that $\rho_1(e) \neq 0$ and $\rho_2(f) \neq 0$. It is easy to check that $\rho = \rho_1 \oplus \rho_2$ does the job. Conversely, if the conditions of b) are satisfied, let $\Lambda_1 = \{e_1, \dots, e_r\}$, where $r \geq 2$. We can assume $e_1(V) \neq 0$. But then $e_1(V)$ generates V as an M -module, and so $e_2(V) = 0$ since $e_1 G e_2 = 0$. Hence $e_2(W) \neq 0$. But now for any $i > 2$, $e_i(V \oplus W) = 0$. So $r = 2$. \square

In this section, we determine Λ and $\lambda : \Lambda \rightarrow 2^\Delta$ in terms of certain invariants (I_+, I_-) and (Δ_+, Δ_-) .

Write

$$\Lambda_1 = \{e_+, e_-\}.$$

Then

$$C = \{e_+, e_-, e_0\},$$

where $e_0 = e_+ \vee e_-$. Let

$$I_+ = \lambda_*(e_+), \quad I_- = \lambda_*(e_-) \text{ and } I_0 = \lambda_*(e_0).$$

Then

$$I_0 = I_+ \cap I_-.$$

By Definition 2.4 c),

$$\Lambda' = \Lambda_+ \sqcup \Lambda_- \sqcup \Lambda_0$$

where $\Lambda_+ = \Lambda_{e_+}$, $\Lambda_- = \Lambda_{e_-}$ and $\Lambda_0 = \Lambda_{e_0}$ and by Definition 2.5 b)

$$\Delta = \Delta_+ \sqcup \Delta_- \sqcup \Delta_0$$

where, $\Delta_+ = \pi^{-1}(e_+)$, $\Delta_- = \pi^{-1}(e_-)$, and $\Delta_0 = \pi^{-1}(e_0)$. Hence

- i) $\alpha \in \Delta_+$ if $e_\alpha \geq e_+$ and $e_\alpha \not\geq e_-$.
- ii) $\alpha \in \Delta_-$ if $e_\alpha \geq e_-$ and $e_\alpha \not\geq e_+$.
- iii) $\alpha \in \Delta_0$ if $e_\alpha \geq e_+$ and $e_\alpha \geq e_-$.

See the paragraph just preceding Definition 2.5 for the definition of e_α . See also Lemma 4.7 for more details.

By Proposition 2.6, our problem is reduced to determining $\lambda^*(\Lambda_+)$, $\lambda^*(\Lambda_-)$ and $\lambda^*(\Lambda_0)$.

Let M be a reductive monoid with zero, and let $\mathcal{U}(M) = \{J \subseteq M \mid J = GeG \text{ for some } e \in \Lambda\}$. The elements of \mathcal{U} are called \mathcal{J} -classes. Associated with each minimal nonzero \mathcal{J} -class $J \in \mathcal{U}(M)$, there is an irreducible representation $\rho : M \rightarrow \text{End}(V)$ such that $\rho(e) \neq 0$ for any $e \in E(J)$. Then let $\{J_1, \dots, J_m\}$ be the minimal nonzero \mathcal{J} -classes of M , and let ρ_i ,

$i = 1, \dots, m$, be the corresponding irreducible representations of M such that $\rho_i(J_i) \neq 0$. Assume that $M_j = \overline{\rho_j(M)}$ is a \mathcal{J} -irreducible monoid of type I_j . Finally, let

$$\rho = (\rho_1, \dots, \rho_m) : M \rightarrow \Pi \text{End}(V_i).$$

Definition 4.2. We define a *multilined closure of type $\mathbf{I}=(I_1, \dots, I_m)$* to be the closure $M(I_1, \dots, I_m)$ of $\rho(M)$ in $\Pi \text{End}(V_i)$. $M(I_1, \dots, I_m)$ is called *the multilined closure associated to M* .

The following proposition was first proved in [3] by the first named author and Zhuo Li. For convenience, we include a proof.

Theorem 4.3. *Let Λ_i (respectively, λ_i) be the cross section lattice (respectively, type map) of M_i . Define the following subset of $\Lambda_1 \times \dots \times \Lambda_m$:*

$$\Lambda_{\mathbf{I}} = \{(e_{Y_1}, \dots, e_{Y_m}) \mid Y_i \subseteq \lambda_j(Y_j), \text{ if } Y_i \neq \emptyset\}.$$

Define

$$\lambda_{\mathbf{I}}(e_{Y_1}, \dots, e_{Y_m}) = \bigcap_{i=1}^m \lambda_i(e_{Y_i}).$$

If Λ is the cross section lattice of $M(\mathbf{I})$, and λ is its type map, then

$$\Lambda = \Lambda_{\mathbf{I}}, \text{ and}$$

$$\lambda = \lambda_{\mathbf{I}}.$$

The inclusion $M \subseteq M(\mathbf{I})$, induces a bijection on minimal nonzero \mathcal{J} -classes.

Proof. By the structure of \mathcal{J} -irreducible monoids [9], $\Lambda \subseteq \Lambda_1 \times \dots \times \Lambda_m$. Let $(e_2, \dots, e_m) \in \Lambda$, $e_i = e_{Y_i}$, $i = 1, \dots, m$. We claim that $e \in \Lambda_{\mathbf{I}}$. Otherwise, for some nonempty Y_i, Y_j , we would have $Y_i \not\subseteq Y_j$. Then for some simple root $\alpha \in \Delta$, $s = s_\alpha \in Y_i$ while $s \notin \lambda_j(Y_j)$. Let $P = P(e)$ and $P^- = P^-(e)$. Then the root subgroup of α ,

$$X_\alpha \subseteq R_u(P_{\lambda_j(Y_j)}) \subseteq R_u(P),$$

since $P \subseteq P_{\lambda_j(Y_j)}$. Hence $X_\alpha e = e$. Since $s \in C_W(e_i)$, $X_\alpha \subseteq C_G(e_i)$. Let

$$H = \{g \in G \mid ge_i = e_i g = e_i\} \subseteq C_G(e_i).$$

since $H \subseteq C_G(e_i)$ is a normal subgroup with $X_\alpha \subseteq H$, $s \in W(H)$, the Weyl group of H . Now $Q = C_B(e_i)H$ is a parabolic subgroup of $C_G(e_i)$, and thus $Q = C_B(e_i)W_K C_B(e_i)$. Since H is normal in $C_G(e_i)$, the component $A \subseteq S$ of s in Y_i is contained in K . Let $\{f_i\}$ be the set of minimal elements of Λ_i . Then $e_i \geq f_i$, and

$$Qf_i \subseteq HBf_i = Hf_iBf_i = f_iBf_i$$

Hence, $Q \subseteq P(f_i) = P_{I_i}$. Then $A \subseteq I_i$, a contradiction. Hence, $e \in \Lambda_{\mathbf{I}}$. Clearly, $\lambda(e) = \lambda_{\mathbf{I}}(e)$.

