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Representations and blocks of algebraic monoids

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Abstract. This paper is a survey on the representations of algebraic monoids. Obviously, there are many similarities with algebraic groups, since the dense unit groups exert a lot of influence on the outcome. The differences, however, are very interesting as it concerns any salient diagram of associative algebras or representations. I have focussed mainly on this issue, and the related results that could be of interest to the participants of the ICRA-X Workshop.

1 Introduction

The ultimate dictionary might define the theory of algebraic monoids as *a branch of algebra that determines the content of mathematical problems relating convexity and positivity to representation theory.* We can regard this imaginary, dictionary definition as the theme of our discussion. What I want to do here is provide the reader with a reasonably self-contained overview of what I know about algebraic monoids and their finite-dimensional representations. Many of the results I discuss have been known for some time, but it appears that no one has ever published a monoid survey aimed at the ICRA faithful. So, it is time.

Let M be an irreducible algebraic monoid over the algebraically closed field K. See [15, 20]. Recall that M is *normal* if it is so as an algebraic variety, and *reductive* if the unit group of M is a reductive group. The single most important result in the representation theory of reductive, normal monoids is the following *extension principal* [22].

We fix notation: Let M be a reductive, normal algebraic monoid with unit group G and let $T \subseteq G$ be a maximal torus of G. Let $Z = \overline{T}$ be the Zariski closure of T in M. If $\alpha : T \longrightarrow End(V)$ is a rational representation of T, let $\Phi(\alpha) \subseteq X(T)$ be the set of weights of T on V.

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Theorem 1.1 *Let $\rho : G \longrightarrow Gl(V)$ be a rational representation of G, and let $\alpha = \rho|T$. The following are equivalent:*

a) *ρ extends to a rational representation $\overline{\rho} : M \longrightarrow End(V)$.*
b) *α extends to a rational representation $\overline{\alpha} : Z \longrightarrow End(V)$.*
c) *$\Phi(\alpha) \subseteq X(Z)$.*

Thus, we can use weights to identify *Rep*(M) as a subset of *Rep*(G), and for irreducible representations, we only need the highest weight (see 2.6 below). This theorem had already been observed by Schur for $M = M_n(K)$. He refers to representations of $M_n(K)$ as *polynomial* representations of $Gl_n(K)$.

In the next section we discuss representations of normal, reductive monoids. Here we find that there is a perfect analogue of many results about reductive groups. In particular, we obtain the desired relationship between the set of irreducible representations of M, and the coordinate ring of the adjoint quotient of M.

In the third section we focus on the special properties of a normal, reductive monoid M in characteristic $p > 0$. The results here are largely due to S. Doty. We find that $K[M]$ has a *good* filtration in the sense of Donkin [6]. Furthermore, the category of rational M-monoids is a *highest weight category* in the sense of Cline, Parshall and Scott [3].

In the fourth and fifth sections we discuss the *blocks* of an algebraic monoid. Blocks are often better behaved for monoids than they are for groups. We discuss the blocks of monoids in two contrasting situations; solvable monoids with zero, and $M = M_n(K)$ when $char(K) = p > 0$. The blocks of a solvable monoid were studied by the author in [26]. The blocks of $M_n(K)$ were calculated by S. Donkin in [7].

In the final section we discuss the enumerative theory of irreducible, modular representations of the finite, reductive monoids $M(\mathbb{F}_{q^r})$, $r > 0$. Here we find a revealing relationship between the modular representations of M and the generating function of the adjoint quotient of M; a striking blend of arithmetic topology, representation theory and combinatorics.

2 Irreducible representations and conjugacy classes of semisimple elements

Semisimple elements play an important rôle in the representation theory of reductive groups. The most fundamental results relate the conjugacy classes of semisimple elements to the characters of irreducible representations, via the ring of class functions on the adjoint quotient. In this section we discuss the analgous results for normal reductive monoids.

Let M be an irreducible algebraic monoid. An element $x \in M$ is *semisimple* if $\rho(x) \in M_n(K)$ is diagonalizable for any rational representation $\rho : M \longrightarrow M_n(K)$ of M.

