MODULAR REPRESENTATIONS OF FINITE MONOIDS OF LIE TYPE

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1 Introduction

Associated with each finite monoid of Lie type is its characteristic. For example, if $M = M_n(F_p)$ then the characteristic of M is p. So it is natural to give special attention to the theory of finite dimensional irreducible representations of M over $\overline{F}_p$, the algebraic closure of F_p. There is already a lot of evidence from [13] to suggest that we are dealing with a remarkable situation. Indeed,

(1) Any irreducible representation $\rho : M \to End(V)$ of M over $\overline{F}_p$ restricts to an irreducible representation $\rho|G$ of the unit group M.

(2) From the work of Richen [3] it follows that the irreducible representation of G can be classified by their *weights* (see section 2 below).

The purpose of this paper is to combine these two results and obtain a detailed identification of the irreducible representations of M in terms of their weights and their associated $\mathcal{J}$-class. Associated with each irreducible representation $\rho : M \to End(V)$ is the weight of $\rho|G$ and also the *apex* of ρ (see section 2 below). So we obtain our complete identification by characterizing the possible combinations of weights and apexes that can actually occur for some irreducible representations of M. This identification brings to light a new combinatorial invariant of monoids of Lie type which, together with the type map (see section 3 below), essentially classifies monoids of Lie type up to central extension. Rather than stating our results in full generality, let us here illustrate the entire situation with a familiar example. We consider the case $M = M_n(F_q)$ where $q = p^k$ for $k > 0$. The first thing to do here is to identify all the relevant invariants of M.

$$
\begin{aligned}
\boldsymbol{\Lambda} &= \left\{ \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \cdots, \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \right\} \\
&= \{e_0, e_1, \dots, e_n\}
\end{aligned}
$$

$$\begin{aligned} \mathcal{U}(M) &= \{J_i | i = 0, \ldots, n\} \quad \text{where} \\ J_i &= \{x \in M | \text{rank}(x) = i\}, \\ S &= \{\alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-1,n}\} \quad \text{where} \end{aligned}$$

$$\alpha_{ij} \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = t_i t_j^{-1}$$

and the type map $\lambda : \mathcal{U}(M) \to 2^S$ is given by

$$\begin{aligned} \lambda(J_0) &= \lambda(J_n) = S \quad \text{and} \\ \lambda(J_i) &= \{\alpha_{12}, \ldots, \alpha_{i-1,i}, \alpha_{i+1,i+2}, \ldots, \alpha_{n-1,n}\} \quad \text{if } 0 < i < n \ . \end{aligned}$$

Recall from [11, 12] that $\lambda(J_i)$ is the type of the parabolic subgroup $P_i = \{g \in G | ge_i = e_i g e_i\}$. One checks that

$$P_i = \left\{ g = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in G\ell_n(F_q) | A \text{ is } i \times i \right\} \ .$$

The new invariant here is

$$v : \mathcal{U}(M) \to 2^S \ .$$

This is defined generally in section 2 below. In the case of $M = M_n(F_q)$, it turns out simply that

$$\begin{aligned} v(J_0) &= S, v(J_n) = \phi \text{ and} \\ v(J_i) &= \{\alpha_{i+1,i+2}, \ldots, \alpha_{n-1,n}\} \text{ if } 0 < i < n \ . \end{aligned}$$

In general, $v(J) \subseteq \lambda(J)$. Furthermore, $\lambda(J)\backslash v(J)$ records the type, as a subset of S, of the unit group of $e_J M e_J$, where $\{e_J\} = \Lambda \cap J$. $v(J)$ records the type as a subset of S of the parabolic subgroup BK, where $B = \{g \in G | ge = ege \text{ for all } e \in \Lambda\}$ and $K = \{g \in G | ge_J = e_J\}$. Notice that K is normal in $P(e_J) = \{g \in g | ge_J = e_J g e_J\}$. So in our example $\lambda(J_i)\backslash v(J_i)$ represents the upper $i \times i$ block, and $v(J_i)$ represents the lower $(n-i) \times (n-i)$ block of P_i. Now given an irreducible represention $\rho : M \to End(V)$ over $\overline{F}_p$ we first recall from [13] that $\rho|G$ is irreducible. By the results of Richen [3], there exists a unique line $Y \subseteq V$ such that $P = \{g \in G | \rho(g)Y = Y\}$ contains B. So P is a parabolic subgroup of type $I(\rho) \subseteq S$. We write $P = P_{I(\rho)}$.

According to Richen's theory [3], ρ corresponds to a *weight*, which is a pair (I, χ) such that $I \subseteq S$ and $\chi : P_I \to \overline{F}_P^*$ is a homomorphism. Furthermore, $\rho|G$ corresponds to $(I(\rho), \chi(\rho))$ where $\rho(g)(y) = \chi(\rho)(g)y$ defines $\chi(\rho) : P \to \overline{F}_p^*$.

On the other hand, ρ determines a $\mathcal{J}$-class $J(\rho) \in \mathcal{U}(M)$. Indeed, by the results of Munn and Ponizovskii [2], there exists a unique smallest $\mathcal{J}$-class $J(\rho) \in \mathcal{U}(M)$, such that $\rho(J(\rho)) \neq 0$. In our example, $J(\rho)$ is the set of matrices x of minimum rank such that $\rho(x) \neq 0$. Say $J(\rho) = J_i$. The main point of this paper is to characterize generally the relationship between $J(\rho)$ and $(I(\rho), \chi(\rho))$.

In our example it turns out that

(i) $v(J(\rho)) \subseteq I(\rho) \subseteq \lambda(J(\rho))$, so that

$$P_{v(J(\rho))} = \left\{ \left(\begin{array}{cc|c} & * & \\ 0 & & * \\ \hline 0 & & * \end{array} \right) \right\} \subseteq P_{I(\rho)} \subseteq P_{\lambda(J(\rho))} = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \right\}$$

(ii) If $P_{I(\rho)} = \left\{ \left(\begin{array}{c|c} & * \\ * & \\ \hline 0 & * \end{array} \right) \right\}$ then $H = \left\{ \left(\begin{array}{cc|c} 1 & 0 & \\ & \ddots & * \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & * \end{array} \right) \right\}$ is in the kernel of $\chi(\rho) : P_{I(\rho)} \to \overline{F}_p^*$.

Conversely, given $J_i \in \mathcal{U}(M)$ and (I, χ) with $\chi : P_1 \to \overline{F}_q^*$, there exists an irreducible representation of M with these data if and only if $v(J_i) \subseteq I \subseteq \lambda(J_i)$, and $\left\{ \left(\begin{array}{c|c} I_i & * \\ \hline 0 & * \end{array} \right) \right\}$ is the kernel of χ.

The main purpose of this paper is to extend the above results to their natural generality (see Theorem 3.1 below). We then go on to discuss a number of related issues, in particular,

(a) Enumerative formulae for the number of irreducible representations.

(b) Conjugacy classes of semisimple elements in reductive monoids.

(c) Betti numbers of compactifications of the adjoint quotient.

2 Background

In this section we briefly summarize some of the background ideas that are required for a proper understanding of our main results. As we have stated in the introduction, our purpose here is to combine the theory of Munn and Ponizovskii on representations of finite semigroups with the theory of Richen on modular representations of finite groups of Lie type. Our point of departure here is the striking fact that irreducible representation of finite monoids of Lie type restrict to irreducible representations of the unit group [13; Corollary 2.7]. The following results are due largely to Munn and Ponizovskii (see [2; Theorem 5.33]).