We now prove the converse, namely that $\lambda_{\mathbf{I}}(e) \subseteq \lambda$. For $Y \subseteq S$, let

$$e_Y = (e_{Y_1}, \dots, e_{Y_m}).$$

where Y_i is the union of the components of Y not contained in I_i . Then for all i, j ,

$$Y_i \subseteq Y \subseteq \lambda_j(Y_j).$$

Hence, $e_Y \in \Lambda_{\mathbf{I}}$. Clearly, $e_Y e_Z = e_{Y \cap Z}$, for all $Y, Z \subseteq S$. Let

$$\widehat{Y} = \{(e_{Z_1}, \dots, e_{Z_m}) \mid Z_i = 0 \text{ or } Y_i\}$$

We claim that

$$\Lambda_{\mathbf{I}} = \cup_{Y \subseteq S} \widehat{Y}$$

Let $0 \neq e = (e_1, \dots, e_m) \in \Lambda_{\mathbf{I}}$, $Y = \lambda(e)$. If $e_i = e_Z \neq 0$, then

$$Z \subseteq Y \subseteq \lambda_i(e_i).$$

Hence, $Y_i = Z$, and so $e \in \widehat{Y}$. This proves the claim.

To finish the proof, we need to show that $\Lambda_{\mathbf{I}} \subseteq \Lambda$, and we do this by induction on m . If $m = 1$, this is just the \mathcal{J} -irreducible case. So assume $m > 1$. By the above discussion, if $(e_1, \dots, e_m) \in \Lambda$ and $e_i = 1$, then

$$e_j = 1 \text{ or } 0 \text{ for all } j = 1, \dots, m. \quad (*)$$

Let P be a standard maximal parabolic subgroup of G . Let P^- be its opposite, and let $L = P \cap P^-$. Let T_0 be the identity component of the center of L . So $\dim(T_0) = 1$. For $i = 1, \dots, m$, let

$$E(\overline{\{\alpha \rho_i(t) \mid t \in T_0\}}) = \{1, e_i, f_i, 0\}$$

with $P(e_i) = P^-(f_i) = P$. By the above discussion (concerning \widehat{Y}), it suffices to show that $(e_1, \dots, e_m) \in \Lambda$. But by the induction hypothesis, $(e_1, \dots, e_{m-1}, 0) \in \overline{E(T_1)}$, where

$$T_1 = \{(\alpha_1 \rho_1(t), \dots, \alpha_m \rho_m(t)) \mid \alpha_i \in K^*, i = 1, \dots, m\}.$$

Clearly,

$$E(\overline{T_1}) \subseteq \oplus_{i=1}^m \{1, e_i, f_i, 0\}.$$

By (*), $1 = (1, \dots, 1)$ covers $(1, \dots, 1, 0)$, and $(1, \dots, 1, 0)$ covers $(e_1, \dots, e_{m-1}, 0)$ in $E(\overline{T_1})$. But $E(\overline{T_1})$ is a relatively complemented lattice. Hence there exists $e \in E(T_1)$ such that $e \neq (1, 1, \dots, 1, 0)$, and such that

$$(e_1, \dots, e_{m-1}, 0) < e < 1$$

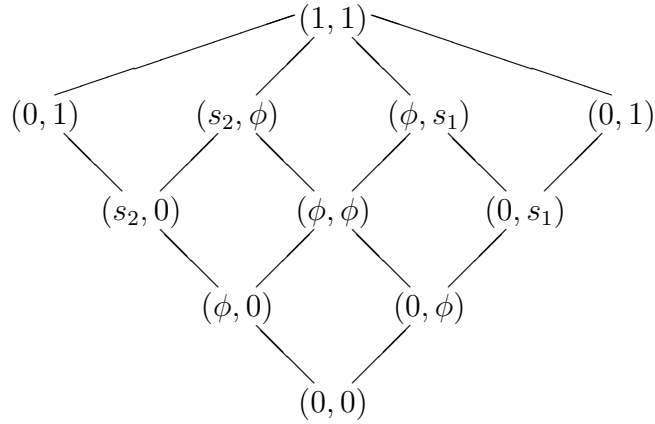
So again by (*), either $e = (e_1, \dots, e_{m-1}, e_m)$, or else $e = (e_1, \dots, e_{m-1}, f_m)$. However, $P(e_1, \dots, e_{m-1}, f_m)$ is not parabolic. Hence, $e = (e_1, \dots, e_{m-1}, e_m)$, completing the proof. \square

Example 4.4. Let $G_0 = Sl_3(K)$ so that $S = \{s_1, s_2\}$. Let $\rho_1 = id : G_0 \longrightarrow Gl_3(K)$, and let $\rho_2 : Sl_3(K) \longrightarrow Gl_3(K)$ be defined by $\rho_2(x) = (x^{-1})^t$. Define $M_1 = \overline{K^* \rho_1(G_0)}$, and $M_2 = \overline{K^* \rho_2(G_0)}$. Thus

$$M_1 \text{ is of type } J_1 = \{s_1\},$$

$$M_2 \text{ is of type } J_2 = \{s_2\}.$$

By Theorem 4.3, $M(J_1, J_2)$ has cross-section lattice as depicted in the diagram below.



Remark 4.5. If M is 2-reducible but not semisimple, then $\dim(Z(G)) = 2$. In this situation,

- a) $\lambda^*(\Lambda_+) = \{X \subseteq \Delta \mid \text{no component of } X \text{ is in } I_+\}$
- b) $\lambda^*(\Lambda_-) = \{X \subseteq \Delta \mid \text{no component of } X \text{ is in } I_-\}$
- c) $\lambda^*(\Lambda_0) = \{X \subseteq \Delta \mid \text{no component of } X \text{ is in } I_0\}$

With these assumptions, M is the special case of the multilined closure where $n = 2$ (Here, n is the number of minimal, nonzero orbits of M). The multilined closure is described in detail, in Theorem 4.3 above. See also Remark 4.6 below for the case $n = 1$, which is referred to as the \mathcal{J} -irreducible case. The semisimple 2-reducible situation is more interesting geometrically than the other 2-reducible cases.