Proposition 2.1 *Let M be irreducible. The following are equivalent.*

a) *$x \in M$ is semisimple.*
b) *$x \in \overline{T}$ for some maximal torus $T \subseteq G$.*

Proof b) clearly implies a). So assume a). Assume $M \subseteq M_n(K)$ as a closed submonoid. So $x \in M_n(K)$ is diagonalizable, and it follows easily that $x \in H_e$, the unit group of eMe. But x is semisimple in this reductive group and so $x \in S$ where

$S \subseteq H_e$ is a maximal torus. However, any maximal torus S of H_e is of the form $S = eT$ for some maximal torus $T \subseteq C_G(e)$. But $eT \subseteq \overline{T}$. So $x \in S = eT \subseteq \overline{T}$. □

Theorem 2.2 *Suppose M is reductive and $x \in M$. Then the following are equivalent:*

a) *x is semisimple.*
b) *$x \in \overline{T}$ for some maximal torus T of G.*
c) *$C\ell(x)$, the conjugacy class of x, is closed in M.*

Proof Suppose $x \in \overline{T}$. Then $txt^{-1} = x$ for $t \in T$. Thus, by 2.13, Corollary 1 of [32] $C\ell(x) \subseteq M$ is closed.

Conversely, assume $C\ell(x) \subseteq M$ is closed. We assume M has a zero element. The general case follows easily from this. By [14] the categorical quotient $\pi : M \longrightarrow X$ of M by $G \times M \longrightarrow M$, $(g, x) \longmapsto gxg^{-1}$, exists and induces a one to one correspondence between closed orbits of M and the points of X. Consider

$$\pi \mid \overline{T} \; : \; \overline{T} \longrightarrow X \; .$$

If $\pi(z) = \pi(y)$ then $gyg^{-1} = z$ for some g. But $g^{-1}Tg$ and T are both contained in $C_G(y)^\circ$. So there exists $h \in C_G(y)^\circ$ such that $hg^{-1}Tgh^{-1} = T$. But then $hg^{-1} \in N_G(T)$ while $(hg^{-1})^{-1}g \; hg^{-1} = gyg^{-1} = z$. Thus, $\pi|\overline{T}$ induces an injective map

$$\theta \; : \; \overline{T}/W \longrightarrow X$$

where $W = N_G(T)/T$. On the other hand θ is known to be birational by 3.4, Corollary 2 [32]. So θ is an open embedding, topologically. Since M has a zero, $\overline{T}/W$ and X are cones. It follows that θ is a finite bijective morphism. If M is normal then so is X and thus, θ is an isomorphism.

Recall now the element $x \in M$ with $C\ell(x) \subseteq M$ closed. By the above, $\pi(x) = \pi(y)$ for some $y \in \overline{T}$. But π separates closed orbits. So $C\ell(x) = C\ell(y)$. □

Corollary 2.3 *Suppose M is reductive and let $T \subseteq G$ be a maximal torus. Then*

$$\begin{aligned} C_M(T) &= \{x \in M \mid xt = tx \text{ for all } t \in T\} \\ &= \overline{T} \; . \end{aligned}$$

Proof If $x \in C_M(T)$ then $C\ell(x) \subseteq M$ is closed, and so by 2.2, x is semisimple. It is then possible to find a Borel subgroup $B \subseteq G$ such that $T \cup \{x\} \subseteq \overline{B}$. From there we embed $\overline{B}$ in $T_n(K)$ as a closed submonoid. It follows from Proposition 15.4 [12] that $T \cup \{x\}$ is contained in a maximal torus of $T_n(K)$. So $T \cup \{x\} \subseteq T_n(K) \underset{\pi}{\longrightarrow} D_n(K)$ is injective, where π is the projection to the diagonal. But from Theorem 2.3 of [23], $\pi|\overline{B}$ factors through the universal morphism of $\overline{B}$ to a D-monoid $\theta : \overline{B} \longrightarrow \overline{T}$ and thus $x \in \overline{T}$. □

