Betti Numbers and Permutations

Lex E. Renner
The University of Western Ontario

Abstract

This is a short introduction to the topic of permutahedra, Weyl groups and Betti numbers. It was used to make a poster for a Faculty of Science show and tell in a big hotel downtown.

1 Introduction

We describe four series of polynomials, each arising naturally from some part of mathematics.

1. The **Eulerian polynomial** $E_n(t)$.
2. The **Ascent polynomial** $A_n(t)$ of the symmetric group S_n.
3. The h**-polynomial** $h_n(t)$ of the permutahedron $\mathcal{P}_n$.
4. The **Poincaré polynomial** $P_n(t)$ of a certain, related, projective variety X_n.

The Eulerian polynomials are defined by power series equations involving rational functions. The polynomial $A_n(t)$ is defined in terms of a set of generators of the symmetric group S_n. An h-polynomial $h_n(t)$ is defined by counting up the vertices, edges, facets and so on, of a certain polytope $\mathcal{P}_n$. The Poincaré Polynomial $P_n(t)$ is defined by calculating the Betti numbers of X_n.

Our intention here is to illustrate, using topological explanations, why these seemingly unrelated collections of polynomials turn out to be the same.

2 Polynomials

2.1 Eulerian Polynomials

We start with the well-known identity

$$\sum_{k\geq 1} x^k = \frac{x}{1-x}.$$

Differentiating repeatedly, and then simplifying, we obtain

$$\sum_{k\geq 1} kx^k = \frac{x}{(1-x)^2}(1).$$

$$\sum_{k\geq 1} k^2x^k = \frac{x}{(1-x)^3}(1+x).$$

$$\sum_{k\geq 1} k^3x^k = \frac{x}{(1-x)^4}(1+4x+x^2).$$

$$\vdots$$

$$\sum_{k\geq 1} k^mx^k = \frac{x}{(1-x)^{m+1}}(\sum_{n=0}^{m-1} E(m,n)x^n).$$

By definition,

$$E_m(t) = \sum_{n=0}^{m-1} E(m,n)x^n$$

is the m-th **Eulerian polynomial**. One can prove that

$$E(m,n) = nE(m-1,n) + mE(m,n-1).$$

The first few Eulerian polynomials.

m	$E_m(t)$
1	1
2	$1+t$
3	$1+4t+t^2$
4	$1+11t+11t^2+t^3$
5	$1+26t+66t^2+26t^3+t^4$
6	$1+57t+302t^2+302t^3+57t^4+t^5$
7	$1+120t+1191t^2+2416t^3+1191t^4+120t^5+t^6$
8	$1+247t+4293t^2+15619t^3+15619t^4+4293t^5+247t^6+t^7$
9	$1+502t+14608t^2+88234t^3+156190t^4+88234t^5+14608t^6+502t^7+t^8$

Notice that the sum of the coefficients of $E_m(t)$ is $m! = m(m-1)(m-2)\cdots(2)(1)$.

2.2 Ascent Polynomials

A *permutation* is a rearrangement of $\underline{n} = \{1, 2, ..., n\}$. We write $\sigma = (p_1p_2...p_n)$ for the permutation that rearranges $\underline{n}$ by sending 1 to p_1, 2 to p_2, 3 to p_3, and so on. We denote

