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RESEARCH ARTICLE

Regular Algebraic Monoids

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1. Introduction

The purpose of this paper is to provide a proper identification of normal, irreducible, regular algebraic monoids (*regular* is defined in §2). The results of [3,4] suggest that we should be able to find a classification of these monoids in terms of their unit groups, and related toroidal data. And that is what we accomplish here.

Assume that M is a normal, regular, algebraic monoid with unit group G. All our algebraic monoids are defined over an algebraically closed field of arbitrary characteristic. Let $e \in M$ be a minimal idempotent, and define

$$G_e = \{g \in G \mid ge = eg = e\}^0.$$

Assume, for simplicity, that G_e is a Levi factor of G. Thus,

$$G \cong G_e \ltimes R_u(G) \quad \text{(semidirect product)}$$

where $U = R_u(G) \lhd G$ is the unipotent radical of G.

Theorem A.

(a) *Let $T \subseteq G$ be a maximal torus and let $\overline{T} \subseteq M$ be the Zariski closure of T in M. So $T \subseteq \overline{T}$ induces $X(\overline{T}) \subseteq X(T)$. Let $\Phi_U \subseteq X(T)$ be the weights of the action $Ad\colon T \to \mathrm{Aut}(\mathcal{L}(U))$ on the Lie algebra of U. Then $\Phi_U \subseteq X(\overline{T}) \cup -X(\overline{T})$.*

(b) *Conversely, suppose we are given an algebraic group $G = G_0 \ltimes R_u(G)$ (where $G_0 \subseteq G$ is a Levi factor) along with a normal torus embedding $T \subseteq \overline{T}$ of the maximal torus $T \subseteq G_0$. Let M_0 be the normal, reductive monoid with 0 and unit group G_0 and maximal D-monoid $\overline{T}$[3]. Consider the action $Ad\colon T \to \mathrm{Aut}(\mathcal{L}(U))$ and assume that $\Phi_U \subseteq X(\overline{T}) \cup -X(\overline{T})$. Then there exists a unique, normal, algebraic monoid M with unit group G and maximal D-monoid $\overline{T} \subseteq M$.*

(c) *Any monoid M, as in (b), has the follwoing structure: Let $e = e^2 \in M$ be the zero element of M_0. Define $U_+ = \{u \in U \mid eu = e\}, U_0 = \{u \in U \mid eu = ue\}$ and $U_- = \{u \in U \mid ue = e\}$. Then $M \cong U_+ \times M_e \times U_0 \times U_-$ and the monoid multiplication of M can be defined explicitly (see Proposition 2.6) with these coordinates.*

The above theorem is an organized summary of Corollary 2.3, Proposition 2.6 and Theorem 3.3.

The reader should notice that Theorem A classifies only those normal regular monoids with unit group G of a particular type (that is, G is related to the monoid in a particular way). The general case is explained in Section 4. It is a relatively minor modification of the above theorem. For convenience we describe it here.

So let M be <u>any</u> normal, irreducible, regular, algebraic monoid with unit group G, and let $e \in E(M)$ be a minimal idempotent. Let $N = \overline{G_e R_u(G)}$ (Zariski closure), and set $H = G_e R_u(G)$. The following theorem is an organized summary of Lemma 4.1 and Theorem 4.2.

Theorem B.

(a) *N is a regular monoid of the type considered in Theorem A. Furthermore, $gNg^{-1} = N$ for $g \in G$.*

(b) *Define $N \times^H G =: \{[x,g] \mid x \in N, g \in G\}$ where $[x,g] = [y,h]$ if there exists $k \in H$ such that $y = xk^{-1}$ and $h = kg$. Then $N \times^H G$ is a regular monoid with multiplication $[x,g][y,h] = [xgyg^{-1}, gh]$. Furthermore,*

$$\varphi\colon\ N \times^H G \to M$$
$$\varphi([x,g]) = xg$$

is an isomorphism of algebraic monoids.

2. Taking it apart

A monoid M is *regular* if for any $x \in M$ there exists $a \in M$ such that $xax = x$.

Let M be a normal, regular, irreducible, algebraic monoid with unit group G, and let $e \in E(M) = \{e \in M \mid e = e^2\}$ be a minimal idempotent. By [1, Theorem 7.4] $G_e = \{g \in G \mid ge = eg = e\}^0$ is a reductive subgroup of G.

Assumption 2.1. $G_e \subseteq G$ is a Levi factor, so that $G = G_e \ltimes R_u(G)$, where $R_u(G) \lhd G$ is the unipotent radical.

