Linear Algebraic Monoids

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Introduction

Let G be an algebraic group defined over an algebraicly closed field k. (See [1].) If $k = \mathbb{C}$, then G is naturally a Lie group.

A *linear algebraic monoid* is an affine, algebraic variety M with associative multiplication $m : M \times M \to M$ with two-sided unit $1 \in M$. (The reader should consult [5], [8] and [12] for systematic information about algebraic monoids.) Evidently, the unit group of M is an algebraic group G, which is open in M, for any reasonable topology on M; in particular the Zariski topology, which is the most natural topology here.

If G is a reductive group and M is irreducible, then we say M is a *reductive*, algebraic monoid. Reductive monoids are intelligibly pieced together from their unit groups G and their set of idempotents $E(M) = \{e \in M | e^2 = e\}$. Many of the standard constructions of Lie theory arise naturally from questions about the idempotent set of a reductive monoid. (See [8; Section 5] and [7; Theorem 4.16] for some illustration.)

On the other hand, any irreducible monoid M is obtainable as the Zariski closure of a finite dimensional rational represention of its unit group:

$$M_\rho = \overline{\rho(G)} \subseteq M_n(k), \quad \text{where}$$

$$\rho : G \to G\ell_n(k)\ .$$

Not surprisingly, many important properties of the representation are reflected in the monoid M_ρ. For example, if G_0 is semisimple with Lie algebra $\mathfrak{g}$ and $G = G_0 \times k^*$ with $\rho : G \to G\ell(\mathfrak{g})$ defined by $\rho(g,t) = t\ Ad(g)$, then there is a one-to-one correspondence between

$$\{J \subseteq M \mid J = GxG \text{ for some } x \in M\}$$

and

$$\left\{ U \subseteq G \;\middle|\; \begin{array}{l} U \text{ is the center of a unipotent radical of a} \\ \text{standard parabolic subgroup} \end{array} \right\}.$$

The lattice of inclusions among these subgroups can then be calculated with the algorithm of [7; Theorem 4.16].

The axiomatic theory of reductive algebraic monoids is in a reasonably complete state, due largely to the efforts of the authors. Contributions to the theory have been made also by some others, notably Okninski, Solomon [12] and Vinberg [14].

The basic notation and facts here are as follows:

M	= an irreducible algebraic monoid,
G	= the unit group of M,
$B \subseteq G$	a Borel subgroup,
$T \subseteq B$	a maximal torus,
$\overline{T} \subseteq M$	the Zariski closure of T in M.

For $X \subseteq M$ we define $E(X) = \{e \in X \mid e^2 = e\}$.

Fact 1 [5]: Let M be reductive. If $\Lambda = \{e \in E(\overline{T}) \mid Be \subseteq eB\}$ then $M = \bigsqcup_{e\in\Lambda} GeG$. Furthermore, $GeG \subseteq \overline{GfG}$ iff $ef = fe = e$.

Fact 2 [9]: Let M be reductive. Let $R = \overline{N_G(T)}$ and $\mathcal{R} = R/T = T\backslash R$. Then $M = \bigsqcup_{r\in\mathcal{R}} BrB$.

Fact 3 [10]: Let M be reductive. If M is a normal variety, then so is $\overline{T}$. Furthermore, $M \longmapsto (\Phi, X(T), X(\overline{T}))$ is a complete invariant of M to within isomorphism. Here, $\Phi \subseteq X(T)$ is the system of roots, and $X(\overline{T}) \subseteq X(T)$ is the set of characters λ of T that can be extended to morphisms $\overline{\lambda} : \overline{T} \longrightarrow k$.

This is only a taste. The interested readers should consult [8] for a survey of other results, [5] for a systematic development up to 1988, and [14] for some recent developments.

The purpose of this article is to discuss four open problems. Each problem "represents" a particular type of problem that is well established.

§1 represents combinatorial orbit problems for reductive monoids.

§2 represents classification problems and regularity conditions for nonreductive monoids.