$$S_n = \{\sigma \mid \sigma \text{ is a permutation of } \underline{n}\}$$

For $\sigma \in S_n$ we define the *length* of σ as

$$l(\sigma) = |\{(i,j) \mid 1 \leq i < j \leq n \text{ and } p_i > p_j\}|.$$

For example, in S_4,

$l(1234) = 0$

$l(2134) = 1$

$l(3214) = 3$

Let $\sigma = (p_1 p_2 ... p_n) \in S_n$. Define

$$A(\sigma) = \{i \mid 1 \leq i \leq n-1 \text{ and } p_i \leq p_{i+1}\},$$

the **ascent set** of σ. For example, in S_4,

$A(1234) = \{1,2,3\}$

$A(1324) = \{1,3\}$

$A(4321) = \phi$

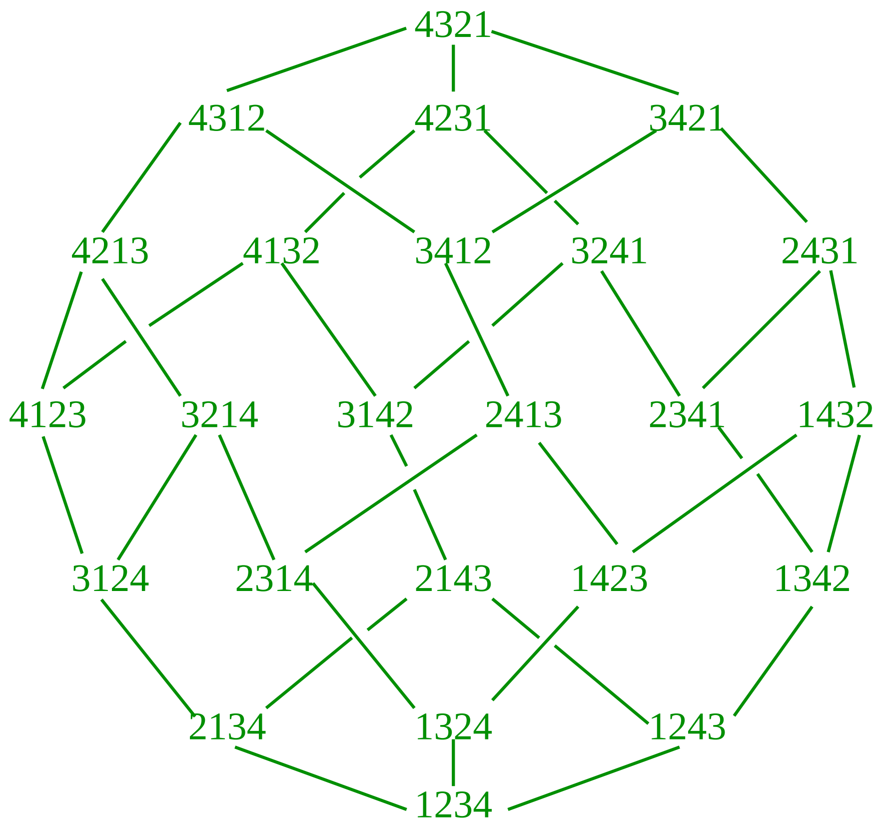
Define

$$a(\sigma) = |A(\sigma)|,$$

and

$$A_n(t) = \sum_{\sigma \in S_n} t^{a(\sigma)}.$$

A_n is the **ascent polynomial** for S_n.



τ is directly above σ, in this picture, if $l(\tau) > l(\sigma)$ and τ is obtained from σ by composing on the left with a simple switcheroo

$$s_i : i \to i+1.$$

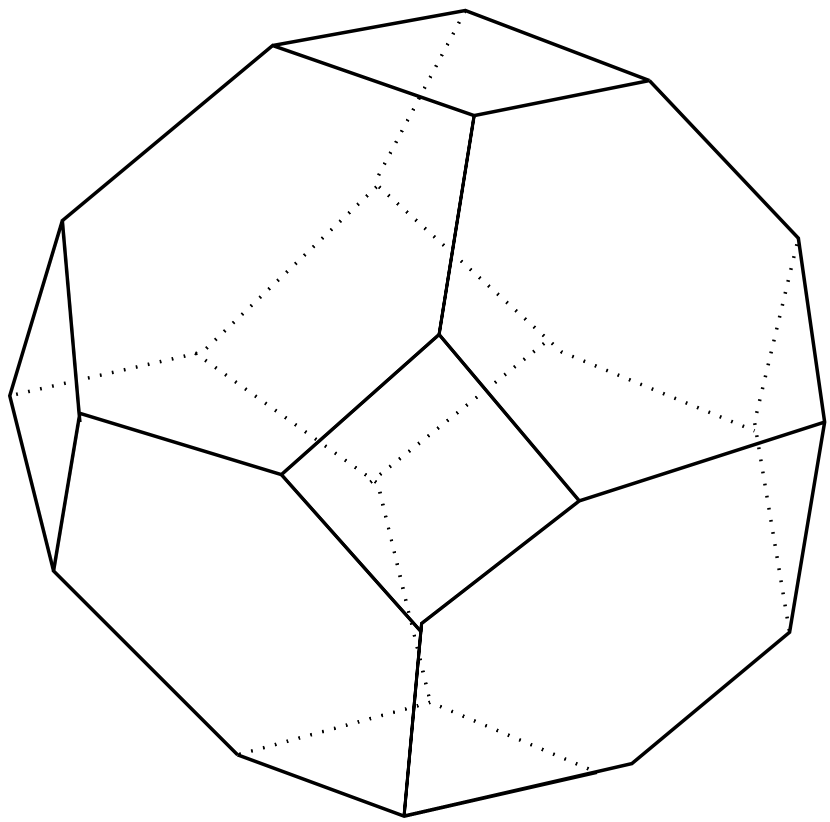
A simple calculation yields that

$$A_4(t) = 1 + 11t + 11t^2 + t^3.$$

2.3 h-polynomials and the Permutahedron

We define a certain n-dimensional, convex polytope $\mathcal{P}_n$, called the *permutahedron*. $\mathcal{P}_n$ is defined to be the convex hull, in $\mathbb{R}^{n+1}$, of the set

$$\{(p_1, p_2, ..., p_{n+1}) \in \mathbb{R}^{n+1} \mid (p_1 p_2 ... p_{n+1}) \in S_{n+1}\}.$$