Let S be a finite regular semigroup and let k be a field. Define an equivalence relation $\mathcal{J}$ on S by declaring $x\mathcal{J}y$ if $SxS = SyS$. The equivalence classes are called *$\mathcal{J}$-classes*. Let $\mathcal{U}(M) = \{J \subseteq M | J \text{ in a } \mathcal{J}\text{-class}\}$. $\mathcal{U}(M)$ is a poset with $J_1 \geq J_2$ if $SJ_1S \supseteq J_2$. For any irreducible representation $\rho : S \to End(V)$ there exists a unique smallest $J \in \mathcal{U}(M)$, called the *apex* of ρ, such that $\rho(J) \neq \{0\}$. $J^0 = J \cup \{0\}$ is a semigroup with $x \cdot y = xy$ if

$xy \in J$ and $x \cdot y = 0$ if $xy \notin J$. Furthermore, ρ determines an irreducible representation $\overline{\rho} : J^0 \longrightarrow End(V)$ via

$$\overline{\rho}(x) = \begin{cases} \rho(x), & x \in J \\ 0, & x = 0 \end{cases}$$

Conversely, given $J \in \mathcal{U}(S)$ and an irreducible representation $\rho_1 : J^0 \longrightarrow End(V)$, there exists an irreducible representation $\rho : S \longrightarrow End(V)$, with apex J such that $\overline{\rho} = \rho_1$. Furthermore, ρ is unique up to equivalence.

The second part of this theory identifies the irreducible representations of J^0 with those of H_e, the $\mathcal{H}$-class (i.e. unit group of eMe) of any $e \in E(J)$. Indeed, if $\rho : J^0 \longrightarrow End(V)$ is irreducible, then $\rho|H : H \longrightarrow End(\rho(e)(V))$ is an irreducible representation of H. Conversely, any irreducible representation $\alpha : H \longrightarrow End(W)$ can be "extended" to an irreducible representation $\rho : J^0 \longrightarrow End(V)$ such that $(\rho|H, \rho(e)(V)) \cong (\alpha, W)$. Here again, the representation ρ is unique up to equivalence. The most interesting open problem here is to relate the dimension of W to the dimension of V. This amounts to computing the rank over k of the image under ρ of the Rees sandwich matrix of J^0.

The other major background ingredient for this paper is Richen's theory which is recorded in detail by Curtis in [3]. Let G be a finite group of Lie type. By definition there exists a reductive group $\underline{G}$ defined over $\overline{F}_p$ and an endomorphism $\sigma : \underline{G} \longrightarrow \underline{G}$ such that $G = \{x \in \underline{G} | \sigma(x) = x\}$. By the results of Richen the irreducible modular representations of G are in one to one correspondence with the set of *weights* $\{(I, \chi) | \chi : L_I \longrightarrow \overline{F}_p^*\}$ where $I \subseteq S$ is a subset of the set of vertices of the Coxeter graph of G and L_I is the corresponding Levi subgroup of G. This correspondence has the following properties:

Given $\rho : G \longrightarrow G\ell(V)$ irreducible and $B \subseteq G$ a Borel subgroup, there exists a unique line $Y \subseteq V$ such that $\rho(B)Y = Y$. Let $P_I = \{g \in G | \rho(g)Y = Y\}$. Then we obtain $\chi : P_I \longrightarrow \overline{F}_p^*$ via $\rho(g)(y) = \chi(g)(y)$. But χ factors through $P_I \longrightarrow P_I/U = L_I$ where $U \subseteq P_I$ is the unipotent radical of P_I. The correspondence mentioned above identifies ρ with (I, χ).

There is one more feature that deserves mention. Suppose $\rho : G \longrightarrow G\ell(V)$ is irreducible. Then there exists an algebraic group $\underline{G}$ defined over $\overline{F}_p$ with endomorphism $\sigma : \underline{G} \longrightarrow \underline{G}$ such that

(i) $G \cong \{x \in \underline{G} | \sigma(x) = x\}$

(ii) There is a representation $\underline{\rho} : \underline{G} \longrightarrow G\ell(V)$ of algebraic groups such that $\underline{\rho}|\underline{G} = \rho$.

See [23; Theorem 13.3]. This will be useful in the next section when we construct finite monoids of Lie type from irreducible representations of finite groups of Lie type.

It is impossible in general to specify the "typical" irreducible represention of a finite group G of Lie type. In fact, the (conjectured) dimension formula for the irreducible module of type (I, χ) has yet to be verified in all cases. For $S\ell_2(F_q)$ one can specify a complete list of modules using Steinberg's tensor product theorem [21] (since the factors are known explicitly in this case). The first step is to obtain the $\otimes$-irreducible modules for $S\ell_2(F_q)$. These are V_0 (the trivial one dimensional module), $V_1 = \overline{F}_q \oplus \overline{F}_q, V_2 = S^2(V_1), V_3 = S^3(V_1), \ldots$ and $V_{p-1} = S^{p-1}(V_1)$, where V_1 has the usual action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax & + & by \\ cx & + & dy \end{pmatrix}$$

and S^i denotes the i^{th} symmetric power. Define

$$\alpha_i : S\ell_2(\overline{F}_p) \to S\ell_2(\overline{F}_p) \text{ via } \alpha_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^\gamma & b^\gamma \\ c^\gamma & d^\gamma \end{pmatrix}$$

where $\gamma = p^i$. Then any module (ρ, V) for $S\ell_2(\overline{F}_p)$ yields other modules $(\rho^{\alpha_1}, V), (\rho^{\alpha_2}, V), \dots$ via $\rho^{\alpha_1}(x)(v) = \rho(\alpha_i(x))(v)$. We denote (ρ^{α_1}, V) by V^{α_1}.

Steinberg's ⊗-product theorem for $S\ell_2$

(a) Any irreducible module of $S\ell_2(\overline{F}_q)$ can be written uniquely as $V_{i_0}^{\alpha_0} \otimes V_{i_1}^{\alpha_1} \otimes \cdots \otimes V_{i_2}^{\alpha_2}$ where $0 \leq i \leq p-1$.

(b) Any irreducible module of $S\ell_2(F_q)$ is the restriction of some irreducible module of $S\ell_2(\overline{F}_q)$. If $V = V_{i_0}^{\alpha_0} \otimes \cdots \otimes V_{i_2}^{\alpha_2}$ where $0 \leq i_j \leq p-1$ and $\sum_{j=0}^{S} p_j^{j_i} \leq q-1$ then V remains irreducible for $S\ell_2(F_q)$.

Thus, the irreducible modules of $S\ell_2(F_q)$ can be indexed by the non-negative integers $m \leq q-1$. To compute $V(m)$, write $m = \sum_{j=0}^{S} p_j^{j_i}$ where $0 \leq i_j \leq p-1$ (p-adic expansion of m). Then $V(m) \cong \bigotimes_{j=0}^{S} V_{i_j}^{a_{ij}}$ and $dim\ V(m) = \prod_{j=0}^{S} (i_j + 1)$. In the terminology of Richen's theory, for $m > 0$, $V(m)$ is the irreducible module of type (ϕ, χ) where $\chi : L_\phi \to F_q^*$ is given by

$$\chi(t) = t^m \text{ and } L_\phi = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| \, t \in F_q^* \right\}$$

is the Levi factor of the upper triangular group (which is the only proper parabolic subgroup of $S\ell_2(F_q)$ up to conjugacy).

We can depict the pattern here using the "3-adic Pascal triangle".