Remark 4.6. We recall that a reductive monoid M with zero is \mathcal{J} -irreducible if $M \setminus \{0\}$ has exactly one minimal $G \times G$ -orbit. See [9] for the systematic study of this important class of reductive monoids. The type map λ of each \mathcal{J} -irreducible monoid is determined by $\lambda(e_0) \subseteq \Delta$, where $e_0 \in \Lambda'$ is the unique minimal element. Indeed, if we let $\lambda(e_0) = J_0$, then by Theorem 4.16 of [9],

$$\Lambda = \{e_I \mid I \subseteq \Delta, \text{ no connected component of } I \text{ lies in } J_0\}.$$

Furthermore, the type map is given by

$$\lambda(e_I) = I \cup \{\alpha \in J_0 \mid s_\alpha s_\beta = s_\beta s_\alpha \text{ for all } \beta \in I\}$$

so that

$$\lambda^*(e_I) = I \text{ and } \lambda_*(e_I) = \{\alpha \in J_0 \mid s_\alpha s_\beta = s_\beta s_\alpha \text{ for all } \beta \in I\}.$$

See [9] for more details, and some interesting examples. We shall freely use the results from [9] about \mathcal{J} -irreducible monoids in the proof of Theorem 4.8. Notice that Proposition 2.6 includes the \mathcal{J} -irreducible case as a special case.

We now return to the 2-reducible case.

Lemma 4.7. $\Delta_+ \neq \phi$ and $\Delta_- \neq \phi$.

Proof. Choose a maximal $e \in \Lambda_+$. Then e is covered by some $f \in \Lambda_0$. f is unique, since if e is also covered by $h \in \Lambda_0$, and $f \neq h$, then $e = fh \geq e_0$, a contradiction. Thus, both fMf and M_f are \mathcal{J} -irreducible, and hence semisimple. Hence, $\lambda(e) = \Delta \setminus \{\alpha\}$ for some $\alpha \in \Delta$. This $e \in \Lambda$ is actually unique with $\lambda(e) = \Delta \setminus \{\alpha\}$ (The connected center Z of $C_G(e)$ is two dimensional. So \overline{Z} has exactly four idempotents $\{e, f, 0, 1\}$. $P(f)$ is the opposite parabolic of $P(e)$. But then $B \not\subseteq P(f)$, so that $f \notin \Lambda$.) In any case, $\alpha \in \Delta_+$. Similarly, $\Delta_- \neq \phi$. \square

We usually refer to this “ e ” in the proof of Lemma 4.7 as e_α . See Definition 2.5 and the preceding remarks.

As we already mentioned, we want to determine $\lambda : \Lambda \rightarrow 2^\Delta$ in terms of I_+, I_-, Δ_+ and Δ_- . By Proposition 2.6, it suffices to determine the sets $\lambda^*(\Lambda_+), \lambda^*(\Lambda_-)$ and $\lambda^*(\Lambda_0)$. Let

$$\mathcal{A}_+ = \{X \subseteq \Delta \mid \text{no component of } X \text{ is contained in } I_+, \Delta_+ \not\subseteq X\}$$

$$\mathcal{A}_- = \{X \subseteq \Delta \mid \text{no component of } X \text{ is contained in } I_-, \Delta_- \not\subseteq X\}$$

$$\mathcal{A}_0 = \{X \subseteq \Delta \mid \text{no component of } X \text{ is contained in } I_0, \text{ and either } \Delta_+ \not\subseteq X \text{ and } \Delta_- \not\subseteq X \text{ or else } \Delta_+ \cup \Delta_- \subseteq X\}$$

Theorem 4.8.

$$a) \lambda^*(\Lambda_+) = \mathcal{A}_+$$

$$b) \lambda^*(\Lambda_-) = \mathcal{A}_-$$

$$c) \lambda^*(\Lambda_0) = \mathcal{A}_0$$

In all cases, λ^* is injective.

Proof. Suppose first, that $X \in \mathcal{A}_+$. Then $\alpha \notin X$ for some $\alpha \notin \Delta_+$. By Theorem 10.20 of [6], there exists $e \in \Lambda_+$ such that $\lambda(e) = \Delta \setminus \{\alpha\}$. So $X \subseteq \lambda(e)$. By Proposition 2.6 a), $X \subseteq \lambda^*(e)$. Now eMe is a \mathcal{J} -irreducible monoid of type $I_+ \cap \lambda^*(e)$. Since no component of X is contained in I_+ , there exists $f < e$ such that $\lambda^*(f) = X$. Clearly, $f \in \Lambda_+$.

Conversely, let $f \in \Lambda_+$. Let $e \in \Lambda_+$ be maximal such that $f \leq e$. By the proof of Lemma 4.7, $|\lambda(e)| = |\Delta| - 1$. So, $\lambda(e) = \Delta \setminus \{\alpha\}$ for some $\alpha \in \Delta_+$. Also, $\lambda_*(f) \subseteq \lambda_*(e) \subseteq \Delta \setminus \{\alpha\}$ by iv) of Remark 2.3. So $\lambda^*(f) \in \mathcal{A}_+$, by Proposition 2.6 a).

Similarly, $\lambda^*(\Lambda_-) = \mathcal{A}_-$.

To prove c), we proceed by induction on $\dim(M)$. Let $f \in \Lambda_0$. Then $f \leq e$ for some maximal $e \neq 1$. So eMe is a 2-reducible monoid. First, suppose that eMe is not semisimple. Then by Remark 4.5, $\lambda(e) = \lambda^*(e) = \Delta \setminus \{\alpha_1, \alpha_2\}$. By Proposition 2.6, $\alpha_1, \alpha_2 \notin \Delta_0$. Suppose $\alpha_1, \alpha_2 \in S_+$. Then there exist $e_1, e_2 \in \Lambda_+$ such that $\lambda(e_1) = \Delta \setminus \{\alpha_1\}$ and $\lambda(e_2) = \Delta \setminus \{\alpha_2\}$.

By Remark 4.5, there exists $h \in \Lambda_+$ such that e covers h and $\lambda(h) \subseteq \Delta \setminus \{\alpha_1, \alpha_2\}$. By Proposition 2.6 b), $h < e_1$ and $f < e_2$. But $\dim(T_h) = 2$, since $\{1, e_2, h\}$ is a maximal chain. On the other hand, $\{1, e_1, e_2, e, h\} \subseteq E(\overline{T}_h)$. This is a contradiction since $|E(\overline{T}_h)| = 4$. Similarly, $\alpha_1, \alpha_2 \in \Delta_-$ leads to a contradiction. So assume that $\alpha_1 \in \Delta_+$ and $\alpha_2 \in \Delta_-$. So by Proposition 2.6 b), $\lambda^*(f) \subseteq \lambda^*(e) = \Delta \setminus \{\alpha_1, \alpha_2\}$. So $\lambda^*(f) \in \mathcal{A}_0$.