The h-**polynomial** of $\mathcal{P}_n$ is

$$h_n(t) = \sum_{i=0}^{n-1} f_i (t-1)^i$$

where

$$f_i\text{=the number } i\text{-dimensional faces of } \mathcal{P}_n.$$

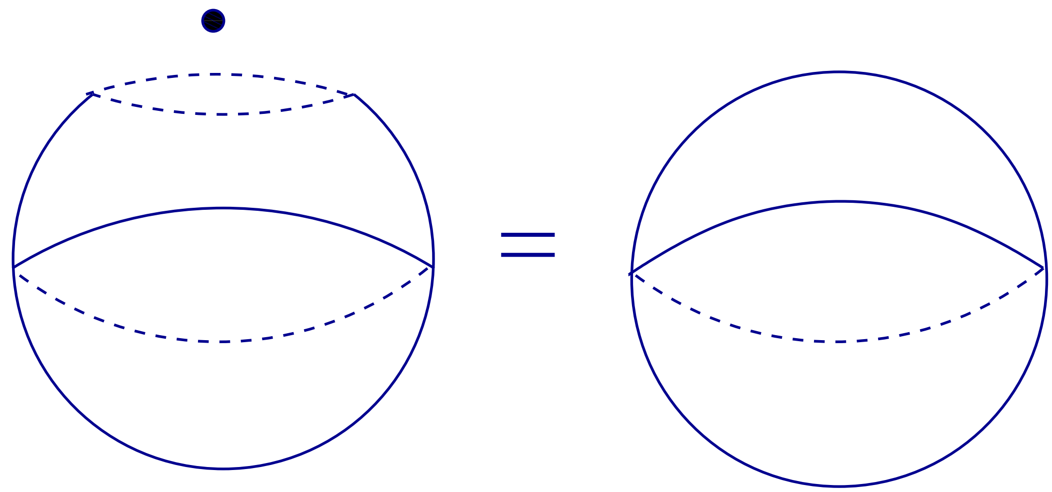
In the above example we obtain

$$h_3(t) = 1 + 11t + 11t^2 + t^3.$$

Notice, in particular, that the coefficients are all positive.

2.4 Betti Numbers

Many topological spaces X of interest can be "assembled" by "glueing" together cells along their boundaries in a controlled way. Such spaces are called **CW-complexes**.



This depicts S^2 (the 2-sphere) as a union of one 0-cell and one 2-cell. This disassembly often results in a procedure to calculate the **homology groups** of X. In the above example, $H_n(S^2) = \mathbb{Z}$, if $n = 0$ or 2, and zero otherwise. This reflects the fact that S^2 has exactly one 3-dimensional "hole" and no others.

More generally, if the space X can be written (as a CW complex) using only even-dimensional cells

$$X = \coprod_{\alpha \in I} C_\alpha,$$

where $C_\alpha \cong \mathbb{R}^{2k_\alpha}$ then, from elementary homological algebra,

$$H_m(X) = \mathbb{Z}^{n_m} \tag{2.0}$$

where n_m is the number of m-dimensional cells. In any case, the **Poincaré polynomial** of X is, by definition,

$$P_X(t) = \sum_{i \geq 0} (-1)^i rank(H_i(X)) t^i.$$

It is of great interest to study and interpret the numbers

$$\beta_i(X) = rank(H_i(X)),$$

the **Betti numbers** of X. If X is a compact, oriented manifold, of real dimension n, then

$$\beta_i(X) = \beta_{n-i}(X).$$

This is known as **Poincaré duality**. If further, X is a nonsingular, projective, algebraic variety over $\mathbb{C}$, of complex dimension d, then by the **Lefschetz Theorem** [5],

$$\beta_i(X) \leq \beta_{i+2}(X)$$

for $0 \leq i < d$.