Module	**Dimension**	
$V(0)$	1	⊙
$V(1)$	2	⊙⊙
$V(2)$	3	⊙ ⊙ ⊙
$V(3)$	2	⊙ • • ⊙
$V(4)$	4	⊙ ⊙ • ⊙ ⊙
$V(5)$	6	⊙ ⊙ ⊙ ⊙ ⊙⊙
$V(6)$	3	⊙ • • ⊙ • • ⊙
$V(7)$	6	⊙ ⊙ • ⊙ ⊙ • ⊙ ⊙
$V(8)$	9	⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙

Each row represents the irreducible module with given label and has the indicated dimension (= the number of $\odot$'s in that row). If we label the positions in the i^{th} row $0, 1, 2, \ldots, n$ then there is a $\odot$ in the i^{th} position if and only if $\begin{pmatrix} n \\ i \end{pmatrix} \not\equiv 0 \bmod (3)$.

We now state the main result of [13]. This is the main reason we are able to obtain so much information about irreducible modular representations of finite monoids of Lie type. Let M be a finite monoid of Lie type with unit group G of characteristic p.

2.1 Theorem [13]. *Suppose* $\rho : M \rightarrow End(V)$ *is an irreducible representation of* M *over* $\overline{F}_p$*. Then* $\rho|G$ *is irreducible.*

The reader is reminded here that there is rarely such a direct and appealing relationship between the irreducible representations of a monoid and those of its unit group.

Irreducible representations of $M_n(F_p)$ show up in a very significant way in stable homotopy theory. This is what motivated Harris and Kuhn to study the irreducible representations of $M_n(F_p)$ in [5]. The main theorem of [5] relates in a precise way the irreducible representations of $M_n(F_p)$ to the indecomposable wedge summands of the stable homotopy type of the classifying space of a finite abelian p-group. The first step here is the observation that $M_n(F_p)$ acts on $B(F_p^n)$, the classifying space of F_p^n, induced from the standard action of $M_n(F_p)$ on F_p^n.

3 The Main Theorem

Let M be a finite monoid of Lie type of characteristic p, and let $\rho : M \rightarrow End(V)$ be an irreducible representation of M defined over $\overline{F}_p$. On the other hand, by the theory of Munn and Ponizovskii, ρ determines an apex, $Apex(\rho) \in \mathcal{U}(M)$. But on the other hand $\rho|G$ is irreducible, and so by the theory of Richen $\rho|G$ is determined by its weight $(I(\rho), \chi(\rho))$. In any case, $\rho : M \rightarrow End(V)$ determines the following data:

(i) $J = Apex(\rho) \in \mathcal{U}(M)$

(ii) $I = I(\rho) \in 2^S$

(iii) $\chi = \chi(\rho) : P_I \rightarrow \overline{F}_p^*$.

We consider the following two questions:

(a) Is ρ uniquely determined up to equivalence of representations by (J, I, χ)?

(b) What are the conditions on a triple (J, I, χ) with $\chi : P_I \rightarrow \overline{F}_p^*$ and $J \in \mathcal{U}(M)$, so that there exists an irreducible representation ρ of M with

  (i) $\rho|G$ of type (I, χ)

  (ii) $Apex(\rho) = J$?

To answer these two questions we need some further notions about monoids of Lie type. The reader should consult [11] for a detailed account of this theory (notice however that in [11], monoids of Lie type are referred to as *regular split monoids*).

Let M be a finite monoid of Lie type with unit group G. Let S be the Coxeter-Dynkin diagram of G. Let $U = \mathcal{U}(M)$ set of regular $\mathcal{J}$-classes of M. It turns out that U is the set of two-sided G-orbits of M. Define $GaG \geq GbG$ if $b \in MaM$. In this way, U becomes a lattice. There is a cross-section of idempotents $\Lambda = \{e_J | J \in U\} \subseteq E(M)$, such that $J = Ge_JG$, and for all $J_1, J_2 \in U, e_{J_1}e_{J_2} = e_{J_2}e_{J_1} = e_{J_1 \wedge J_2}$. Λ is called a *cross-section lattice*. Furthermore, $E(J) = \{ge_Jg^{-1} | g \in G\}$. There is a *type map*

$$\lambda : U \to 2^S$$

such that for all $J \in U$

$$P(e_J) = P_{\lambda(J)}.$$

Here $P(e_J) = C_G^r(e) =: \{G \in G | ge = ege\}$, and $P_I \subseteq G$ denotes the *parabolic subgroup of type I* (see [1; Section 2.5]).

We now introduce a new invariant. Let Λ be a cross-section lattice and let $e \in \Lambda$. Then $\{e\} = \Lambda \cap J$ and $C_G^r(e) = P_{\lambda(J)}$, where $\lambda(J) \subseteq S$. If $B \subseteq C_G^r(e)$ is a Borel subgroup and $H = \{g \in G | ge = e\}$ then BH is a parabolic subgroup containing B. Hence, we define

$$v(J) \in 2^S \text{ via } BH = P_{v(J)}.$$

Notice also, if we let $K_J = \{g \in G | ge = eg = e\}$ then $P_{v(J)} = BK_J$. We can now state the main theorem (in particular, answering questions a) and b) above).

3.1 Theorem. *Let $I \in 2^S$ and $J \in \mathcal{U}(M)$. Assume that $v(J) \subseteq I \subseteq \lambda(J)$. Define*

$$\alpha_{I,J} = \{\chi : L_I \to \overline{F}_q^* | \chi(g) = \chi(h) \text{ if } e_Jg = e_Jh\}.$$

Then there is a one to one correspondence between the irreducible representations of M and the set

$$\bigsqcup_{\substack{I \in 2^S, J \in \mathcal{U}(M) \\ v(J) \subseteq I \subseteq \lambda(J)}} \alpha_{I,J}.$$

Under this correspondence $\chi \in \alpha_{I,J}$ corresponds to the unique irreducible representation $\rho : M \to End(M)$ such that

(i) $Apex(\rho) = J$

(ii) There is a line $Y \subseteq V$ such that $\{g \in G | \rho(g)Y = Y\} = P_I$.

(iii) If $g \in P_I$ and $y \in Y$ then $\rho(g)(y) = \chi(g)y$.

Proof. Let $\rho : M \to End(V)$ be irreducible with apex $J \in \mathcal{U}(M)$. So V is also an irreducible J^0-module, and for $e \in E(J)$, $e(V)$ is an irreducible H_e-module. But $H_e = eC_G(e)$, and so $e(V)$ is also an irreducible $C_G(e)$-module. Now $C_G(e) \subseteq G$ is the Levi factor of $P_{\lambda(J)} = C_G^r(e)$, and so it is also a finite group of Lie type. Hence, Richen's theory applies to the $C_G(e)$-module $e(V)$. Thus, for any Borel subgroup $B_0 \subseteq C_G(e)$ there exists a unique line $Y \subseteq e(V)$ such that $\rho(B_0)Y = Y$. ☐

Claim: Let $H = \{g \in G | \rho(g)Y = Y\}$. Then $H \subseteq C^r_G(e)$ and H contains a Borel subgroup of G.