Next assume that eMe is semisimple. Then $\lambda(e) = \lambda^*(e) = \Delta \setminus \{\beta\}$ for some $\beta \in \Delta_0$. Correspondingly, in eMe , let

$$\Delta \setminus \{\beta\} = \Delta'_+ \sqcup \Delta'_0 \sqcup \Delta'_-.$$

Let λ_1 denote λ in eMe . We claim that $\Delta_+ = \Delta'_+$. Let $\alpha \in \Delta_+$. Since eMe is a semisimple monoid, there exists $e_1 < e$ such that $\lambda_1(e_1) = \Delta \setminus \{\alpha, \beta\}$. If $\lambda(e_1) = \Delta \setminus \{\alpha\}$, then $e_1 \in \Lambda_+$, and hence $\alpha \in \Delta'_+$. So assume $\lambda(e_1) = \Delta \setminus \{\alpha, \beta\}$. Now $\lambda(e_2) = \Delta \setminus \{\alpha\}$ for some $e_2 \in \Lambda_+$. However,

$$\beta \notin \lambda^*(e_2) \implies \lambda^*(e_2) \subseteq \Delta \setminus \{\beta\} \implies e_2 \leq e$$

and

$$e_2 \leq e \implies \lambda_1(e_2) = \Delta \setminus \{\alpha, \beta\} \implies e_1 = e_2 \implies \alpha \in \Delta'_+.$$

So let $\beta \in \lambda^*(e_2)$. Since $e_2 \in \Lambda_+$, e_2Me_2 is \mathcal{J} -irreducible, and hence semisimple. Let λ_2 denote λ for e_2Me_2 . There exists $e_3 < e_2$ such that $\lambda_2(e_3) = \lambda_2(e_2) \setminus \{\beta\}$. So

$$\Delta \setminus \{\alpha, \beta\} = \lambda_*(e_2) \cup (\lambda^*(e_2) \setminus \{\beta\}) \subseteq \lambda_2(e_3) \cup \lambda_*(e_2) \subseteq \lambda(e_3).$$

If $\lambda(e_3) = \Delta \setminus \{\beta\}$, then $e_3 = e \in \Lambda_0$, a contradiction. Hence, $\lambda(e_3) = \Delta \setminus \{\alpha, \beta\}$. By Proposition 2.6 b), $e_3 < e$. So $\alpha \in \Delta'_+$. Thus, $\Delta_+ \subseteq \Delta'_+$. Similarly, $\Delta_- \subseteq \Delta'_-$.

Suppose that $\alpha \in \Delta'_+$, $\alpha \notin \Delta_+$. Then $\alpha \notin \Delta_-$ since $\Delta_- \subseteq \Delta'_-$. So $\alpha \in \Delta_0$. There exists $e_1 \in \Lambda_+$ with $e_1 < e$ such that $\lambda_1(e_1) = \Delta \setminus \{\alpha, \beta\}$. Since $\alpha \notin \Delta_+$, $\lambda(e_1) \neq \Delta \setminus \{\alpha\}$. So $\lambda(e_1) = \Delta \setminus \{\alpha, \beta\}$. Now $\lambda(e_2) = \Delta \setminus \{\alpha\}$ for some $e_2 \in \Lambda_0$, since $\alpha \in \Delta_0$. By Proposition 2.6 b), $e_1 < e_2$. So $e_1 < ee_2$. By ii) of Remark 2.3,

$$\Delta \setminus \{\alpha, \beta\} = (\Delta \setminus \{\alpha\}) \cap (\Delta \setminus \{\beta\}) \subseteq \lambda(ee_2).$$

By Proposition 2.6 b), $\lambda(ee_2) \neq \Delta \setminus \{\alpha\}$ or $\Delta \setminus \{\beta\}$. So $\lambda(ee_2) = \Delta \setminus \{\alpha, \beta\}$. So $\lambda_1(e_1) = \lambda_1(ee_2) = \Delta \setminus \{\alpha, \beta\}$. Since eMe is semisimple, $e_1 = ee_2 \in \Lambda_0$, a contradiction. Hence, $\Delta'_+ \subseteq \Delta_+$. So $\Delta'_+ = \Delta_+$. Similarly, $\Delta'_- = \Delta_-$. By the induction hypothesis, $\lambda^*(f) \in \mathcal{A}_0$. Thus, $\lambda^*(\Lambda_0) \subseteq \mathcal{A}_0$.

Conversely, let $X \in \mathcal{A}_0$. Suppose first that $\Delta_+ \cup \Delta_- \subseteq X$, $X \neq \Delta$. Then $X \subseteq \Delta \setminus \{\beta\}$ for some $\beta \in \Delta_0$. There exists $f \in \Lambda_0$ such that $\lambda(f) = \Delta \setminus \{\beta\}$. If fMf is semisimple, then $\Delta'_+ = \Delta_+$ and $\Delta'_- = \Delta_-$ as above; and by the induction hypothesis, $\lambda^*(f') = X$ for some $f' \in \Lambda_0$, $f' \leq f$. If fMf is not semisimple, then the same is true by Remark 4.6.

Suppose next that $\Delta_+ \not\subseteq X$ and $\Delta_- \not\subseteq X$. Let $\alpha \in \Delta_+, \beta \in \Delta_-$ be such that $X \subseteq \Delta \setminus \{\alpha, \beta\}$. We first show that there exists $f \in \Lambda_0$ such that $\lambda(f) = \Delta \setminus \{\alpha, \beta\}$. Now there exists $e \in \Lambda_+$ such that $\lambda(e) = \Delta \setminus \{\alpha\}$. Then M_e and eMe are both semisimple. Suppose $\beta \in \lambda_*(e)$. Then there exists $f > e$ such that $\lambda_1(f) = \lambda^*(e) \setminus \{\beta\}$, where λ_1 is λ for M_e . So in M (using iv) of Remark 2.3),

$$\Delta \setminus \{\alpha, \beta\} = (\lambda_*(e) \setminus \{\beta\}) \cup \lambda^*(e) \subseteq \lambda(f).$$

Since $f > e \geq e_+$, $f \notin \Lambda_-$. Hence, $\lambda(f) \neq \Delta \setminus \{\beta\}$. So $\lambda(f) = \Delta \setminus \{\alpha, \beta\}$. So e is central in fMf , and thus fMf is not \mathcal{J} -irreducible. Hence, $f \notin \Lambda_+$. Thus $f \in \Lambda_0$.