§3 represents $B \times B$-orbit problems.

§4 represents connections (as yet unexplored) with other parts of Lie$^+$ theory.

1 Types

Let M be a reductive monoid with unit group G. Define

$$U(M) = \{J \subseteq G \mid J = GxG \text{ for some } x \in M\}.$$

$U(M)$ is a finite lattice, with $J_1 \leq J_2$ if $J_1 \subseteq \overline{J}_2$ (Zariski closure). We regard $U(M)$ as a primary combinatorial invariant of M. An alternate description of $U(M)$ is the following:

Let $T \subseteq B \subseteq G$ and $\overline{T}$ be as above. Define the *cross-section lattice* of (T, B)

$$\Lambda = \{e \in \overline{T} \mid e^2 = e, Be \subseteq eB\}\ .$$

If we let $E(\overline{T}) = \{e \in \overline{T} \mid e^2 = e\}$ then the Weyl group W acts on $E(\overline{T})$ by conjugation. $\Lambda \subseteq E(\overline{T})$ is a set of representatives for the orbits of this action. $E(\overline{T})$ can be identified with the bare lattice of a convex, rational polytope $\mathcal{C}$ in the root space of W. Λ then corresponds to those faces of $\mathcal{C}$ whose interior meets the closure of the Weyl chamber.

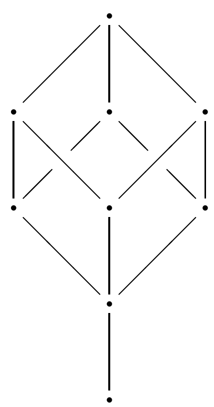
By [5; Theorem 9.3] $\Lambda \cong U(M)$ via $e \longleftrightarrow J = J_e$ if $e \in J$. Furthermore, $ef = fe = f$ iff $\overline{J}_e \supseteq J_f$.

Λ is tied to the Coxeter-Dynkin diagram as follows: Define the *type map*

$$\lambda \,:\, \Lambda \longrightarrow 2^S$$

by $\lambda(e) = \{s \in S \mid se = es\}$, where $S \subseteq W$ is the set of simple reflections relative to T and B.

For example, if $\varphi : S\ell_4 \longrightarrow G\ell(V)$ where $V = W \otimes \Lambda^2 W \otimes \Lambda^3 W$ and $W = k^4$, let $M_\varphi = \varphi(S\ell_4)k^* \subseteq End(V)$. Then (Λ, λ) can be depicted as follows:



$E(\overline{T})$ is a truncated octahedron. The most general result here is as follows:

Theorem 1 [7]: *Suppose M is a reductive monoid with zero such that $\Lambda\backslash\{0\}$ has a unique minimal element. Then there exists a unique subset $J_0 \subseteq S$ such that*

(a) $\Lambda\backslash\{0\} = \left\{ e_Y \;\middle|\; \begin{array}{l} Y \subseteq S \textit{ and no connected component of } Y \\ \textit{is contained in } J_0 \end{array} \right\}.$

(b) $\lambda : \Lambda\backslash\{0\} \longrightarrow 2^S$ *is defined by* $\lambda(e_Y) = Y \cup \{s \in J_0 \mid st = ts \textit{ for all } t \in Y\}$.

The type map (generally) is remarkably decisive. By the results of [6; Theorem 3.8], any two reductive monoids with the same unit group G and the same type (Λ, λ) differ (abstractly) by a kind of "central extension".

The reader is referred to [7] for the systematic development of Theorem 1. The monoids M of Theorem 1 are characterized by the fact that there exists an irreducible representation $\rho : M \longrightarrow End(V)$ such that $\rho^{-1}(0_V) = \{0_M\}$. $J_0 \subseteq S$ is the set of simple reflections that stabilize the highest weight vector of ρ.

A reductive monoid M is *semisimple* if $0 \in M$ and $dim(Z(G)) = 1$. (All monoids discussed in Theorem 1 are semisimple.)