Corollary 2.4 *Let $\pi : M \longrightarrow X$ be the categorical quotient induced by conjugation of G on M. Then in the following diagram θ is an isomorphism*

$$\begin{array}{ccc} \overline{T} & \hookrightarrow & M \\ \big\downarrow & & \big\downarrow \pi \\ \overline{T}/W & \underset{\theta}{\longrightarrow} & X \; . \end{array}$$

Proof This is included in the proof of 2.2. □

Example 2.5 Let M be a normal monoid with 0 and unit groups $G\ell_2(K)$. By [21], $M \cong M_r$, $r \in \mathbb{Q}^+$, where M_r is the unique semisimple monoid with

$$X(\overline{T}) \cong \{(a,b) \in \mathbb{Z}^2 \;\left|\; \left|\frac{b-a}{b+a}\right| \leq r\right\} \cup \{(0,0)\} \;.$$

Assume further that $r = 1/n$ with $(2,n) = 1$. Then

$$K[\overline{T}] \cong K[x,u,v]/(x^n - uv)$$

and the non-trivial element σ of the Weyl group $W = \{1, \sigma\}$ acts by

$$\begin{array}{rcl} \sigma(x) & = & x, \\ \sigma(u) & = & v \quad \text{and} \\ \sigma(v) & = & u \;. \end{array}$$

Hence, the ring of invariants is

$$K[X] \cong K[x,u,v]^W = K[x,\; u+v] \;.$$

Thus,

$$\pi \;:\; M_r \longrightarrow X \cong K^2$$

is a flat morphism.

The irreducible representations of a normal monoid can be calculated using Theorem 2.2. Let M be normal and reductive. We obtain

$X(\overline{T}) \subseteq X(T)$, characters of $\overline{T}$, and
$X(\overline{T})^+ \subseteq X(\overline{T})$, the set of dominant weights of $\overline{T}$.

$X(\overline{T})^+$ is obtained by intersecting the set $X(T)^+$ of dominant weights of $X(T)$ with $X(\overline{T})$. As expected, $X(\overline{T})^+$ is a fundamental domain for the action of W on $X(\overline{T})$.

Theorem 2.6 *Let M be reductive and normal. Then there is a canonical one to one correspondence between $X(\overline{T})^+$ and the set of irreducible representations of M.*

Proof Assume $\rho : M \longrightarrow End(V)$ is irreducible. Then $\rho|G$ is irreducible since $G \subseteq M$ is dense. So $\rho|G$ is identified by its highest weight $\lambda \in X(T)^+$. But, $\rho|G$ came from ρ, and so $\lambda \in X(\overline{T})$. So $\lambda \in X(\overline{T})^+$. Conversely, given $\lambda \in X(\overline{T})^+$ we start with $\rho_\lambda : G \longrightarrow G\ell(V_\lambda)$, the unique irreducible representation with highest weight λ. But all the other weights of ρ_λ are in the convex hull of $W \cdot \lambda$, which is contained in $X(\overline{T})$. So $\rho_\lambda|T$ extends over $\overline{T}$. Thus, by Theorem 1.1, ρ_λ extends to $\rho_\lambda : M \longrightarrow End(V_\lambda)$. □

The reader who wants more detailed information about rational representations should consult [8, 24].

3 Rep(M) according to Doty

In this section we discuss some results of S. Doty [8] concerning the structure of $K[M]$ as a $G \times G$-module. This is not much of an issue if $char(K) = 0$ since in that case, $K[M] = \bigoplus_{\lambda \in L(M)} K[M]_\lambda$, and each $K[M]_\lambda$ is $G \times G$-irreducible. Furthermore, $L(M)$ is just the set of high weights of G that come from representations of M. If M is normal, we see that $L(M) = X(\overline{T})$, as in Theorem 2.6.

On the other hand, if $char(K) = p > 0$, then the situation is more complex. First of all, it is no longer sufficient to consider just the simple $G \times G$-modules. This leads us naturally to the theory of *highest weight categories*. From there we can better understand $K[M]$ in terms of filtrations.