Assume next that $\beta \in \lambda^*(e)$. Then there exists $e_1 < e$ such that $\lambda_1(e_1) = \lambda_*(e) \setminus \{\beta\}$, where λ_1 is λ for eMe . So by iv) of Remark 2.3,

$$\Delta \setminus \{\alpha, \beta\} = (\lambda^*(e) \setminus \{\beta\}) \cup \lambda_*(e) \subseteq \lambda(e_1).$$

Since $e_1 < e$, $e_1 \notin \Lambda_-$. Hence $\lambda(e_1) \neq \Delta \setminus \{\beta\}$. So $\lambda(e_1) = \Delta \setminus \{\alpha, \beta\}$. Hence, e is central in M_{e_1} . So M_{e_1} has at least four central idempotents. So let f be a central idempotent of M_{e_1} such that $f \notin \{1, e, e_1\}$. Then $\Delta \setminus \{\alpha, \beta\} \subseteq \lambda(f)$ by iv) of Remark 2.3. Since $f > e_1$, $f \notin \Lambda_-$. So, $\lambda(f) \neq \Delta \setminus \{\beta\}$. Since $f \neq e$, $\lambda(f) \neq \Delta \setminus \{\alpha\}$. Thus, $\lambda(f) = \Delta \setminus \{\alpha, \beta\}$. If $f \in \Lambda_+$, then fMf is \mathcal{J} -irreducible, and e_1 is a central idempotent; a contradiction. So $f \in \Lambda_0$.

There exists $f \in \Lambda_0$ such that $\lambda(f) = \Delta \setminus \{\alpha, \beta\}$. So either M_f is not semisimple, or fMf is not semisimple. Suppose M_f is not semisimple. There exists $f' > f$, $f' \neq 1$, such that f' is central in M_f . By iv) of Remark 2.3, $\Delta \setminus \{\alpha, \beta\} \subseteq \lambda(f')$. Since $f' \in \Lambda_0$, $\alpha \in \Delta_+$, $\beta \in \Delta_-$, $\lambda(f') \neq \Delta \setminus \{\alpha\}$ and $\lambda(f') \neq \Delta \setminus \{\beta\}$. So $\lambda(f') = \Delta \setminus \{\alpha, \beta\}$. Thus by iv) of Remark 2.3, and Proposition 2.6 b), $f = f'$; a contradiction. So M_f is semisimple. But then fMf is not semisimple. Since $X \subseteq \Delta \setminus \{\alpha, \beta\}$, $X \subseteq \lambda_*(f)$. By Remark 4.6, $\lambda^*(f') = X$ for some $f' \in \Lambda_0$, $f' \leq f$. Thus, $\mathcal{A} \subseteq \lambda^*(\Lambda_0)$. This concludes the proof. □

Corollary 4.9. *The partial order on Λ is determined as follows. Let $e, f \in \Lambda$. Then the following are equivalent:*

- a) $e \leq f$;
- b) i) $\lambda^*(e) \subseteq \lambda^*(f)$, and
ii) $e, f \in \Lambda_+$; $e, f \in \Lambda_-$; $e, f \in \Lambda_0$; $e \in \Lambda_+$, $f \in \Lambda_0$; or $e \in \Lambda_-$, $f \in \Lambda_0$.

Proof. This is straightforward using Proposition 2.6 and Theorem 4.8. □

5 Calculating the Type Map Geometrically

In the previous section we found the exact combinatorial description of the type map

$$\lambda : \Lambda \rightarrow 2^\Delta.$$

of a 2-reducible monoid by first identifying the necessary combinatorial invariants (I_+ , I_-) and (Δ_+ , Δ_- , Δ_0). In this section we determine some geometric refinements of that situation by calculating the decomposition

$$\Delta = \Delta_+ \sqcup \Delta_- \sqcup \Delta_0$$

in terms of the coordinates of $\Lambda_1 = \{e_+, e_-\}$, thought of as vertices of the polytope \mathcal{P} of M . The problem here is to determine which decompositions of Δ are possible for a 2-reducible

monoid M of type (I_+, I_-) . This is no longer a purely combinatorial problem. It depends on the “shape” of the Cartan matrix.

Let M be a 2-reducible, semisimple monoid, and let T, \bar{T}, Λ , etc. have the usual meanings. As above, let \mathcal{P} be the polytope of M . By Theorem 3.3, we have a canonical bijection

$$\iota : \Lambda_1 \rightarrow \{x, y\}.$$

We write $\iota(e_+) = x$ and $\iota(e_-) = y$ where $\{x, y\}$ is the set of vertices of \mathcal{P} that are contained in $X(\bar{T})_+ \otimes \mathbb{Q}^+$. Let $Bd(\mathcal{P})$ be the boundry of \mathcal{P} . For $\alpha \in \Delta$ let

- i) $H_\alpha = \text{Span}_{\mathbb{Q}}(\Delta \setminus \{\alpha\})$
- ii) $H_\alpha^+ = \text{Cone}_{\mathbb{Q}^+}(\Delta \setminus \{\alpha\})$

For $\alpha \in \Delta$, let $\omega_\alpha \in X(\bar{T})_+ \otimes \mathbb{Q}^+$ be the fundamental dominant weight that is orthogonal to H_α .

Lemma 5.1. *For any $\alpha \in \Delta$ there is a unique $z_\alpha \in \mathbb{Q}^+\omega_\alpha$ such that*

$$(z_\alpha + H_\alpha) \cap \mathcal{P} = (z_\alpha + H_\alpha) \cap Bd(\mathcal{P}) \neq \emptyset$$

Furthermore,

- i) $z_\alpha \in Bd(\mathcal{P})$;
- ii) $(z_\alpha + H_\alpha) \cap \mathcal{P}$ is the face F of \mathcal{P} corresponding to e_α .

Proof. Let $e = e_\alpha$ be the unique idempotent such that $\lambda(e_\alpha) = \Delta \setminus \{\alpha\}$ (see Lemma 3.4). Let $F \in \mathcal{F}$ be the face of \mathcal{P} corresponding to $e \in \Lambda$. So

$$\mathbb{Q}_+\omega_\alpha \subseteq \mu_e,$$

and thus, $\mathbb{Q}^+\omega_\alpha \cap F = \{z_\alpha\}$ (since F is a subset of $z_\alpha + H_\alpha$, it must be orthogonal to $\mathbb{Q}\omega_\alpha$). Clearly, $F \subseteq Bd(\mathcal{P})$.