Problem 1.1. Suppose G_0 is a simple algebraic group. Find all possible type maps of semisimple monoids with unit group $G_0 \times k^*$. Indeed, is this even plausible?

Given any semisimple monoid M, there is a completely reducible representation $\rho = \bigoplus_{i=1}^{m} \rho_i : M \longrightarrow End\left(\bigoplus_{i=1}^{m} V_i\right)$ such that $\rho^{-1}(0) = \{0\}$, and $U(M)\backslash\{0\}$ has exactly m minimal elements. So $\rho(M) \subseteq End(\oplus V_i)$ is closed and has the same type (Λ, λ), as M. So, we assume that $M = \rho(M)$. Thus, $M \subseteq M(\rho)$, where

$$M(\rho) = \overline{\left\{ \bigoplus_{i=1}^{m} \alpha_i \rho_i(x) \in End(\oplus V_i) \mid x \in M,\ \alpha_i \in k \right\}} .$$

Although, $M(\rho)$ is not semisimple, $(dim(Z(G)) = m)$, we can determine its type map (and this could be useful in Problem 1.1, since $M \subseteq M(\rho)$).

So consider $M = M(\rho)$ as above, where ρ_i is an irreducible representation of type J_i. Let (Λ_i, λ_i) by the type of

$$M_i = \overline{\{\alpha_i \rho_i(g) \mid y \in G,\ \alpha_i \in K\}} \ \subseteq\ End(V_i)$$

as in Theorem 1. Define

$$\mathbf{\Lambda} = \{(e_{Y_1}, \ldots, e_{Y_m}) \in \Lambda_1 \times \cdots \times \Lambda_m \mid Y_i \subseteq \lambda_i(Y_i) \text{ if } Y_i \neq \phi\}$$

and let

$$\boldsymbol{\lambda} : \mathbf{\Lambda} \longrightarrow 2^S$$

be defined by

$$\boldsymbol{\lambda}(e_{Y_i}, \ldots, e_{Y_m}) = \bigcap_{i=1}^{m} \lambda_i(e_{Y_i})$$

Theorem 2: *The type map of $M(\rho)$ is $(\mathbf{\Lambda}, \boldsymbol{\lambda})$.*

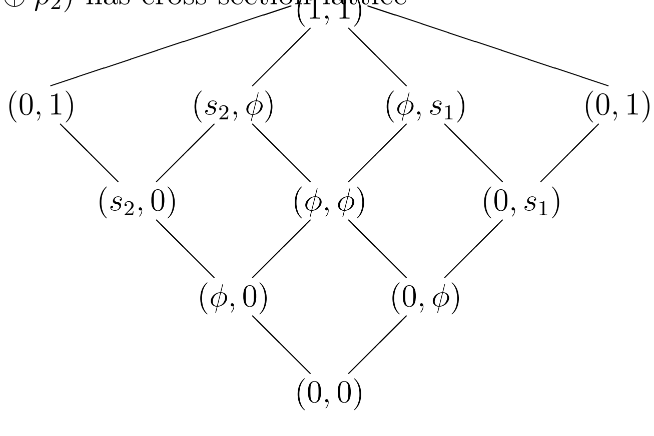
This is proved by elaborating the proof of Theorem 1.

Example: Let $G_0 = S\ell_3(k)$ so that $S = \{s_1, s_2\}$. One checks that if $\rho_1 = id : G_0 \longrightarrow G\ell(k^3)$ and $\rho_2 : S\ell_3(k) \longrightarrow (G\ell_3(k)$ is given by $\rho_2(x) = (x^{-1})^t$, then

$$M_1 \text{ is of type } J_1 = \{s_1\}, \text{ and}$$

$$M_2 \text{ is of type } J_2 = \{s_2\} .$$

By Theorem 2, $M(\rho_1 \oplus \rho_2)$ has cross-section lattice



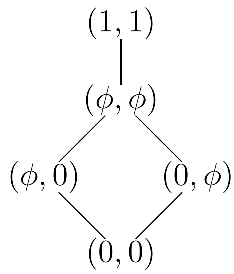
Associated with any positive, one-parameter subgroup $\mu = (\alpha^k, \alpha^\ell)$ of $Z(G(M(\rho_1 \oplus \rho_2))$, we obtain a semisimple monoid

$$M_\mu = \overline{\{\alpha^k \rho_1(g) \oplus \alpha^\ell \rho_2(g) \in M(\rho_1 \oplus \rho_2) \mid \alpha \in k, g \in G_0\}}$$