Proof of Claim: First of all H contains $B = B_0 R_u(C^u_G(e))$, which is a Borel subgroup of G. So let $g \in H$, and let $v = ev \in Y$ be a non-zero element. Then for some $\gamma \in \overline{F}^*_p$ $0 \neq gv = gev = \gamma ev = egev$. In particular, $egev \neq 0$. But $ege \underset{J}{\leq} e$ while if $z \underset{J}{<} e$ then $zv = 0$ since J is the apex of ρ. Hence, $ege\mathcal{H}e$. But then by [9; Theorem 3] $g \in C^\ell_G(e)C^r_G(e) = R_u(C^\ell_G(e))C^r_G(e)$. Hence, $B \subseteq H \subseteq R_u(C^\ell_G(e))C^r_G(e)$. Suppose $H \nsubseteq C^r_G(e)$. Then there exists $\alpha \in \Phi$ such that $U_\alpha \subseteq H$ yet $U_\alpha \nsubseteq C^r_G(e)$. But $B \subseteq C^r_G(e)$ and so $U_\alpha \nsubseteq B$. But then $U_{-\alpha} \subseteq B \subseteq H$. Hence, $G_\alpha = \langle U_\alpha, U_{-\alpha} \rangle \subseteq H$, and so $s_\alpha \in H$. But $s_\alpha \notin C^r_G(e)$ since $U_{-\alpha} \subseteq C^r_G(e)$ and $U_\alpha = s_\alpha U_{-\alpha} s_\alpha \nsubseteq C^r_G(e)$. Consider the double coset decomposition

$$G = \bigsqcup_{w \in W/W(C^r_G(e))} R_u(C^\ell_G(e)) w C^r_G(e)$$

Since the union is disjoint, $s_\alpha \in R_u(C^\ell_G(e))C^r_G(e)$ implies that $s_\alpha \in W(C^r_G(e))$. But as above this is not possible. This establishes the claim.

Observe that $K_J \subseteq H$, and so $P_{v(J)} = BK_J \subseteq H \subseteq P_{\lambda(J)} =: C^r_G(e)$. So we can now summarize the relevant properties of an irreducible representation $\rho : M \to End(V)$ with apex $J \in \mathcal{U}(M)$.

(a) Let $H = \{g \in G | \rho(g)Y = Y\}$. Then $H = P_I$ is parabolic and $v(J) \subseteq I \subseteq \lambda(J)$.

(b) If $g \in K_J$ then $\rho(g)y = y$ for all $y \in Y$.

(c) $\rho|G$ is the irreducible representation of type (I, χ), where χ is defined via $\rho(g)(y) = \chi(g)y$ for $g \in L_I$.

On the other hand, let $\rho' : M \to End(V')$ be an irreducible representation with apex J, and $H' = P_I$ with character χ. Then by Richen's results, $(\rho'|C_G(e), \rho'(e)(V)) \cong (\rho|C_G(e), \rho(e)(V))$ since they have the same (I, χ). But then by Munn-Ponizovskii, ρ and ρ' are equivalent. Thus, the correspondence

$$\rho \rightarrowtail (I, J, \chi)$$

is injective. To complete the proof, it remains only to be shown that all possible invariants (I, J, χ) actually arise from irreducible representations of M. But this is now a counting problem. It is easy to check, using Richen's results, that the number of irreducible representations of H_e is

$$\sum_{v(J) \subseteq I \subseteq \lambda(J)} |\alpha_{I,J}|$$

Thus, by Munn-Ponizovskii, there are exactly

$$\sum_{J \in \mathcal{U}(M)} \sum_{v(J) \subseteq I \subseteq \lambda(J)} |\alpha_{I,J}|$$

irreducible representations of M. This concludes the proof. □

3.2 Example. *In [13] the author and M. Putcha construct for each finite group G of Lie type, a certain canonical monoid $M(G)$ having the following properties:*

(a) G is the unit group of $M(G)$.

(b) The type map $\lambda : \mathcal{U}(M(G)) \to 2^S$ of $M(G)$ satisfies

  (i) $\lambda : \mathcal{U}(M(G)) \backslash \{0\} \to 2^S$ is bijective

  (ii) $\lambda(J_e \wedge J_f) = \lambda(J_e) \cap \lambda(J_f)$, where $\mathcal{U}(M(G))$ has been identified with a cross-section lattice Λ of $M(G)$.

(c) For each $e \in \Lambda \{g \in G | ge = eg = e\} = \{1\}$.

By the results of [11], $M(G)$ is determined up to isomorphism by these properties. This monoid also enjoys a number of other useful properties that were important in the proof of Theorem 2.1. In any case, if $J \in \mathcal{U}(M(G))$ then by (c), $v(J) = \phi$. Furthermore, $\alpha_{I,J} = Hom(L_I, \overline{F}_q^*)$ for any $I \subseteq \lambda(J)$. So $\alpha_{I,J}$ is independent of J if it is non-empty.

Define, for any finite monoid M of Lie type,

$$IR(M) = \{\rho : M \to End(V) | \rho \text{ is irreducible}\} / \sim$$

where "$\sim$" denotes equivalence of representations. Thus, by Theorem 3.1

$$\begin{aligned} |IR(M(G))| &= \sum_{I \subseteq \lambda(J)} |\alpha_{I,J}| \\ &= \sum_{I \subseteq S} \sum_{J \in \mathcal{U}(S)} |\alpha_{I,J}| \\ &= \sum_{I \subseteq S} 2^{|S \backslash I|} \alpha(I) \end{aligned}$$

where $\alpha(I)$ is the common value of $|\alpha_{I,J}|$ for $I \subseteq \lambda(J)$. This agrees with the formula (1) of Theorem 2.2 of [13].

If $G = S\ell_{n+1}(F_q)$ then $|S| = n$ and for $|I| = i, \alpha(I) = (q-1)^{n-i}$. Thus

$$\begin{aligned} |IR(M(G))| &= \sum_{I \subseteq S} 2^{|S \backslash I|} \alpha(I) \\ &= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} 2^{n-i} (q-1)^{n-i} \\ &= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} (2q-2)^{n-i} \\ &= (2q-1)^n \end{aligned}$$

This corrects the calculation error in example 2.3 of [13].

3.3 Example. *Using the results of [11] it is possible to construct, for each group G of Lie type a monoid $M(G)^*$ with the following properties:*

(a) If λ is the type map then $\lambda : \mathcal{U}(M(G)^*)\backslash\{G\} \to 2^S$ is bijective, and $\lambda(J_e \wedge J_f) = \lambda(J_e) \cup \lambda(J_f)$.

(b) For each $e \in \Lambda\backslash\{1\}$, say $\lambda(J_e) = I, H_e = eL_I \cong L_I/L_I'$, where L_I' is the subgroup of L_I generated by the elements of order p^n for some $n > 0$. $M(G)^*$ is called the dual canonical monoid *of G. By [11; Theorem 3.8(ii)] $M(G)^*$ is uniquely determined up to isomorphism by (a) and (b). It is easily verified that, for each $J \in \mathcal{U}(M(G)^*), v(J) = \lambda(J)$. Combining this observation with properties (a) and (b), and applying Theorem 3.1 yields the following remarkable property of $M(G)^*$.*

Let $\rho : G \to G\ell(V)$ be a nontrivial irreducible representation. Then there exists a unique representation $\rho^* : M(G)^* \to End(V)$ such that

(i) $\rho^*|G = \rho$

(ii) $\rho^*(M(G)^*\backslash G) \neq \{0\}$.

Furthermore, if ρ is of type (I, χ) then the apex of ρ^* is the unique $J \in \mathcal{U}(M(G)^*)\backslash\{G\}$ such that $\lambda(J) = I$.