Let $I = \Delta \setminus \{\alpha\}$. Then F is W_I -invariant. Thus, $F - z_\alpha$ is also W_I -invariant. But, $\mathbb{Q}\omega_\alpha \cap (F - z_\alpha) = \{0\}$, and so $(F - z_\alpha)^{W_I} = \{0\}$. Thus, $F - z_\alpha \subseteq H_\alpha$. Hence, $F \subseteq H_\alpha + z_\alpha$. \square

The authors would like to thank Hugh Thomas for the proof of the following Lemma.

Lemma 5.2. *Let $\alpha \in \Delta$. Then the following are equivalent.*

- a) $x \in z_\alpha + H_\alpha$.
- b) $x \in z_\alpha + H_\alpha^+$.
- c) $e_+ \leq e_\alpha$.

The corresponding result holds with x replaced by y and e_+ replaced by e_- .

Proof. For $\alpha \in \Delta$, let $C_1 = \text{Cone}(\{\omega_\alpha\})$ and $C_2 = \text{Cone}((\Delta \setminus \{\alpha\}) \cup \{\omega_\alpha\})$. We claim that $C_1 \subseteq C_2$. It suffices to show that $\omega_\beta \in C_2$ for any $\beta \in \Delta \setminus \{\alpha\}$. Now

$$X(T) \otimes \mathbb{Q} = H_\alpha \oplus \mathbb{Q}\omega_\alpha,$$

an orthogonal decomposition. So let

$$\omega_\beta = x + c\omega_\alpha.$$

It suffice to show that

- i) $c \geq 0$, and
- ii) $x \in \text{Cone}(\Delta \setminus \{\alpha\})$.

To get i), we use the inner product. Since $\omega_\beta = x + c\omega_\alpha$, we obtain

$$\langle \omega_\beta, \omega_\alpha \rangle = \langle x, \omega_\alpha \rangle + c \langle \omega_\alpha, \omega_\alpha \rangle.$$

But $\langle x, \omega_\alpha \rangle = 0$, so that

$$c = \langle \omega_\beta, \omega_\alpha \rangle / \langle \omega_\alpha, \omega_\alpha \rangle,$$

and it is well known that this is non negative.

To get ii), first notice that

$$\langle \beta, x \rangle = \langle \beta, \omega_\beta - c\omega_\alpha \rangle = \langle \beta, \omega_\beta \rangle = 1.$$

But if $\gamma \neq \beta, \alpha$, we obtain

$$\langle \gamma, x \rangle = \langle \gamma, \omega_\beta - c\omega_\alpha \rangle = 0.$$

So x is the dual of β in the root system $(H_\alpha, \Delta \setminus \{\alpha\})$. But it is well known that for any root system, the cone generated by the fundamental weights is contained in the cone generated by the positive weights, since the inverse of the Cartan matrix has positive entries. This proves the claim.

Now let $\mathcal{C} = \text{Cone}(\{\omega_\alpha | \alpha \in \Delta\})$. We claim now that

$$(z_\alpha + H_\alpha) \cap \mathcal{C} = (z_\alpha + H_\alpha^+) \cap \mathcal{C}$$

From our first claim,

$$\mathcal{C} = \bigcup_{r \geq 0} (rz_\alpha + H_\alpha^+) \cap \mathcal{C} = \mathcal{C} = \bigcup_{r \geq 0} (rz_\alpha + H_\alpha) \cap \mathcal{C}.$$

But $(rz_\alpha + H_\alpha) \cap (sz_\alpha + H_\alpha) = \emptyset$ if $r \neq s$. So,

$$(z_\alpha + H_\alpha) \cap \mathcal{C} \subseteq (z_\alpha + H_\alpha^+) \cap \mathcal{C},$$

and this establishes the second claim.

Now assume that $x \in z_\alpha + H_\alpha$. Then since $x \in \mathcal{P} \cap \mathcal{C}$, we get from the claim that $x \in z_\alpha + H_\alpha^+$. So clearly, a) and b) are equivalent. a) and c) are equivalent, since from Lemma 5.1, $(z_\alpha + H_\alpha) \cap \mathcal{P}$ is the face of \mathcal{P} corresponding to $e_\alpha \in \Lambda$; while $x \in \mathcal{P}$ is the vertex of \mathcal{P} corresponding to e_+ . This completes the proof. \square

Corollary 5.3. For each $\alpha \in \Delta$, either $x \in z_\alpha + H_\alpha$, or else $y \in z_\alpha + H_\alpha$.

Proof. $\{e_\alpha, e_+, e_-\} \subseteq \Lambda'$, while $\Lambda_1 = \{e_+, e_-\}$. Thus, $e_\alpha \geq e_+$ or else $e_\alpha \geq e_-$. □

Theorem 5.4. Write $x - y = \sum_{\alpha \in \Delta} r_\alpha \alpha$, where $r_\alpha \in \mathbb{Q}$.

a) The following are equivalent:

- i) $r_\alpha > 0$
- ii) $e_\alpha \in \Lambda_+$
- iii) $x \in z_\alpha + H_\alpha^+$, $y \notin z_\alpha + H_\alpha^+$.

b) The following are equivalent:

- i) $r_\alpha < 0$
- ii) $e_\alpha \in \Lambda_-$
- iii) $y \in z_\alpha + H_\alpha^+$, $x \notin z_\alpha + H_\alpha^+$.

c) The following are equivalent:

- i) $r_\alpha = 0$
- ii) $e_\alpha \in \Lambda_0$
- iii) $x \in z_\alpha + H_\alpha^+$, $y \in z_\alpha + H_\alpha^+$.

Proof. In each case, it suffices to show that i) and ii) are equivalent, since by Lemma 5.2, ii) and iii) are equivalent. By Corollary 5.3, exactly one of a) iii), b) iii) or c) iii) occurs.

In case a), $x \in z_\alpha + H_\alpha^+$ and $y \notin z_\alpha + H_\alpha^+$. So

$$x = z_\alpha + \sum_{\beta \neq \alpha} a_\beta \beta$$

and

$$y = z_\alpha + \sum_{\beta \in \Delta} b_\beta \beta.$$

But $b_\alpha < 0$, since y lies in the bounded part of $\mathcal{C} \setminus (z_\alpha + H_\alpha)$, thus “below” the plane $z_\alpha + H_\alpha$. Hence,

$$x - y = \sum_{\beta \neq \alpha} a_\beta \beta - \sum_{\beta \in \Delta} b_\beta \beta = \sum_{\beta \neq \alpha} (a_\beta - b_\beta) \beta - b_\alpha \alpha.$$

Thus $r_\alpha = -b_\alpha > 0$ here. Case b) is similar to case a).