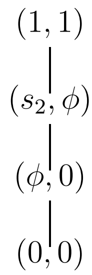
with cross-section lattice Λ_μ.

Examples: μ Λ_μ

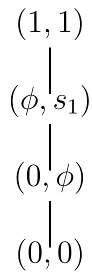
$\alpha\rho_1 \oplus \alpha\rho_2$



$\alpha\rho_1 \oplus \alpha^2\rho_2$



$\alpha^2\rho_1 \oplus \alpha\rho_2$



Problem 1.2. Find the cross-section lattice Λ_μ and type maps $\lambda_\mu : \Lambda_\mu \longrightarrow 2^s$ of the semisimple monoids M_μ arising from representations of the form $\alpha^{k_1}\rho_1 \oplus \cdots \oplus \alpha^{k_m}\rho_m$.

2 Classification and Regularity Conditions

It follows from [10; Theorem 6.5] that a normal, reductive monoid M is determined by the diagram

$$\overline{T} \supseteq T \subseteq G\,.$$

So, once we know the closure of a maximal torus in M, we can identify M to within isomrophism. Furthermore, any $\overline{T} \supseteq T \subseteq G$, as above, for which

a) the Weyl group action on T extends over $\overline{T}$,

b) $\overline{T}$ is a normal affine variety occurs for some M,

occurs for some M.

The main reason for this rigidity in the classification is the *Extension Property* (EP) enjoyed by all normal reductive monoids:

Any morphism $\alpha : G \longrightarrow N$ of algebraic monoids which extends over $\overline{T}$ via

$$\begin{array}{ccc} T & \xrightarrow{\alpha|T} & N \\ \cap & \nearrow & \\ \overline{T} & & \end{array} ,$$

can be extended to a unique morphism $\beta : M \longrightarrow N$ of algebraic monoids. This extension property results largely from the fact that reductive monoids are *regular*. In fact, $M = \bigcup_{e \in \Lambda} GeG$, a condition that should make one suspect that M has the EP.

So what about nonredutive monoids? It is easy to see from simple examples that

(1) Not every M is regular.

(2) Not every M satisfies EP.

(3) (Unlike the case of reductive monoids) there exist morphisms of nonreductive monoids
$\varphi : M' \longrightarrow M$ such that
$\varphi : \overline{T}' \underset{\cong}{\longrightarrow} \overline{T}$,
$\varphi : G' \underset{\cong}{\longrightarrow} G$, and
φ is not a finite-to-one morphism.

To illustrate (3) define

$$M' = \{(s, t, u) \mid s, t, u \in k\} \text{ with}$$

$$(s, t, u)(k, \ell, v) = (sk, t\ell, \ell u + sv)$$

and

$$M = \{(s, t, u) \mid s, t, u \in k\} \text{ with}$$
$$(s, t, u)(k, \ell, v) = (sk, t\ell, k\ell u + s^2 v)$$

Finally, define $\varphi : M' \longrightarrow M$ by

$$\varphi(s, t, u) = (s, t, su) \ .$$

So there are some very significant differences. However, there are some very compelling open questions here, and they are all related.

Problem 2.1. Given an irreducible monoid M_1, does there exist an irreducible monoid M and a morphism $\varphi : M \longrightarrow M_1$ such that

a) $\varphi : G \underset{\cong}{\longrightarrow} G_1$,

b) $\varphi : \overline{T} \underset{\cong}{\longrightarrow} \overline{T}_1$ and

c) M satisfies EP relative to $\overline{T} \supseteq T \subseteq G$?