3 The Theorem

Theorem 3.1. *Let $E_n(t)$ be the n-th Eulerian polynomial, let $h_n(t)$ be the h-polynomial of $\mathcal{P}_n$, and let $A_n(t)$ be the ascent polynomial of S_{n+1}. Then*

$$E_{n+1}(t) = h_n(t) = A_{n+1}(t)$$

Proof. We focus mainly on proving that $h_n(t) = A_{n+1}(t)$ since one can show directly (not easily) that $E_{n+1}(t) = A_{n+1}(t)$ using the fact that they satisfy the same recurrence relations.

To obtain that $h_n(t) = A_{n+1}(t)$ we use topology. It turns out that there is a "naturally occuring" nonsingular, projective, algebraic variety X_n, constructed directly from $\mathcal{P}_n$, such that

i) $X_n = \bigsqcup_{\sigma \in S_{n+1}} C_\sigma$

ii) $C_\sigma \cong \mathbb{C}^{a(\sigma)}$, where $\sigma = (p_1 p_2 ... p_n) \in S_n$ and $a(\sigma) = |\{i \mid 1 \leq i \leq n-1 \text{ and } p_i \leq p_{i+1}\}|$

Thus, from Equation 2.0,

$$P_{X_n}(t) = \sum_{\sigma \in S_n} t^{2a(\sigma)} = A_{n+1}(t^2).$$

We now explain why $P_{X_n}(t) = h_n(t^2)$. As above,

$$X_n = \bigsqcup_{\sigma \in S_{n+1}} C_\sigma$$

where $C_\sigma \cong \mathbb{C}^{a(\sigma)}$. Using the method of [1] one can show that

$$C_\sigma = \bigsqcup_{\alpha \subset A(\sigma)} Z_\sigma^\alpha$$

in such a way that Z_σ^α corresponds to the unique k-dimensional face (where $k = |\alpha|$) of the polytope $\mathcal{P}_n$ which is labelled by the pair (σ, α), with $\sigma \in S_{n+1}$, and $\alpha \subset A(\sigma)$. It turns out that the faces of $\mathcal{P}_n$ correspond bijectively to pairs of elements (σ, α) where $\sigma \in S_{n+1}$ and $\alpha \subset A(\sigma)$. We say that F **likes** σ if F is corresponds to (σ, α) for some $\alpha \subset A(\sigma)$. In any case,

$$t^{2a(\sigma)} = ((t^2-1)+1)^{a(\sigma)} = \sum_{k=0}^{a(\sigma)} \binom{a(\sigma)}{k} (t^2-1)^k = \sum_{F \text{ likes } \sigma} (t^2-1)^{dim(F)},$$

where the last sum is taken over all the faces F that like σ. Hence,

$$A_{n+1}(t^2) = \sum_{\sigma \in S_{n+1}} t^{2a(\sigma)} = \sum_{\sigma} (\sum_{F \text{ likes } \sigma} (t^2-1)^{dim(F)}) = \sum_{\text{all faces of } \mathcal{P}_{n+1}} (t^2-1)^{dim(F)}$$

But the last summation is equal to $h_n(t^2)$ by definition of the h-polynomial. □

Corollary 3.2. *We obtain the following results about the h-polynomial* $h_n(t) = \sum_{i=0}^{n} h_i t^i$ *of the polytope* $\mathcal{P}_n$.

1. **Dehn-Sommerville Equations**: $h_i = h_{n-i}$ *for all* $i \leq [\frac{n}{2}]$.

2. **Unimodality**: $1 = h_0 \leq h_1 \leq ... \leq h_m$ *where* $m = [\frac{n}{2}]$.

Proof. The most elegant proof here uses Poincaré Duality and The Hard Lefschetz Theorem. See [5]. □

4 Vista

This relationships between S_{n+1} and $\mathcal{P}_n$ persists more generally. Indeed, associated with each **Weyl group** (W, S) [3] is an ascent polynomial $A(t) = \sum_{\sigma \in W} t^{a(\sigma)}$, where $a(\sigma) = |\{s \in S \mid l(\sigma) < l(s\sigma)\ \}|$. But also there exists a non-singular, projective variety $X = X(W, S)$ which can be decomposed as a CW-complex in such a way that

i) $X = \bigsqcup_{\sigma \in W} C_\sigma$, and

ii) $C_\sigma \cong \mathbb{C}^{a(\sigma)}$.

$X(W, S)$ is, in fact, a projective **toric variety** [2]. Such varieties have an orbit structure that mirrors the face lattice of a certain polytope (as we saw here in the case of the permutahedron). In [6] it is shown exactly how to construct $X(W, S)$ from (W, S), and how to relate the ascent polynomial of (W, S) to the Poincaré polynomial of $X(W, S)$.

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