3.4 Example. *Let $M = M_n(F_q)$. By the theory of Munn-Ponizovskii*

$$|IR(M)| = \sum_{k=0}^{n} |IR(G\ell_k(F_q))|$$

while by the formulas of Richen (for $k > 0$)

$$\begin{aligned} |IR(G\ell_k(F_q))| &= \sum_{\sum_{i=1}^{s} k_i = h} |Hom(\prod G\ell_{k_i}(F_q), F_q^*)| \\ &= \sum_{\sum_{i=1}^{s} k_i = h} (q-1)^s \\ &= \sum_{i=0}^{k-1} \left(\begin{array}{c} k-1 \\ i \end{array} \right) (q-1)^{i+1} \\ &= (q-1)q^{k-1} \end{aligned}$$

Combining these formulas

$$|IR(M)| - 1 = (q-1)\sum_{k=1}^{n} q^{k-1} = q^n - 1$$

But we can also write

$$\begin{aligned}
|IR(M)| - 1 &= \sum_{k=1}^{n} |IR(G\ell_k(F_q))|, \text{ as above} \\
&= \sum_{k=1}^{n} \sum_{i=0}^{k-1} \left(\begin{array}{c} k-1 \\ i \end{array} \right) (q-1)^{i+1} \\
&= \sum_{i=0}^{n} \left(\sum_{k=i+1}^{n} \left(\begin{array}{c} k-1 \\ i \end{array} \right) \right) (q-1)^{i+1} \\
&= \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i+1 \end{array} \right) (q-1)^{i+1}
\end{aligned}$$

We can interpret this coefficient as follows. An elementary calculation shows that if $J \subseteq M$ is a J-class and $v(J) \subseteq I \subseteq \lambda(J)$ then

$$|\alpha_{I,J}| = (q-1)^{|\lambda(J)\backslash I|+1}$$

We conclude that

$$\left(\begin{array}{c} n \\ i+1 \end{array} \right) = \left| \left\{ (I,J) \in 2^S \times \mathcal{U}(M) \,\middle|\, \begin{array}{c} v(J) \subseteq I \subseteq \lambda(J) \\ |\lambda(J)\backslash I| = i \end{array} \right\} \right| .$$

We consider now the relationship between the irreducible modular representations of M and the conjugacy classes of M. This is particularly interesting for reductive monoids. See Section 4.

3.5 Definition. *Let M be a finite monoid of Lie type with unit group G of characteristic p. We say $x \in M$ is* semisimple *if*

- *(i) $x \in H_e$ for some $e \in E(M)$.*
- *(ii) $x^k = e$ for some $k, (k,p) = 1$.*

Let $M_{ss} = \{x \in M \mid x \text{ is semisimple}\}$ and let $M_{ss}/\sim$ denote the set of conjugacy classes of semisimple elements of M.

3.6 Theorem. *$|IR(M)| = |M_{ss}/\sim|$.*

Proof. By the theory of Munn and Ponizovskii, $|IR(M)| = \sum_{e\in\Lambda} |IR(H_e)|$, while by [19; Theorem 42] $|IR(H_e)| = (H_e)_{ss}/\sim$. Hence, it suffices to show that two elements of H_e are H_e-conjugate if and only if they are G-conjugate.

If $x, y \in H_e$ and $gxg^{-1} = y$ with $g \in G$, then $geg^{-1} = e$. But then $h = ge \in H_e$ and $hxh^* = gexeg^{-1} = gxg^{-1} = y$, where $h^* \in H_e$ is the inverse of h in H_e.

Conversely, if $x, y \in H_e$ and $h \in H_e$ is such that $hxh^{-1} = y$, then we recall from [11; Corollary 3.4] that $C_G(e) \to H_e$, $g \rightarrowtail ge$ is surjective. So let $g \in C_G(e)$ be such that $ge = h$. It follows easily that $gxg^{-1} = y$. ☐

4 Finite Reductive Monoids

In this section we consider reductive algebraic monoids $\underline{M}$ defined over a finite field F_q. We readily obtain finite monoids of Lie type as follows:

Let $M = \underline{M}(F_q)$ be the finite monoid of F_q-rational points of $\underline{M}$. By standard facts about finite fields and the Galois theory of algebraic varieties [8; Chapter II, Section 4] there exists an F_q-automorphism $\sigma : \underline{M} \to \underline{M}$ of algebraic monoids such that $M = \{x \in \underline{M} | \sigma(x) = x\}$. But then by [13; Theorem 4.3], M is a finite monoid of Lie type. Actually, one obtains a sequence of finite monoids $M_r = \{x \in \underline{M} | \sigma^r(x) = x\}$, $r = 1, 2, 3, \ldots$. The purpose of this section is to establish interesting and useful formulae for $|IR(M_r)|$. The basic problem here is to consider formulae of the form

$$|IR(M_r)| - 1 = (q-1)\sum_{i=1}^{n} a_i q^{ri}$$

where $a_i \in \mathbb{Z}$ and is independent of r. Whenever this can be done, it is particularly interesting to interpret the a_i's.

4.1 Definition.

(a) *By a* finite reductive monoid M, *we mean* $M = \underline{M}(F_q)$ *where* $\underline{M}$ *is a reductive monoid defined over* F_q. *Notice that such a monoid is a finite monoid of Lie type.*

(b) *We say* M *is* locally simply connected *if for all* $e \in E(M)$, H_e *is a simply connected reductive group. Recall from [24; Remark 2.13] that means that* $\underline{H}'_e = (\underline{H}_e, \underline{H}_e)$ *is a simply connected, semisimple group.*

We remind the reader that the Levi subgroups of any simply connected group are also simply connected [24; Lemma 2.17]. Our purpose in introducing this notion is made clear in Theorem 4.2 below.

If $\underline{M}$ *is a reductive monoid, we let* $\underline{X}$ *denote the adjoint quotient of* $\underline{M}$. *There is a canonical quotient morphism* $\pi : \underline{M} \longrightarrow \underline{X}$ *of algebraic varieties. By the results of [16]* $\underline{X}$ *parametrizes the conjugacy classes of semisimple elements of* $\underline{M}$.

4.2 Theorem. *Let* $\underline{M}$ *be a locally simply connected reductive monoid defined over* F_q *and let* $M = \underline{M}(F_q)$. *Then*

(a) *The canonical map* $M_{ss}/\sim \longrightarrow \underline{X}(F_q)$ *is bijective.*

(b) $|IR(M)| = \sum_{e\in\Lambda} q^{r(e)}|\underline{H}(e)_{ab}|$ *where* $r(e)$ *is the semisimple rank of* $\underline{H}(e)$ *and* $\underline{H}(e)_{ab}$ *is the abelianization of* $\underline{H}(e)$.

(c) *If dim* $Z(\underline{G}) = 1$ *and* $0 \in \underline{M}$ *then for each* $e \in \Lambda$ $|\underline{H}(e)_{ab}(F_q)| = (q-1)^a(q+1)^b(q^2+q+1)^c$ *for appropriate integers* $a, b, c \geq 0$.

(d) *If* $\underline{M}$ *is split over* F_q *then* $|\underline{H}(e)_{ab}(F_q)| = (q-1)^a$ *for some* $a \geq 0$.

Proof. For the proof of (a) we recall two pertinent facts. First of all, as in the proof of 3.6, if $e \in E(M)$ then $x, y \in H_e$ are G-conjugate if and only if they are H_e-conjugate. Secondly, as pointed out above, $\underline{X}$ parametrizes the $\underline{G}$-conjugacy classes of semisimple elements of $\underline{M}$. But now we can apply [22; Theorem 10.3]. This says in our situation that for each $e \in E(M)$, $\underline{H}(e)(F_q) \ / \sim \ \to \underline{X}(H(e))(F_q)$ is bijective. Combined with the above facts we obtain our result.