Let $\lambda : B \longrightarrow K^*$ be a character of the Borel subgroup B of G. Define

$$H^0(\lambda) = ind_B^G(K_\lambda) = \left\{ \; f \in K[G] \; \middle| \; \begin{array}{l} f(bg) = \lambda(b)f(g) \text{ for all} \\ b \in B, \, y \in G \end{array} \right\} .$$

$H^0(\lambda)$ is naturally a G-module via $g \cdot f(x) = f(xg)$ for $x, g \in G$. It is well known that $H^0(\lambda)$ is nonzero if and only if λ is dominant. In this case, the unique, maximal completely reducible submodule of $H^0(\lambda)$ is

$$V_\lambda \subseteq H^0(\lambda),$$

the irreducible G-module with highest weight λ.

Now let M be normal and reductive. Define

$$ind\frac{M}{B}(K_\lambda) = \left\{ \; f \in K[M] \; \middle| \; \begin{array}{l} f(xy) = \lambda(x)f(y) \text{ for all} \\ x \in \overline{B}, \, y \in M \end{array} \right\} .$$

It follows from Theorem 2.6 that $ind\frac{M}{B}(K_\lambda) \neq (0)$ if and only if $\lambda \in X(\overline{T})^+$, the set of dominant weights of $X(\overline{T})$.

Definition 3.1 a) A *good filtration* $(0) = V_0 \subseteq V_1 \subseteq V_2 \subseteq \dots$ of the rational G-module $V = \bigcup_{i \geq 0} V_i$ is one for which $V_{i+1}/V_i \cong H^0(\lambda_i)$ for each i, where λ_i is a dominant weight.

b) Let $X(T)^+$ be the set of dominant weights of T. If $\pi \subseteq X(T)^+$ and V is a rational G-module we say V *belongs to* π if any G-module V_λ of the Jordan-Hölder series of V has $\lambda \in \pi$. In any case we let

$$O_\pi V \subseteq V$$

be the maximal submodule of V belonging to π.

Proposition 3.2 *Let* $\pi = X(\overline{T})^+$.

a) $O_\pi H^0(\lambda) = \begin{cases} H^0(\lambda) & \textit{if } \lambda \in \pi \\ (0) & \textit{if } \lambda \notin \pi \, . \end{cases}$

b) *Let* $I(\lambda)$ *(resp.* $Q(\lambda)$*) be the injective hull of* V_λ *in the category of rational* G*-modules (resp.* M*-modules). Then*

$$O_\pi(I(\lambda)) = \begin{cases} Q(\lambda) & , \quad \lambda \in \pi \\ (0) & , \quad \lambda \notin \pi \, . \end{cases}$$

Furthermore, $Q(\lambda)$ *has a good filtration with all factors* V_μ *having* $\mu \geq \lambda$*.*

c) $O_\pi K[G] = K[M]$*. Furthermore,* $K[M]$ *has a good filtration with each* $H^0(\mu)$*,* $\mu \in \pi$*, occurring exactly dim* $H^0(\mu)$ *times.*

Proof a) Any submodule of $H^0(\lambda)$ contains V_λ. so if $\lambda \notin \pi$, then $O_\pi H^0(\lambda) = (0)$. Now suppose $\lambda \in \pi$. Then $H^0(\lambda)$ lifts (by 3.1.2) to become an M-module. So $O_\pi H^0(\lambda) = H^0(\lambda)$.

b) For b), first notice that $O_\pi(-)$ takes injectives to injectives by (1.1d) of [6]. So the formula for $O_\pi(I(\lambda))$ follows. $Q(\lambda)$ has a good filtration by Theorem 8 of [6].

c) This is proved by "Frobenius" reciprocity. See Theorem 4.4 of [8].

□

Corollary 3.3 *$K[M]$ as an $M \times M$-module, has a good filtration with composition factors of the form $H^0(\lambda) \otimes H^0(\lambda^*)$.*

Proof This follows from c) using (2.2a) of [6]. □

We now explain the key features of the category $Rep(M)$ of rational M-modules. The ingredients are

a) $X(\overline{T})^+ = X(T)^+ \cap X(\overline{T})$ the *poset* of dominant weights of M.

b) $\{V_\lambda \mid \lambda \in X(\overline{T})^+\}$ the *simple* objects of $Rep(M)$.

c) $\{H^0(\lambda) \mid \lambda \in X(\overline{T})^+\}$ the *standard* objects of $Rep(M)$.