In case c) we can write

$$x = z_\alpha + \sum_{\beta \neq \alpha} a_\beta \beta$$

and

$$y = z_\alpha + \sum_{\beta \neq \alpha} b_\beta \beta.$$

Then

$$x - y = \sum_{\beta \neq \alpha} (a_\beta - b_\beta) \beta$$

and thus, $r_\alpha = 0$ in this case. □

6 Monoids with $I_+ = \Delta \setminus \{\alpha\}$ and $I_- = \Delta \setminus \{\beta\}$

In this section we exhibit some explicit calculations of the type maps of 2-reducible monoids. We restrict our attention to certain monoids with group $G = Gl_{n+1}(K)$ where K is an algebraically closed field. The general problem here is to determine all possible $(+, -, 0)$ -decompositions of Δ that can actually occur for the given I_+ and I_- . We do not yet have a general solution to this intriguing problem. However, the calculations indicate that it has something to do with linear programming problems involving the inverse of the Cartan matrix.

So let $G = Gl_{n+1}(K)$, and let us consider 2-reducible, semisimple monoids M with unit group G . Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots of G , and $\Delta^\vee = \{\omega_1, \dots, \omega_n\}$ the set of fundamental, dominant weights. Then, it is well known that for $i = 0, \dots, n-1$,

$$(n+1)\omega_{i+1} = (n-i)\alpha_1 + 2(n-i)\alpha_2 + \dots + (i+1)(n-i)\alpha_{i+1} + (i+1)(n-i-1)\alpha_{i+2} + \dots + (i+1)\alpha_n.$$

So for convenience, we let

$$x_{i+1} = (n+1)\omega_{i+1}.$$

Let M be a 2-reducible, semisimple monoid with unit group G and assume that $I_+ = \Delta \setminus \{\alpha_1\}$, $I_- = \Delta \setminus \{\alpha_{i+1}\}$. The polytope \mathcal{P} of M is the convex hull of the W -orbit of $\{x, y\} \subseteq X(T_0) \otimes \mathbb{Q}^+$. So x is a rational multiple of x_1 , and y is a rational multiple of x_{i+1} . Without loss of generality, $x = x_1$ and $y = rx_{i+1}$ for some $r > 0$. By the results of Theorem 5.4, we need to calculate

$$x - y = \sum_{i=1}^n r_i \alpha_i.$$

But that is elementary, and we obtain,

- i) $r_j = n - j + 1 - j(r(n - i))$ if $j \leq i$
- ii) $r_j = (1 - (i + 1)r)(n - j + 1)$ if $j > i$

By Corollary 5.3, we must have

- i) $n - r(n - i) > 0$, and
- ii) $(1 - (i + 1)r) < 0$.

So

$$1/(i + 1) < r < n/(n - i).$$

For certain special values of r , r_j can be zero. These values are

$$r = (n - j)/(j + 1)(n - i).$$

In any case, it is an elementary calculation. We summarize our results as follows.

Theorem 6.1. *Let M be a 2-reducible, semisimple monoid with unit group $Gl_{n+1}(K)$, and assume $I_+ = \Delta \setminus \{\alpha_1\}$, $I_- = \Delta \setminus \{\alpha_{i+1}\}$. Write $x = x_1$, $y = rx_{i+1}$ as above. Then*

a) $1/(i + 1) < r < l/(l - i)$;

b) if $1 \leq j \leq i - 1$ and $r = (n - j)/(j + 1)(n - i)$ then

$$\Delta_+ = \{\alpha_1, \dots, \alpha_j\};$$

$$\Delta_- = \{\alpha_{j+2}, \dots, \alpha_n\};$$

c) If $0 \leq j \leq i - 1$ and $(n - j - 1)/(j + 2)(n - i) < r < (n - j)/(j - i)(n - i)$ then

$$\Delta_+ = \{\alpha_1, \dots, \alpha_{j+1}\};$$

$$\Delta_- = \{\alpha_{j+2}, \dots, \alpha_n\}.$$

It is now possible to calculate Λ and λ in each case using Theorem 4.8. The details are left to the reader.

7 Monoids with $I_+ = \phi$ and $I_- = \phi$

It is easy to characterize the pairs (I_+, I_-) that can actually occur as $(\lambda_*(e_+), \lambda_*(e_-))$ for some 2-reducible semisimple monoid M with $\Lambda_1 = \{e_+, e_-\}$. Indeed, let $A, B \subseteq \Delta$ be any two proper subsets. Then $(A, B) = (I_+, I_-)$ for some semisimple, 2-reducible monoid M if and only if either

i) $A \neq B$, or else

ii) $A = B$ and $|\Delta \setminus A| \geq 2$.

In particular, $I_+ = I_- = \phi$ is possible; in fact generic. Notice, this is equivalent to $\{x, y\}$ being a subset of \mathcal{C}^0 , the interior of \mathcal{C} .

Theorem 7.1. *The following are equivalent.*

a) *There exists a 2-reducible, semisimple monoid M with $I_+ = I_- = \phi$ and $(\Delta_+, \Delta_-) = (U, V)$.*

b) *$U \neq \phi, V \neq \phi$ and $U \cap V = \phi$.*

Proof. Obviously, a) implies b). So assume $U, V \subseteq \Delta$ satisfy b). Define

$$\delta = \sum_{\alpha \in U} \alpha - \sum_{\beta \in V} \beta.$$

It is then easy to find $x, y \in \mathcal{C}^0$ so that $x - y = \delta$. So apply Theorem 5.4. \square

8 (\mathcal{J}, σ) -irreducible Monoids Revisited

In this section we use the results of Theorems 4.8 and 5.4 to study the orbit structure of certain reductive monoids M with involution $\sigma : M \rightarrow M$.

Definition 8.1. Let M be an reductive monoid with zero, and suppose that $\sigma : M \rightarrow M$ is a bijective morphism of algebraic monoids. We say that (M, σ) is (\mathcal{J}, σ) -irreducible if the map induced by σ is transitive on the set of minimal $G \times G$ -orbits of $M \setminus \{0\}$.

(\mathcal{J}, σ) -irreducible monoids were studied systematically by Z. Li and the authors in [2, 3, 4]. In all cases, except the ones that contain D_4 as a component, σ^2 induces the identity morphism on the set of $G \times G$ -orbits of M . In such cases, M is a 2-reducible monoid precisely when $M \setminus \{0\}$ has exactly two minimal $G \times G$ -orbits and σ exchanges these orbits. In this section, we discuss several examples where M is 2-reducible and semisimple, and σ is actually an automorphism of M of order two. The purpose of Theorems 4.8 and 5.4 is to identify the minimal information (i.e. Δ_+ and Δ_-) needed to get the type map of M .