For (b), consider $e \in \Lambda$ and the corresponding $\mathcal{H}$-class of e, $\underline{H}(e) \subseteq \underline{M}$. Let $\underline{H}(e)' \subseteq \underline{H}(e)$ be the commutator subgroup. By [23; Theorem 14.11], $|IR(H(e)')| = q^{r(e)}$ where $H(e)' = \underline{H}(e)'(F_q)$ and $r(e)$ is the semisimple rank of $\underline{H}(e)'$. On the other hand, by Richen's theory

$$|IR(H(e))| = \sum_{I \subseteq S(e)} |X(L_I)|$$

where $S(e)$ is the set of simple roots associated with $H(e)$ and $\{L_I | I \subseteq S(e)\}$ is a set of representatives for the Levi subgroups of $H(e)$. $X(L_I) = Hom(L_I, \overline{F}_q^*)$. Now $H(e)/H(e)'$ is an abelian group of order prime to q, and so $|X(H(e)/H(e)')| = |H(e)/H(e)'|$ by the well known duality for finite abelian groups. But $L_I' = L_I \cap H(e)'$ is the corresponding Levi subgroup of H_e'. Hence, again by Richen $|IR(H(e)')| = \sum_{I \subseteq S} |X(L_I')|$. But then $|X(L_I)| = |X(L_I')|\ |X(L_I/L_I')| = |X(L_I')|\ |X(H(e)/H(e)')| = |X(L_I')|\ |H(e)/H(e)'|$. That $H(e)/H(e) = (\underline{H}(e)/\underline{H}(e)')(F_q)$ follows from the "σ-process" of [20] since $\underline{H}(e)'$ is a connected algebraic group. Putting things together we obtain

$$\begin{aligned}
|IR(H(e))| &= \sum_{I \subseteq S(e)} = |X(L_I)| \\
&= \sum_{I \subseteq S(e)} |X(L_I')|\ |H(e)/H(e)'| \\
&= |IR(H(e)')|\ |H(e)/H(e)'| \\
&= q^{r(e)} |H(e)/H(e)'|
\end{aligned}$$

Hence, by the Munn-Ponizovskii theory, $|IR(M)| = \sum_{e \in \Lambda} q^{r(e)} |H(e)/H(e)'|$. To establish the formula in (c), we observe that for semisimple monoids $\underline{M}$, the action of σ on an appropriate maximal torus $\underline{T} \subseteq \underline{G}'$ is of the form $\sigma^*(\rho(\alpha)) = q\alpha$ (on $X(T)$) where $\rho : \Delta \to \Delta$ is a disjoint union of automorphisms of order two or three. This follows from the results of section 11.6 of [23]. Since the center of $\underline{G}$ is one dimensional, $\gamma = \sigma | Z(\underline{G})^0$ has the form $\gamma(t) = t^q$, by the comment following Theorem 5.1 of [18]. Thus, $det(\sigma^* - 1) = \prod_{i=1}^{m} (q^{a_1} - 1)$ for some $a_i \leq 3$ and some $m > 0$. Hence, restricted to $X(\underline{H}(e)/\underline{H}(e)') \subseteq X(T \cdot Z(\underline{G})^0)$, the determinant is a factor of $det(\sigma^* - 1)$. But by the formula of [18; Proposition 6.1(d)], this factor equals $|H(e)/H(e)'|$.

The proof of (d) is the same as for (c) taking into account that "split" means simply that $\rho : \Delta \to \Delta$ is the identity automorphism (so that $det(\sigma^* - 1) = (q - 1)^r$). This completes the proof. ☐

4.3 Theorem. *Suppose M is locally simply connected and split over F_q. Then*

(a) $|IR(M)| - 1 = (q-1)\sum_{i\geq 0} b_i q^i$ *for some integers b_i.*

(b) If M has only one nonzero minimal $\mathfrak{J}$-class then $|IR(M)| - 1 = (q-1)\sum_{i\geq 0} a_i(q-1)^i$ *where*

$$a_i = \left| \left\{ (I,J) \in 2^S \times \mathfrak{U}(M) \;\middle|\; \begin{array}{c} v(J) \subseteq I \subseteq \lambda(J) \\ |\lambda(J)\backslash I| = i \end{array} \right\} \right| .$$

In particular, the b_i's of (a) are independent of q.

(c) If $\underline{M}$ has one non-zero minimal $\mathfrak{J}$-class then $\underline{X} = \bigsqcup_{e\in\Lambda} C_e$ where each $C_e \cong G_m \times \mathbb{A}^{n\cdot}$.

In particular, the b_i's are the Betti numbers of $(\underline{X}\backslash 0)/G_m$.

Proof. The formula of (a) holds for any locally simply connected monoid $\underline{M}$, using 4.2(b). The formula of (b) follows directly from Theorem 3.1 since for reductive, split monoids with one non-zero minimal $\mathfrak{J}$-class $|\alpha_{I,J}| = (q-1)^{|\lambda(J)\backslash I|+1}$.

For (c), recall that the $\underline{H}(e)$-conjugacy classes of $\underline{H}(e)$ are the same as the $\underline{G}$-conjugacy classes of $\underline{H}(e)$. Furthermore, $rk_{SS}\underline{H}(e) = rk\underline{H}(e) - 1$, since by [10; Proposition 6.27] $e\underline{M}e$ is also $\mathfrak{J}$-reducible. Thus, applying [22; Theorem 1.6] to $\underline{H}(e)$ we obtain these "cells" C_e. ☐

Recall [10; Definition 15.2] that a reductive monoid $\underline{M}$ is *$\mathfrak{J}$-coirreducible* if $\underline{M}\backslash\underline{G}$ has a unique maximal $\mathfrak{J}$-class. The *divisor class group* $C\ell(\underline{X})$ of a normal variety $\underline{X}$ is the free abelian group generated by the irreducible codimension one subvarieties of $\underline{X}$ modulo the principal divisors. One basic theorem here states that $C\ell(\underline{X}) = (0)$ if and only if $k[\underline{X}]$ is a uniqe factorization domain. See [4] for the whole story on class groups.

4.4 Theorem. *Let $\underline{M}$ be reductive*

(a) If $C\ell(\underline{M}) = (0)$ then $\underline{M}$ is locally simply connected.

(b) If $\underline{M}$ is $\mathfrak{J}$-coirreducible then there exits $\underline{M}' \to \underline{M}$, finite and dominant such that $C\ell(\underline{M}') = (0)$. Furthermore, $C\ell(\underline{M})$ is finite.

(c) For any such $\underline{M}$ there exists $\pi : \underline{M}' \to \underline{M}$ such that

  (i) $C\ell(\underline{M}') = (0)$ and

  (ii) π induces a bijection of $\mathfrak{U}^1(\underline{M}') \to \mathfrak{U}^1(\underline{M})$, where $\mathfrak{U}^1(\underline{M})$ denotes the set of maximal $\mathfrak{J}$-classes of $\underline{M}\backslash\underline{G}$.

Proof. Notice first that, given $\underline{X}$ such that $C\ell(\underline{X}) = (0)$, then $C\ell(\underline{U}) = (0)$ for any open set $\underline{U} \subseteq \underline{X}$. On the other hand, an algebraic group $\underline{G}$ is simply connected if and only if $C\ell(\underline{G}) = (0)$ [6; Proposition 3.2]. Thus, to prove (a) it suffices to show that $C\ell(e\underline{M}e) = (0)$ for any $e \in E(M)$. We apply [4; Theorem 10.6]. To do this we must show that $A = k[\underline{M}] = \bigoplus_{n\geq 0} A_n$ with

$A_0 = k[e\underline{M}e]$. But we know from [10; Corollary 6.10(ii)] that there exists a one parameter subgroup $\lambda : G_m \to \underline{G}$ such that $t \underset{lim}{\longrightarrow} 0\lambda(t) = e$. Consider the action $\mu : G_m \times \underline{M} \to \underline{M}$ given by $\mu(t, x) = \lambda(t)x\lambda(t)$. μ induces a rational action $\rho : k^* \to Aut(k[\underline{M}])$. If we let $A_n = \{f \in k[\underline{M}] | \rho(t)(f) = t^n f\}$ then $k[\underline{M}] = \bigoplus_{n \geq 0} A_n$ is the desired $\oplus$-decomposition.