The following theorem is recorded in [8]

Theorem 3.4 a) *Socle $(H^0(\lambda)) = V_\lambda$ and the composition series of $H^0(\lambda)/V_\lambda$ has only factors of the form V_μ with $\mu < \lambda$.*

b) *Each V_λ has an injective hull $V_\lambda \subseteq Q(\lambda)$ so that $Q(\lambda)$ has a good filtration $(0) = Q(\lambda)_0 \subseteq Q(\lambda)_1 \subseteq \dots$ with $Q(\lambda)_1 = H^0(\lambda)$ and $Q(\lambda)_{i+1}/Q(\lambda)_i = H^0(\mu_i)$ with $\mu_i > \lambda$ for $i > 0$.*

Proof a) is well known, and b) follows from 3.2b). □

We have thus identified a key result about reductive normal monoids. $Rep(M)$ is a *highest weight category* in the sense of Cline, Parshall and Scott [3]. We cannot pursue all the important consequences of this result, but we shall give one striking illustration. Let M be reductive and normal, and suppose M has a zero element. So we can write uniquely

$$K[M] = \bigoplus_{\chi \in Y} K[M]_\chi$$

where $Y = X(\overline{ZG^0})$, and each $K[M]_\chi$ is the subcoalgebra of $K[M]$ defined by

$$K[M]_\chi = \left\{ f \in K[M] \;\middle|\; \begin{array}{l} f(gx) = \chi(g)f(x) \text{ for} \\ g \in \overline{ZG^\circ}, \text{ and } x \in M \end{array} \right\}.$$

It follows that each $K[M]_\chi$ is finite dimensional. Thus, for each $\chi \in Y$,

$$S(M)_\chi = Hom_K(K[M]_\chi, K)$$

is a finite dimensional K-algebra. It follows from Theorem 3.4 that

$$S(M)_\chi \text{ is quasihereditary.}$$

One could also prove this using Donkin's work, since, in Donkin's notation, $S(M)_\chi$ is the generalized Schur algebra $S(\pi_\chi)$ where

$$\pi_\chi = i^{*-1}(\chi) \cap X(\overline{T})^+$$

and $i : \overline{Z(G)^\circ} \longrightarrow \overline{T}$ is the inclusion. Notice that $\pi_\chi \subseteq X(\overline{T})^+$ is a saturated subset of $X(\overline{T})$ with the given ordering.

4 The blocks of $M_n(K)$ when $char(K) = p > 0$

A *block* can be thought of as an equivalence class of irreducible representations. The equivalence relation in this setup is generated by declaring irreducible representations (ρ, U) and (φ, W) in the same block if there exists an indecomposable representation (ψ, V) such that (ρ, U) and (φ, W) occur as factors in a composition series of (ψ, V). In this section we describe Donkin's calculation [7], of the blocks of $M_n(K)$. We end this section with a related, general conjecture about the blocks of irreducible, reductive monoids in characteristic $p > 0$.

Let $M = M_n(K)$, and let $T \subseteq G\ell_n(K)$ be the maximal torus of diagonal matrices. Then

$$\begin{aligned} X(T) &= \mathbb{Z}^n = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z}\}, \\ X(\overline{T}) &= \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \geq 0\} \text{ and} \\ X(T)^+ &= \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}\ . \end{aligned}$$

So

$$X(\overline{T})^+ = \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\},$$

and from Theorem 2.6 we can identify the set of irreducible representations of M with $X(\overline{T})^+$ via the usual identification using highest weights $(V, \rho) = (V_\lambda, \rho_\lambda)$. So let $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(\overline{T})^+$. Define

$$\sum_{i=1}^{n} \lambda_i = r, \text{ the } \textit{degree} \text{ of } \lambda, \text{ and}$$

$$d(\lambda) = max \left\{ \ d \geq 0 \ \middle| \ \begin{array}{l} \lambda_i - \lambda_{i+1} \equiv -1 \bmod(p^d) \text{ for} \\ \text{all } i = 1, 2, \ldots n-1 \end{array} \right\}.$$

Assume $char(K) = p > 0$.