Example 8.2. Let M be a 2-reducible, semisimple monoid with unit group $Gl_6(K)$. Assume that there is an automorphism $\sigma : M \rightarrow M$ such that $\sigma^2 = id$ and $\sigma|_{Gl_6(K)}$ is transpose-inverse.

Let $F = \{\lambda_1, \dots, \lambda_5\}$ be the set of fundamental dominant weights of $Sl_6(K)$. Then σ induces the following involution σ^* on F :

$$\sigma^*(\lambda_i) = \lambda_{6-i}$$

From Table 2 on page 295 of [5] we obtain

$$\lambda_1 - \lambda_5 = \frac{1}{6}(4\alpha_1 + 2\alpha_2 - 2\alpha_4 - 4\alpha_5),$$

and

$$\lambda_2 - \lambda_4 = \frac{1}{6}(2\alpha_1 + 4\alpha_2 - 4\alpha_4 - 2\alpha_5).$$

Now any 2-reducible, semisimple monoid M has a representation $\rho : M \rightarrow End(V \oplus W)$, as in Proposition 4.1. If V is the irreducible M -module with highest weight $\lambda \in X(T)_+$, then W is the irreducible M -module with highest weight $\sigma^*(\lambda) \neq \lambda$. Write

$$\lambda = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_4 + a_5\lambda_5$$

where $a_i \geq 0$, and either $a_1 \neq a_5$ or else $a_2 \neq a_4$ (so that $\sigma^*(\lambda) \neq \lambda$). In any case,

$$\begin{aligned} \lambda - \sigma^*(\lambda) &= \frac{1}{6}([4(a_1 - a_5) + 2(a_2 - a_4)]\alpha_1 + [2(a_1 - a_5) + 4(a_2 - a_4)]\alpha_2) \\ &\quad - \frac{1}{6}([2(a_1 - a_5) + 4(a_2 - a_4)]\alpha_4 + [4(a_1 - a_5) + 2(a_2 - a_4)]\alpha_5). \end{aligned}$$

Now

$$\begin{aligned} I_+ &= \{\alpha_i \mid a_i \neq 0\} \\ I_- &= \{\alpha_i \mid a_{6-i} \neq 0\}. \end{aligned}$$

Notice that in all cases, $\Delta_- = \{\alpha_{6-i} \mid \alpha_i \in \Delta_+\}$, while $\alpha_3 \notin \Delta_+ \sqcup \Delta_-$. So it suffices to calculate the possibilities for Δ_+ in terms of λ .

1. $\Delta_+ = \{\alpha_1, \alpha_2\}$ if $2(a_1 - a_5) + (a_2 - a_4) > 0$ and $(a_1 - a_5) + 2(a_2 - a_4) > 0$.
2. $\Delta_+ = \{\alpha_1, \alpha_4\}$ if $2(a_1 - a_5) + (a_2 - a_4) > 0$ and $(a_1 - a_5) + 2(a_2 - a_4) < 0$.
3. $\Delta_+ = \{\alpha_1\}$ if $2(a_1 - a_5) + (a_2 - a_4) > 0$ and $(a_1 - a_5) + 2(a_2 - a_4) = 0$.
4. $\Delta_+ = \{\alpha_2\}$ if $2(a_1 - a_5) + (a_2 - a_4) = 0$ and $(a_1 - a_5) + 2(a_2 - a_4) > 0$.

All other feasible data are obtained by reversing the rôles of λ and $\sigma^*(\lambda)$. But we obtain no new monoids. The potential cases with $\Delta_+ = \{\alpha_1, \alpha_5\}$ or $\{\alpha_2, \alpha_4\}$ are not possible. Also, any situation where $|\Delta_+| \geq 3$ is not possible.

We see from Theorems 4.8 and 5.4 that the type map of M is now determined in each case.

Example 8.3. Let M be a 2-reducible, semisimple monoid with unit group $K^*SO_{2n}(K) \subseteq Gl_{2n}(K)$. Assume that there is an automorphism $\sigma : M \rightarrow M$ such that $\sigma^2 = id$ and $\sigma|_{SO_{2n}(K)}$ is transpose-inverse.

Let $F = \{\lambda_1, \dots, \lambda_{n-2}, \lambda_{n-1}, \lambda_n\}$ be the set of fundamental dominant weights of $SO_{2n}(K)$. Then σ induces the following involution σ^* on F :

$$\sigma^*(\lambda_i) = \lambda_i \text{ if } i \leq n-2,$$

and

$$\sigma^*(\lambda_{n-1}) = \lambda_n, \sigma^*(\lambda_n) = \lambda_{n-1}.$$

From Table 2 on page 296 of [5] we obtain

$$\lambda_m - \lambda_{n-1} = \frac{1}{2}(-\alpha_{n-1} + \alpha_n).$$

As in the previous example, any 2-reducible, semisimple monoid M has a representation $\rho : M \rightarrow End(V \oplus W)$, according with Proposition 4.1. If V is the irreducible M -module with highest weight $\lambda \in X(T)_+$, then W is the irreducible M -module with highest weight $\sigma^*(\lambda) \neq \lambda$. Write

$$\lambda = a_1\lambda_1 + a_2\lambda_2 + \dots + a_{n-2}\lambda_{n-2} + a_{n-1}\lambda_{n-1} + a_n\lambda_n$$

where $a_n \neq a_{n-1}$, (so that $\sigma^*(\lambda) \neq \lambda$). Then

$$\lambda - \sigma^*(\lambda) = \frac{a_n - a_{n-1}}{2}(-\alpha_{n-1} + \alpha_n)$$

Now

$$\begin{aligned} I_+ &= \{\alpha_i \mid a_i \neq 0\} \\ I_- &= \{\alpha_i \mid \bar{a}_i \neq 0\}. \end{aligned}$$

where $\bar{a}_n = a_{n-1}$, $\overline{a_{n-1}} = a_n$ and $\bar{a}_i = a_i$ if $i \leq n-1$. Notice again, that in all cases, $\Delta_- = \{\bar{\alpha} \mid \alpha \in \Delta_+\}$, so we only need to consider the possibilities for Δ_+ in terms of λ . There are just two cases.

1. $\Delta_+ = \{\alpha_{n-1}\}$ if $a_{n-1} \geq a_n$.
2. $\Delta_+ = \{\alpha_n\}$ if $a_n \geq a_{n-1}$.

Again we see from Theorems 4.8 and 5.4 how the type map of M is completely determined in each case. The details are left to the reader. Notice that the two cases yield the same monoid M , since σ^* exchanges α_{n-1} and α_n .

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