As pointed out in the introduction, the general case can easily be derived from this one. We adhere strictly to Assumption 2.1 except in Section 4.

Proposition 2.2. *Let T be a maximal torus and define $N = \overline{TR_u(G)} \subseteq M$. Then N is regular.*

Proof. Since M is regular, G_e is reductive for any minimal idempotent e of M. So $G_e \cap R_u(G) = \{1\}$. Thus, $(TR_u(G))_e \cap R_u(G) \subseteq G_e \cap R_u(G) = \{1\}$. So $(TR_u(G))_e$ has no unipotent elements other than the identity. So it must be a torus. By [1, Theorem 7.4], N is regular. ∎

Corollary 2.3. *Let $\Phi_U \subseteq X(T)$ be the weights of Ad: $T \to \mathrm{Aut}(\mathcal{L}(U))$ on the Lie algebra $\mathcal{L}(U)$ of $U = R_u(G)$. Then $\Phi_U \subseteq X(\overline{T}) \cup -X(\overline{T})$.*

Proof. Since $\overline{T}$ has a zero, this follows from [4, Corollary 2.4]. ∎

Proposition 2.4. *Let $U = R_u(G)$ and let*

$$\begin{aligned} U_+ &= \{u \in U \mid eu = e\}, \\ U_0 &= \{u \in U \mid eu = ue\} \text{ and} \\ U_- &= \{u \in U \mid ue = e\}. \end{aligned}$$

Then

(a) $U = U_+U_0U_- \cong U_+ \times U_0 \times U_-$

(b) $G_e \subseteq N_G(U_+) \cap C_G(U_0) \cap N_G(U_-)$.

Proof. (a) follows from [4; Formula (3)]. For (b) notice first that $G_e \subseteq C_G(e)$. So if $u \in U_+$ and $g \in G_e$, then $egug^{-1} = g(eu)g^{-1} = geg^{-1} = e$. So $gug^{-1} \in U_+$. Similarly $G_e \subseteq N_G(U_-)$.

Now $G_e \subseteq N_G(U_0)$, by an argument similar to the above. But we can prove a little more for U_0. Indeed, let $T \subseteq G_e$ be a maximal torus and let $u \in U_0$. Then for $t \in T, etut^{-1} = eut^{-1} = uet^{-1} = ue = eu$. So $etut^{-1}u^{-1} = e$, which implies that $tut^{-1}u^{-1} \in U_+$. But $etut^{-1}u^{-1} \in U_0$ since $T \subseteq N_G(U_0)$. So $tut^{-1}u^{-1} \in U_0 \cap U_- = \{1\}$, so that $ut = tu$. But then $T \subseteq C_G(U_0)$ for any maximal torus $T \subseteq G_e$. On the other hand, $\bigcup_{T \subseteq G} T \subseteq G_e$ is Zariski dense. Thus, $G_e \subseteq C_G(U_0)$. ∎

Proposition 2.5. *Let $M_e = \overline{G_e} \subseteq M$. Then M_e is normal.*

Proof. Consider

$$\varphi\colon\ M_e \hookrightarrow M \to M/\!/R_u(G)$$

where $M/\!/R_u(G)$ is as in [2; Theorem 4.2]. Now $M/\!/R_u(G)$ is normal and $\mathcal{O}(M/\!/R_u(G)) = \mathcal{O}(M)^{R_u(G)}$. By [2, Theorem 4.2], φ induces an isomorphism on $\overline{T}$, so by [3, Corollary 4.5] φ is an isomorphism. ∎

Proposition 2.6. *$M \cong U_+ \times C_M(e)^0 \times U_-$ and $C_M(e)^0 \cong M_e \times U_0$.*

Proof. Define φ: $U_+ \times C_M(e)^0 \times U_- \to M$ by $\varphi(x,y,z) = xyz$. We define a monoid structure on $U_+ \times C_M(e)^0 \times U_-$ so that φ is a morphism, and $U_+ \times C_M(e)^0 \times U_-$ is regular. From there it follows that φ is surjective and birational. But M is normal, so φ is an isomorphism.