To prove (b) we may assume without loss of generality that $0 \in \underline{M}$. Notice that this implies that $dim\ Z(\underline{G}) = 1$. Now $\underline{M} \backslash \underline{G}$ is irreducible of codimension one in $\underline{M}$. So it determines a divisor class $D \in C\ell(\underline{M})$. Since $\underline{M} \backslash \underline{G}$ is irreducible we have by [4; Corollary 7.2] an exact sequence

$$0 \to \mathbb{Z} \cdot D \to C\ell(\underline{M}) \to C\ell(\underline{G}) \to 0 \tag{1}$$

By [14; Theorem 3.4] there exists $\chi : \underline{M} \to k$ such that $\chi^{-1}(0) = \underline{M} \backslash \underline{G}$. Hence, in $C\ell(\underline{M})$, D has finite order. We conclude that $C\ell(\underline{M})$ is finite since from [6; Corollary 2.8], $C\ell(\underline{G})$ is finite. To find $\underline{M}'$ we first assume $C\ell(\underline{G}) = (0)$. For if $C\ell(\underline{G}) \neq 0$ then we consider $\zeta : \underline{G}_1 \to \underline{G}$, the universal cover of $\underline{G}$, and apply [15; Lemma 7.1.1] to obtain a reductive monoid $\underline{M}_1$ with unit group $\underline{G}_1$ and a finite dominant morphism $\underline{M}_1 \to \underline{M}$ extending ζ. With another application of [15; Lemma 7.1.1] we may assume $\underline{G} = \underline{G}_0 \times k^*$ where $\underline{G}_0$ is semisimple and simply connected. From the exact sequence in (1) we obtain $C\ell(\underline{M}) = \mathbb{Z} \cdot D$, a finite cyclic group. Before we construct $\underline{M}'$ we need to determine exactly what controls the order of D in $C\ell(\underline{M})$. Let $e \in \Lambda \backslash \{1\}$ be the unique maximal element, and let $T_e = T \cup eT = \overline{T}$. Then $T_e \subseteq \overline{T}$ is an open submonoid. Furthermore, by [15; Theorem 4.4] there exist opposite Borel subgroups B, B^- containing T such that $m : B_u^- \times T_e \times B_u \to \underline{M}$, $m(x, y, z) = xyz$, is an open embedding. Let $R = k[\underline{M}]$ and $S = k[B_u^- \times T_e \times B_u]$ we obtain $R \subseteq S$. Since $T_e \cong (k^*)^{r-1} \times k$ as varieties, we see from [4; Corollary 7.2, Theorem 8.1] that S is a UFD. Let $\mu = \{f \in \Re | f | \underline{M} \backslash \underline{G} = 0\}$. Clearly $|C\ell(\underline{M})| = inf\{n | \mu^{(n))}$ is a principal ideal$\}$ where $\mu^{(n)}$ denotes the n^{th} symbolic power of the ideal μ. So write $\mu^{(n)} = (\chi)$ where $n = |C\ell(\underline{M})|$. We may assume $\chi : \underline{M} \to k$ is a morphism of algebraic monoids, adjusting the initial χ with a non-zero scalar if necessary. Consider $\chi \circ m \in S$. From our remarks above we see that $(\chi \circ m) = p^n$ where $p = \mu \cdot S$. Using the isomorphism $T_e \cong (k^*)^{r-1} \times k$ and the fact that $\chi \circ m$ factors through $p_2 : B_u^- \times T_e \times B_n \to T_e$, $(x, y, z) \rightarrowtail y$, we obtain the following diagram

$$\begin{tikzcd} k \arrow[r, hook, "j"'] & T_e \arrow[r, hook, "i"'] & B_u^- \times T_e \times B_u \arrow[r, "\chi\circ m"'] & k \end{tikzcd}$$

j is the unique inclusion with the property $j(0) = e$. It follows that $n =$ degree $\chi \circ m \circ i \circ j)$. Now let (X, ϕ, C) be the *polyhedral root system* of $\underline{M}$ [15; Definition 3.6] and let $v : C \to N$ be the "valuation" determined by j (notice that $v^{-1}(0) \subseteq C$ is the facet of C determined by e). Let $\chi \in C$ denote the restriction of χ to $\overline{T}$. We can construct a new polyhedral root system (X', ϕ', C') as follows.

Since $\underline{G} = \underline{G}_0 \times k^*, X = X_0 \oplus \mathbb{Z}$. Furthermore, $C \subseteq X_0 \oplus \mathcal{N}$ and $\chi = (0, 1) \in C$. We define

$$\begin{aligned} X' &= X_0 \oplus \frac{1}{n}\mathbb{Z} \\ \phi' &= \phi \\ C' &= \{\zeta \in X' | m\zeta \in C \text{ for some } m > 0\}. \end{aligned}$$

It is easily checked that $X', \phi', C')$ is the polyhedral root system of the reductive monoid $\underline{M}'$ obtained via [15; Lemma 7.1.1] from the map $\zeta : \underline{G} \to \underline{G} \subseteq \underline{M}$ given by $\zeta(g, \alpha) = (g, \alpha^n)$. Furthermore, $v : C \to \mathbb{N}$ extends uniquely to $v' : C' \to \mathbb{N}$ via $v'(a, b/n) = v(a, 0) + b = v(a, 0) + 1/n \ v(0, b)$. Notice that if $\chi' = (0, \ 1/n)$ then $v'(\chi') = 1$. But from our above calculation applied to $\underline{M}'$, $|C\ell(\underline{M}')| = v'(\chi')$. Hence, $C\ell(\underline{M}') = (0)$.

For (c) we assume $\underline{M}$ has a zero element. The general case is not essentially different. Let $e \in \Lambda^1$ be a maximal idempotent of $\Lambda \backslash \{1\}$. As above, there exists $v : C \to \mathbb{N}$ which extends to $v : X \to Z$. Let

$$\begin{aligned} H &= \{x \in X | v(x) \geq 0\} \quad \text{and let} \\ C_e &= \sum_{w \in W} w^*(H) \subseteq X \end{aligned}$$

It is easily checked that $X, \phi, C_e)$ is a polyhedral root system with $j : (X, \phi, C) \hookrightarrow (X, \phi, C_e)$. Let $\underline{M}_\alpha$ be the associated reductive monoid, where $\alpha = \underline{G} \in \underline{G} \in \mathcal{U}(\underline{M})$. By construction $\underline{M}_\alpha$ is $\mathcal{J}$-coirreducible. Now from [15; Theorem 8.l(a)] there exists a birational morphism $\zeta_\alpha : \underline{M}_\alpha \to M$ inducing j above. Applying part (b) above we can modify $\underline{M}_\alpha$ slightly, if necessary, and assume that $C\ell(\underline{M}_\alpha) = (0)$. The unique maximal $\mathcal{J}$-class of $\underline{M}_\alpha$ gets mapped to α. After ordering $\mathcal{U}'(\underline{M})$, define

$$\zeta : \prod_{\alpha \in \mathcal{U}^1(\underline{M}_\alpha)} \underline{M}_\alpha \to \underline{M}$$

by $\zeta(x_1, \ldots, x_m) = \zeta_{\alpha_1}(x_1) \ \cdot \ \ldots \ \cdot \ \zeta_{\alpha_m}(x_m)$. Consider the action of $\underline{H} = \underline{G}_0 \times \underset{m-1}{\cdots} \times \underline{G}_0$ on $\underline{N} = \prod \underline{M}_\alpha$ given by

$$(g_1, \ldots, g_{m-1}) \ \cdot \ (x_1, \ldots, x_m) = (x_1 g_1, g_1 x_2 g_2^{-1}, \ldots, g_{m-2} x_{m-1} g_{m-1}^{-1}, g_{m-1} x_m)$$