Theorem 4.1 (Donkin's Theorem) *Let $(\rho_\lambda, V_\lambda)$ and (ρ_μ, V_μ) be irreducible representations of $M_n(K)$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$. Then the following are equivalent:*

a) *λ and μ are in the same block.*

b) i) *λ and μ have the same degree.*
   ii) *$d(\lambda) = d(\mu)$.*
   iii) *There exists $\pi \in S_n$ such that $\lambda_i - i \equiv \mu_{\pi(i)} - \pi(i) mod(p^{d+1})$ for all $i = 1, \ldots, n$.*

There is a more conceptual way to state Donkin's Theorem, that leads to an interesting conjecture. Let λ and μ be as above. By the results of [5], λ and μ are in the same block for $Gl_n(K)$ if and only if the three conditions of 4.1b) above are satisfied. Furthermore, the blocks of any reductive group can be determined in the spirit of the above result for $Gl_n(K)$ using Theorem 5.8 of [5].

Let M be a reductive, normal monoid and let $\lambda, \mu \in X(\overline{T})^+$ represent irreducible representations of M (as in Theorem 2.6). Let

$Bl_G(\lambda) = \{\nu \in X(T)^+ \mid \lambda \text{ and } \nu \text{ are in the same } G\text{-block}\}$ and
$Bl_M(\lambda) = \{\nu \in X(\overline{T})^+ \mid \lambda \text{ and } \nu \text{ are in the same } M\text{-block}\}$.

Conjecture 4.2 *$Bl_M(\lambda) = Bl_G(\lambda) \bigcap X(\overline{T})^+$. In particular, λ and μ are in the same M-block if and only if they are in the same G-block.*

5 The blocks of solvable algebraic monoids

It turns out that there is a striking description of the blocks of certain solvable algebraic monoids. While this is a special case, our basic idea here is expected to yield some decisive results for a large class of (nonsolvable) algebraic monoids in characteristic zero.

Let M be an irreducible, algebraic monoid with unit group G and zero element $0 \in M$.

Definition 5.1 a) M is *solvable* if G is a solvable algebraic group.

b) M is *polarizable* if $0 \in \overline{Z(G)}$.

c) M is *polarized* if we are given $\theta : K^* \longrightarrow Z(G)$ such that θ extends to $\overline{\theta} : K \longrightarrow M$ with $\overline{\theta}(0) = 0$. θ is then called the *polarization*, and (M, θ) is a polarized monoid.

It is easy to check that M is polarizable if and only if there exists a polarization $\theta : K^* \longrightarrow Z(G)$, of M.

Let (M, θ) be polarized and solvable. Then $K^* \times M \longrightarrow M$, $(\alpha, x) \longmapsto \theta(\alpha)x$, induces a direct sum decomposition

$$\mathcal{O}(M) = \bigoplus_{n=0}^{\infty} \mathcal{O}_n(M)$$

where

$$\mathcal{O}_n(M) = \left\{ f \in \mathcal{O}(M) \;\middle|\; \begin{array}{l} f(\theta(\alpha)x) = \alpha^n f(x) \text{ for all} \\ \alpha \in K^*,\, x \in M \end{array} \right\}.$$

It follows easily that

$$\Delta(\mathcal{O}_n(M)) \subseteq \mathcal{O}_n(M) \otimes \mathcal{O}_n(M)$$

where Δ is the coalgebra structure on $\mathcal{O}(M)$.

Let (M, θ), (N, ϕ) be polarized monoids. A *θ-morphism* between (M, θ) and (N, ϕ) is a morphism $\varphi : M \longrightarrow N$ of algebraic monoids such that $\varphi(0) = 0$ and $\varphi(\text{ Image } (\theta)) \subseteq \text{ Image } (\phi)$. The *$\theta$-degree* of φ is the degree of $\varphi | \mathit{image}(\theta)$. So $\varphi(\theta(\alpha)) = \phi(\alpha^n)$ if φ is of θ-degree n.