Problem 2.2. Given $\overline{T} \supseteq T \subseteq G$ so that the W action on T extends over $\overline{T}$, does there exist an M realizing these data?

Problem 2.3. Are the following equivalent for M normal?

a) M has the EP relative to $\overline{T} \supseteq T \subseteq G$.

b) M is quasiregular.

We say an irreducible monoid M is *quasiregular* if there exists a closed two-sided ideal $N \subseteq M$ such that

(i) $dim(N) \leq dim(M) - 2$

(ii) $M \backslash N \subseteq \bigcup_{e \in E(M)} GeG$

Exercise: Check that $T_n(k)$, the monoid of uppertriangular matrices, is quasiregular.

Problem 2.4. If M has the EP, is M quasiregular?

Remark: Notice that it is not always possible to find a regular monoid with given $\overline{T} \supseteq T \subseteq G$. (See [11] for the precise restrictions when G is solvable.) For example, no regular monoid shares $D_n(k) \supseteq D_n(k)^* \subseteq T_n(k)^*$ with $T_n(k)$. It is hoped that quasiregularity is the exact general condition that allows us to extend known results about reductive monoids to the general case.

3 Bruhat-Chevalley Order

Let G be a semisimple algebraic group over $\mathbb{C}$ with W, T and B as usual. It is well known that

$$G = \bigsqcup_{w \in W} BwB$$

We say $w \leq v$ if $BwB \subseteq \overline{BvB}$. It is also well known (see [3]) that there is a description of this order relation completely in terms of the theory of Coxeter groups. We have one basic question here (which we pose now, and later).

Problem 3.1. Can the theory of algebraic monoids be used to obtain new information about $\leq$?

Example: $G = G\ell_n(\mathbb{C}), W = S_n = \{(\epsilon_1, \ldots, \epsilon_n) \mid 1 \leq \epsilon_i \leq n, \epsilon_i \neq \epsilon_i \text{ if } i \neq i\}$. It turns out that

$$x = (\epsilon_1, \ldots, \epsilon_n) \leq (\delta_1, \ldots, \delta_n) = y$$

if and only if there exist $\sigma_i \in Aut\{1, 2, \ldots, i\}$ such that

$$\left.\begin{array}{l} \epsilon_1 \leq \delta_{\sigma(1)} \\ \epsilon_2 \leq \delta_{\sigma(2)} \\ \vdots \\ \epsilon_i \leq \delta_{\sigma(i)} \end{array}\right\} \; (*) \quad (\leq \text{ as integers})$$

Thus, we can describe the relation "$x \leq y$" in terms of a sequence of simpler order relations on the "truncations" $e_i x = (\epsilon_1, \ldots, \epsilon_i)$ and $e_i y = (\delta_1, \ldots, \delta_i)$ of x and y respectively.

This example suggests that the Bruhat-Chevalley order of a reductive group G can be described in terms of the system of idempotents of *any* reductive monoid M with unit group G (and zero element).

Problem 3.2. Let M be a reductive monoid with G, T, B and W as usual. Let

$$\Lambda = \{e \in E(\overline{T}) \mid Be \subseteq eB\}$$

so that $M = \bigsqcup_{e \in \Lambda} GeG$. Define $w \leq v$, for $w, v \in W$, if $BwB \subseteq \overline{BvB}$. Can this relation be characterized in terms of a collection of simpler relations on $\{eW | e \in \Lambda\}$? What is the analogue, in eW, of the relation described in $(*)$ of the above example?

The Bruhat-Chevalley order has an analogue for reductive monoids. This has been described in [4].