By Corollary 2.3 and the comments following Corollary 2.4 of [4], $\Phi_{U_+} \subseteq X(\overline{T})$ and $\Phi_{U_-} \subseteq -X(\overline{T})$. So we obtain $\overline{T} \to \mathrm{End}(U_+)$ extending $T \to \mathrm{Aut}(U_+), g \mapsto \mathrm{int}(g)$; and $\overline{T} \to \mathrm{End}(U_-)$ extending $T \to \mathrm{Aut}(U_-), g \mapsto \mathrm{int}(g^{-1})$. So the sought after multiplication on $U_+ \times C_M(e)^0 \times U_-$ can be defined as in (4) on Page 296 of [4]. That is

$$(u,x,v)(a,y,b) = (u\zeta_+(v,a)^x, x\zeta_0(u,v)y, \zeta_-(v,a)^{\overline{y}}b)$$

where ζ_+, ζ_0 and ζ_- are defined by

$$\zeta_+\colon\ U_- \times U_+ \underset{m}{\longrightarrow} U_+U_0U_- \underset{p_1}{\longrightarrow} U_+,$$

$$\zeta_0\colon\ U_- \times U_+ \underset{m}{\longrightarrow} U_+U_0U_- \underset{p_2}{\longrightarrow} U_0, \text{ and}$$

$$\zeta_-\colon\ U_- \times U_+ \underset{m}{\longrightarrow} U_+U_0U_- \underset{p_3}{\longrightarrow} U_-.$$

The action of $x \in \overline{T}$ on $u \in U_+$ is denoted u^x, and $y \in \overline{T}$ on $v \in U_-$ by $v^{\overline{y}}$. ■

3. Putting it together

In this section we start with the pieces, and show how to construct a regular monoid.

Definition 3.1 Setup. Let M_0 be a normal, reductive monoid with 0, and let U be a connected, unipotent group with regular action $\rho\colon G_0 \to \mathrm{Aut}(U)$ such that $\Phi_U \subseteq X(\overline{T}) \cup -X(\overline{T})$.

In the situation of 3.1 we can write

$$U = U_+U_0U_-$$

where

$$\begin{aligned}
\mathcal{L}(U_+) &= \bigoplus_{\alpha \in X(\overline{T})} \mathcal{L}(U)_\alpha, \\
\mathcal{L}(U_0) &= C_{\mathcal{L}(U)}(T) \text{ and} \\
\mathcal{L}(U_-) &= \bigoplus_{\alpha \in -X(\overline{T})} \mathcal{L}(U)_\alpha.
\end{aligned}$$

Proposition 3.2. *U_+, U_0 and U_- are stabilized by G_0 under ρ.*

Proof. Let $\lambda\colon k^* \to Z(G_0) \subseteq T$ be a 1-psg such that $\lim_{t\to 0} \lambda(t) = 0$. Such a λ exists because G_0 is reductive. Then $\lambda^*(X(T)) \subseteq Z = X(k^*)$. One checks that $\lambda^*(X(\overline{T}) \setminus \{0\} \subseteq \mathbb{Z}^+$ and $\lambda^*(-X(\overline{T}) \setminus \{0\}) \subseteq \mathbb{Z}^-$. Thus,

$$\begin{aligned} U_+ &= \{u \in U \mid \lim_{t\to 0} \lambda(t)u\lambda(t)^{-1} = 1\} \\ U_- &= \{u \in U \mid \lim_{t\to 0} \lambda(t)^{-1}u\lambda(t) = 1\} \text{ and} \\ U_0 &= C_U(\lambda(k^*)). \end{aligned}$$

But $\lambda(k^*) \subseteq G_0$ is central. Thus, U_+, U_0, and U_- are stabilized by G_0 under ρ. ■

Theorem 3.3. *Let M_0, ρ and U be as in 3.1. Then $U_+ \times M_0 \times U_0 \times U_-$ has the unique structure of a regular, algebraic monoid extending the group law on $U_+ \times G_0 \times U_0 \times U_- \overset{\longrightarrow}{\cong} G \ltimes U, (u,g,v,w) \longmapsto (g, uvw)$.*

Proof. By 3.2, ρ: $G \to \mathrm{Aut}(U)$ stabilizes U_+, U_0 and U_-. By definition, $\rho|T$: $T \to \mathrm{Aut}(U_+)$ extends over $\overline{T}, \rho^{-1}|T$: $T \to \mathrm{Aut}(U_-)$ extends over $\overline{T}$. Thus, by [3; Corollary 4.5] there exist unique ρ_+: $M_0 \to \mathrm{End}(U_+)$ extending ρ: $G_0 \to \mathrm{Aut}(U_+)$ and unique ρ_-: $M_0 \to \mathrm{End}(U_-)$ extending ρ^{-1}: $G_0 \to \mathrm{Aut}(U_-)$.