Let A be a finite-dimensional associative K-algebra. The *blocks* of A are the obvious summands in the decomposition

$$A = \bigoplus_{e \in Z} eAe$$

where Z is the set of primitive, central idempotents of A. We denote the blocks of A by $B\ell(A)$ and identify them with Z.

If (M, θ) is a polarized monoid we define the *blocks of M* to be

$$B\ell(M) = \bigsqcup_{n \geq 0} B\ell(S_n(M))$$

where $S_n(M) := Hom_K(\mathcal{O}_n(M), K)$ has the algebra structure induced from the canonical coalgebra structure of $\mathcal{O}_n(M)$.

It is easy to check that this definition agrees with the one given by Green in 1.6b) of [11]. Indeed, he proves that any coalgebra (R, Δ) has a unique expression $R = \bigoplus_{\rho \in \mathcal{B}} R_\rho$ such that

i) $\Delta(R_\rho) \subseteq R_\rho \otimes R_\rho$ for all $\rho \in \mathcal{B}$.

ii) For any other direct sum decomposition $R = \bigoplus_{\gamma \in \Lambda} A_\gamma$ with $\Delta(A_\gamma) \subseteq A_\gamma \otimes A_\gamma$ each A_γ is a sum of some R_ρ's.

Proposition 5.2 *Let (M, θ) and (N, ϕ) be polarized monoids and let $\varphi : M \longrightarrow N$ be a dominant θ-morphism. Then φ induces a map of sets $B\ell(\varphi) : B\ell(N) \longrightarrow B\ell(M)$. This is a contravariant functor.*

Proof If φ has θ-degree k then the induced morphism $\varphi^* : \mathcal{O}_n(N) \longrightarrow \mathcal{O}_{kn}(M)$ dualizes to obtain a surjective morphism of K-algebras

$$\varphi_n : S_{kn}(M) \longrightarrow S_n(N) \ .$$

For each primitive, central idempotent e of $S_n(N)$ there is a unique primitive, central idempotent $f \in S_{kn}(M)$ such that $\varphi_n(f)e = e$. So define $B\ell(\varphi)(e) = f$. $\square$

We now describe $B\ell(M)$ for a polarized monoid (M, θ) with solvable unit group. As it turns out, there is a straightforward description of $B\ell(M)$ in terms of weight spaces. As above we have $S_n(M) = Hom_K(\mathcal{O}_n(M), K)$. Define

$$\rho_n : M \longrightarrow S_n(M)$$

by $\rho_n(x)(f) = f(x)$. ρ_n is the universal θ-morphism of θ-degree n to a K-algebra. It follows easily that $\rho_n(M) \subseteq S_n(M)$ spans, and $S_n(M)$ is a solvable K-algebra. If $T \subseteq G$ is a maximal torus one checks that $D_n = span(\rho_n(T))$ is a maximal toral subalgebra of $S_n(M)$.

Proposition 5.3 *Let $T \subseteq G$ be a maximal torus and define $\mu_n : T \times T \times S_n(M) \longrightarrow S_n(M)$ by $\mu_n(s, t, x) = \rho_n(s) x \rho_n(t)$. Then there is a bijective correspondence between the nonzero weight spaces*

$${}^{\alpha}S_n^{\beta} = \{x \in S_n(M) \mid \rho_n(s) x \rho_n(t) = \alpha(s)\beta(t)x \ \textit{for all}\ s, t \in T\}$$

and the pairs of primitive idempotents $(e, f) \in E(D_n) \times E(D_n)$ with $eS_n(M)f \neq (0)$. (e, f) corresponds to the unique ${}^{\alpha}S_n^{\beta}$ with $eS_n(M)f = {}^{\alpha} S_n^{\beta}$.