4 Arithmetic Semigroups

Let G be an algebraic group defined over $\mathbb{Q}$, and let $\rho : G \longrightarrow G\ell(V)$ be a rational representation of G, also defined over $\mathbb{Q}$. Let $L \subseteq V$ be a full $\mathbb{Z}$-lattice and let $C = L \cap C_0$, where $C_0 \subseteq V$ is a rational polyhedral cone with $C_0 - C_0 = V$ (so that C_0 has nonempty interior). Define

$$S = \{g \in G \mid \rho(g)(C) \subseteq C\} .$$

We call any such S an *arithmetic semigroup*.

For example, let $\rho = id : G\ell_n(\mathbb{Q}) \longrightarrow G\ell_n(\mathbb{Q})$ and let $L = \mathbb{Z}^n \subseteq \mathbb{Q}^n$, $C_0 = \{(x_1, \ldots, x_n) \in \mathbb{Q}^n \mid x_i \geq 0 \text{ for all } i\}$. Then

$$S =: \{g \in G \mid \rho(g)(C) \subseteq C\} \cong \{(a_{ij}) \in M_n(\mathbb{Z}) \mid a_{ij} \geq 0 \text{ for all } i, j\} .$$

Similar definitions make sense over the p-adic numbers $\mathbb{Q}_p$. However, it appears that $C = L$ because of the nonarchimedean situation.

Example: Let X be a simply connected finite CW complex, and define

$$WE(X) = \left\{ f : X \to X \,\middle|\, \begin{array}{l} f \text{ is continuous} \\ f_* : H_*(X;\mathbb{Q}) \underset{\cong}{\longrightarrow} H_*(X;\mathbb{Q}) \end{array} \right\}$$

and

$$WE[X] = \pi_0(WE(X)) = WE(X) \ / \ \text{ homotopy equivalence.}$$

The results of Sullivan [13; Theorems 10.2 and 10.3] strongly suggest that $WE[X]$ is (commensurable in some sense to) an arithmetic semigroup with $C = L$.

We recall that groups G and H are *commensurable* if there exists a diagram $G \supseteq K \subseteq H$ of inclusions of groups such that $(G : K) < \infty$ and $(H : K) < \infty$. This notion is actually part of the definition of an arithmetic group [2]. A similar definition should be possible so as to yield a proper definition of an arithmetic semigroup.

Notice that $C_{\mathbb{R}^+} = \mathbb{R}^+ \cdot C \subseteq \mathbb{R} \otimes_{\mathbb{Q}} V$ is a cone of the type familiar in Lie semigroups and elsewhere, so

$$S_{\mathbb{R}} = \{g \in G(\mathbb{R}) \mid \rho(g)(C_{\mathbb{R}^+}) \subseteq C_{\mathbb{R}^+}\}$$

should be a very nice topological semigroup.

There are many interesting problems here.

Problem 4.1. Does $\mathcal{J} = \mathcal{D}$ in the sense of Green's relations?

Problem 4.2. In the case $C = L$ do we obtain $\left\{ g \in G \,\middle|\, \begin{array}{l} gG(S)g^{-1} \cap G(S) \subseteq G(S) \\ \text{has finite index} \end{array} \right\} = G$?

Problem 4.3. Let $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the ring of p-adic integers, and consider

$$S_p = \{g \in G(\mathbb{Q}_p) \mid \rho(g)(L_p) \subseteq L_p\}$$

where $G(\mathbb{Q}_p)$ is the set of $\mathbb{Q}_p$-points of G. Is it true that "for all but a finite number of primes p, $G(S_p)$ is a maximal compact subgroup of $G(\mathbb{Q}_p)$"?

Problem 4.4. Given $S_\rho = \{g \in G(\mathbb{Q}) \mid \rho(g)(C) \subseteq C\}$ as above, with $\rho : G \longrightarrow G\ell(V)$. Consider also $M_\rho(\mathbb{Q})$ where $M \subseteq End(V \otimes \mathbb{C})$ is the Zariski closure of $\rho(G) \cdot \mathbb{C}^* \cdot id_v$. How are $M(\mathbb{Q})$ and S related? Can $M(\mathbb{Q})$ be constructed from $S \subseteq G$ without reference to ρ?

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