Define $\underline{M}' = \underline{N}/\underline{H}$, the geometric invariant theory quotient of $\underline{\text{N}}$ by $\underline{H}$. By standard results of geometric invariant theory [7; Theorem 1.10] there exists a unique morphism $\pi : \underline{M}' \to \underline{M}$ such that

$$\begin{tikzcd} \underline{N} \arrow[dd] \arrow[dr, "e"] & \\ & M \\ \underline{M}' \arrow[ur, "\pi"'] & \end{tikzcd}$$

commutes. Based on [17; Proposition 3.3] we see that $\underline{M}'$ is a reductive algebraic monoid with unit group $\underline{G}_0 \times k^* \times \underset{(m)}{\cdots} \times k^*$. Furthermore $C\ell(\underline{M}) = (0)$ since $C\ell(\underline{N}) = (0)$ and $\underline{G}_0 \times \cdots \times \underline{G}_0$ is a semisimple group. To complete the proof we must show that π identifies the maximal $\mathcal{J}$-classes of $\underline{M}'$ with those of $\underline{M}$. But π is surjective by construction and so for each $\alpha \in \mathcal{U}^1(\underline{M})$ there exists $J \in \mathcal{U}^1(\underline{M}')$ such that $\pi(J) = J_\alpha$. But also, $C\ell(\underline{M}') = (0)$, so $|\mathcal{U}^1(\underline{M}')| \leq dim(Z(\underline{G}')) = m$. Thus, $\mathcal{U}^1(\underline{M}') \to \mathcal{U}^1(\underline{M})$ is bijective by the world's oldest counting argument. □

If M is a $\mathcal{J}$-coirreducible monoid then the lattice of $\mathcal{J}$-classes $U = \mathcal{U}(M)$, and the type map $\lambda : U \longrightarrow 2^S$ are both determined by $\lambda(J)$ where $J \in U\backslash\{G\}$ is the unique maximal element. Indeed, if $\lambda(J) = I$, then

$$U = \left\{ \; X \in 2^S \;\middle|\; \begin{array}{l} \text{no component of } X \\ \text{is contained in } I \end{array} \; \right\} \cup \{1\}$$

where $U\backslash\{1\}$ is ordered by reverse inclusion and $1 \in U$ is the largest element. Furthermore, $\lambda : U \longrightarrow 2^S$ is defined by

$$\lambda(X) \; X \sqcup C_I(X)$$

if $X \neq \phi$ or 1, and $\lambda(\phi) = I$ and $\lambda(1) = S$. Here, $C_I(X) = \{\alpha \in I | \sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha$ for all $\beta \in x\}$.

4.5 Theorem. *Suppose $\underline{M}$ is $\mathcal{J}$-coirreducible of type I, split over F_q and with $C\ell(\underline{M}) = 0$. Then*

$$|IR(M)| = \sum_{x \in U} q^{|C_I(X)|}(q-1)^{|S\backslash\lambda(X)|} + q^{|S|}(p-1)$$

Proof. First notice that $|S\backslash X| - |C_I(X)| = |S\backslash\lambda(X)|$ since $\lambda(X) = X \cup C_I(X)$ is a disjoint union whenever no component of X lies in I. But then for each $X \in U$, $IR(H(e_x)) = q^{|C_I(X)|}(q-1)^{|S\backslash\lambda(X)|}$ since $\underline{H}(e_x)$ has rank $|S\backslash X|$ and semisimple rank $|C_I(X)|$. ☐

4.6 Theorem. *Suppose $\underline{M}$ is reductive, split over F_q and locally simply connected. Then*

$$|IR(M)| = \sum_{i=0}^{|S|+1} a_i(q-1)^i$$

where $a_i = |\{(I, J) \in 2^S \times \mathcal{U}(M) | rank(J) - |I| + |v(J)| = i\}|$.

Proof. Recalling Theorem 3.1, it suffices to prove that given our assumptions on $\underline{M}$, $|\alpha_{I,J}| = (q-1)^i$ where $i = rank(J) - |I| + |v(J)|$. But $\alpha_{I,J}$ is the character group of $L_{I\backslash v(J)}(H(e))$, the Levi subgroup of H_e of type $I\backslash v(J)$. This has rank equal to rank(J) and semisimple rank $|I| - |v(J)|$. This yields the formula for a_i. ☐

References

[1] Carter, R., *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*, New York: Wiley, 1985

[2] Clifford, A. and G. Preston, "The algebraic theory of semigroups," *Math Surveys* 7 vol. 1, 1961, Providence.

[3] Curtis, R., "Modular representations of finite groups with split (B, N)-pairs," *Lecture Notes in Math,* 131, 57–95, Springer, 1970.

[4] Fossum, R., *The Divisor Class of a Krull Domain.* New York: Springer, 1973.

[5] Harris, J. and N. Kuhn, "Stable decompositions of classifying spaces of finite abelian p-groups," *Math Proc. Camb. Phil Soc.* 103 (1988), 427–449.

[6] Iversen, B., "The geometry of algebraic groups," *Advances in Math* 20 (1976), 57–85.

[7] Mumford, D. and J. Fogarty, *Geometric Invariant Theory.* Berlin: Springer, 1982, second edition.

[8] Mumford, D., *The Red Book of Varieties and Schemes.* Berlin: Springer, 1988.

[9] Putcha, M., "Determinant functions on algebraic monoids, *Comm. in Alg.* 11 (1983), 695–710.

[10] Putcha, M., *Linear Algebraic Monoids.* Cambridge: Cambridge University Press, 1988.

[11] Putcha, M., "Monoids on groups with BN-pairs," *J. Alg.* 120 (1989), 139–169.

[12] Putcha, M. and L. Renner, "The system of idempotents and the lattice of $\mathcal{J}$-classes of reductive algebraic monoids," *J. Alg.* 116 (1988), 385–399.

[13] Putcha, M. and L. Renner, "The canonical compactification of a finite group of Lie type," *Trans. Amer. Math. Soc.* 337 (1993), 305–319.

[14] Renner, L. "Quasi-affine algebraic monoids," *Semigroup Forum* 30 (1984), 167–176.

[15] Renner, L. Classification of semisimple algebraic monoids, *Trans. Amer. Math. Soc.* 292 (1985), 193–223.

[16] Renner, L. "Conjugacy classes of semisimple elements and irreducible representations,", *Comm. in Alg.* 16 (1988), 1933–1943.

[17] Renner, L., "Classification of semisimple varieties," *J. Alg.* 122 (1989), 275–287.

[18] Renner, L., "Finite monoids of Lie type" in *Monoids and Semigroups with Applications* (J. Rhodes ed.), *World Scientific* (1991), 278–287.

[19] Serre, J., "Linear representations of finite groups," *Graduate Texts in Math* 42. Berlin: Springer, 1977.

[20] Springer, T. and R. Steinberg. "Conjugacy classes," *Lecture Notes in Math* 131 (1970), 167–266, Springer.

[21] Steinberg, R., "Representations of algebraic groups," *Nagoya Math J.* 22 (1963), 33–56.

[22] Steinberg, R., "Reguar elements of semisimple algebraic groups," *Publ. Math.* I.H.E.S. 25 (1965), 49–80.

[23] Steinberg, R., "Endomorphisms of linear algebraic groups," *Mem. Amer. Math. Soc.* 80 (1968).

[24] Steinberg, R., "Torsion in reductive groups," *Advances in Math* 15 (1975), 63–92.

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