Using formula (4) on p. 296 of [4] we can define the desired multiplication on $U_+ \times M_0 \times U_0 \times U_-$, just as we did in Proposition 2.6 above. ■

4. The general case

In this section we consider normal regular monids, but without the restrictions of Assumption 2.1. So let M be normal and regular. If $e \in E(M)$ is a minimal idempotent define

$$N = \overline{G_e R_u(G)}.$$

Lemma 4.1.

(a) $gNg^{-1} \subseteq N$ for $g \in G$.

(b) N is a regular monoid of the type considered in Assumption 2.1.

Proof. If $g \in G$ then $gG_eg^{-1} = G_{geg^{-1}}$. But from [1; Theorem 6.30] it follows that $geg^{-1} = heh^{-1}$ for some $h \in G_eR_u(G)$. But then $gG_eg^{-1} = hG_eh^{-1}$, and so $gG_eR_u(G)g^{-1} = gG_eg^{-1}gR_u(G)g^{-1} = gG_eg^{-1}R_u(G) = hG_eh^{-1}R_u(G) = hG_eh^{-1}hR_u(G)h^{-1} = hG_eR_u(G)h^{-1} = G_eR_u(G)$ since $h \in G_eR_u(G)$. By continuity, $gNg^{-1} \subseteq N$.

For (b), notice that G_e is reductive by [1; Theorem 7.4]. But $(G_eR_u(G))_e = G_e$ and so, again by [1; Theorem 7.4], N is regular. Furthermore, $G_e \times R_u(G) \to G$ is bijective. But we need a little more in positive characteristic.

So let $k^* \subseteq Z(G_e)$ be such that $e \in \overline{k^*}$, as in the proof of Proposition 3.2. So $G_e \subseteq C_G(k^*) = G_eU_0 = U_0G_e$. But also $\mathcal{L}(G) = \mathcal{L}(G)_+ \oplus \mathcal{L}(G_eU_0) \oplus \mathcal{L}(G)_-$, because global and infinitesimal centralizers correspond for torus actions. But from the proof of Proposition 3.2, $\mathcal{L}(U_+) \subseteq \mathcal{L}(G)_+$ and $\mathcal{L}(U_-) \subseteq \mathcal{L}(G)_-$. Thus, $\mathcal{L}(U_+) = \mathcal{L}(G)_+$ and $\mathcal{L}(U_-) = \mathcal{L}(G)_-$, since dim $G = \dim(U_+) + \dim(U_-) + \dim(G_e U_0)$, while $U_+ \times G_e U_0 \times U_- \to G$ is bijective. Hence, $U_+ \times G_e U_0 \times U_- \overset{\longrightarrow}{\cong} G$. But then $G_e \cap R_u(G) = G_e \cap U_0$. But from 2.4(b), $G_e \subseteq C_G(U_0)$. So $G_e \cap U_0$ is a central, unipotent subgroup scheme of G_e. On the other hand, it is well known that $Z(G_e)$ is a diagonalizable group (possibly nonreduced, in general). In any case $G_e \cap U_0 = G_e \cap R_u(G)$ must be the trivial group scheme. Thus, $G_e \times R_u(G) \to G$ is separable, and therefore an isomorphism. ■

Let $H = G_e R_u(G)$ and define $N \times^H G = \{(x,g) \mid x \in N, g \in G\}/\sim$ where $(x,g) \sim (xh^{-1}, hg)$ if $h \in H$. Define φ: $N \times_H G \to M$ by

$$\varphi([x,g]) = xg.$$

Theorem 4.2. *φ is an isomorphism.*

Proof. From the proof of 4.1, H is a normal subgroup of G_-. Define a multiplication on $N \times^H G$ by $[x,g][y,h] = [xgyg^{-1}, gh]$. One checks that this is well defined. Furthermore, φ is a morphism of algebraic monoids.

Now φ is birational since $G(N \times^H G) = G = G(M)$. But also, $G\varphi(N)G = M$, since by [1, Proposition 6.27], N intersects every $\mathcal{J}$-class of M. So φ is surjective and birational, while M is normal. Thus, φ is an isomorphism. ■

Theorem 4.2 tells us how regular monoids, in general, are constructed from those that satisfy Assumption 2.1.

Indeed, let N be a normal regular monoid with unit group H, and assume $H = H_e R_u(H)$ (as in 2.1). Assume $H \lhd G$ and G/H is reductive. Then we can define a regular monoid M with unit group G

$$M = N \times^H G \text{ with multiplication}$$
$$[x,g][y,h] = [xgyg^{-1}, gh].$$

By Theorem 4.2, all normal regular algebraic monoids are obtained this way.

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