Proof We leave the details to the reader. The proof hinges on identifying the set of primitive idempotents of D_n with characters of T. See Proposition 2.3 of [26]. $\square$

So we define

$$S = \left\{(\alpha, \beta) \in X(T) \times X(T) \mid^{\alpha} S_n^{\beta} \neq 0 \text{ for some } n \geq 0\right\}$$

$$\Delta(\overline{T}) = \left\{(\alpha, \beta) \in X(\overline{T}) \times X(\overline{T}) \mid \alpha = \beta\right\} \ .$$

One checks that

$$\Delta(\overline{T}) \subseteq S \subseteq X(\overline{T}) \times X(\overline{T}) \ .$$

We use S to define an equivalence relation on $X(\overline{T})$. For $\alpha, \beta \in X(\overline{T})$ we define

$$\begin{array}{lll} \alpha \longrightarrow \beta & \text{if} & (\alpha, \beta) \in S, \text{ and} \\ \alpha \longleftarrow \beta & \text{if} & (\beta, \alpha) \in S \ . \end{array}$$

Lemma 5.4 a) *The equivalence relations on $X(\overline{T})$ generated by $\longrightarrow$ is the same as the equivalence generated by $\longleftarrow$. It can be described as follows:*

$\alpha \sim \beta$ if there exist $\gamma_1, \ldots, \gamma_{2m-1} \in X(\overline{T})$ such that $\alpha = \gamma_1 \longrightarrow \gamma_2 \longleftarrow \ldots \longleftarrow \gamma_{2m-1}$.

b) *Suppose $\alpha \sim \beta$ and $\lambda \sim \delta$. Then $\alpha\lambda \sim \beta\delta$.*

Proof a) is a straightforward calculation. For b) one uses the fact that S is a semigroup, together with the fact that the "m" for $\alpha \sim \beta$ can be chosen equal to the "m" for $\lambda \sim \delta$. □

Theorem 5.5 *$(X(\overline{T})/\sim\,) \cong B\ell(M)$ via $\alpha \longmapsto [\alpha]$. So $X(\overline{T}) \longrightarrow B\ell(M)$ is a surjective morphism of monoids.*

Proof $B\ell(M) = \bigsqcup_{n\geq 0} B\ell(S_n(M))$. So let $D_n = Span(\rho_n(T))$ be the maximal toral subalgebra as discussed above. Using standard facts about associative algebras we see that

$$B\ell(S_n(M)) = E_1(D_n)/\sim_\circ$$

where $\sim_\circ$ is the equivalence relation on $E_1(D_n)$ generated by declaring

$$e \sim_\circ f \quad \text{if} \quad eS_n(M)f \neq 0 \quad \text{or} \quad fS_n(M)e \neq 0\ .$$

But $E_1(D_n)$ is identified, via 5.3, with $S_n = \{(\alpha,\beta) \in S \mid^\alpha S_n(M)^\beta \neq 0\}$. So the two equivalence relations correspond. Thus $X(\overline{T})/\sim\, \cong \left(\bigsqcup_{n\geq 0} E_1(D_n)\right)/\sim_\circ$. □

Now $X(\overline{T}) \longrightarrow X(\overline{T})/\sim$ determines a subscheme $Y \subseteq \overline{T}$ via

$$Y = Spec(K[X(\overline{T})/\sim]) \qquad \text{(monoid algebra)}$$

and one obtains, from 3.2 of [26], that $Y \subseteq Z(M)$ the center of M. Furthermore, the surjection, $K[M] \longrightarrow K[Y]$, identifies $S_n(Y)$ with the maximal toral subalgebra of $Z(S_n(M))$. In particular,

$$Y = \bigcap_{g\in G} g\overline{T}g^{-1}\ .$$

From these comments, and a little more calculation (3.5 of [26]) we obtain the following theorem.

Theorem 5.6 *There is a canonical bijection*

$$B\ell_n(M,\theta) \cong X_n(Y)$$

where $X_n(Y) = \{\chi : Y \longrightarrow K \mid \chi$ has θ-degree $n\}$.

We conclude the chapter with three examples and a conjecture.

Example 5.7 We define polarizable, solvable monoids M and M' as follows:

$M = \{(u,(r,s)) \mid u,r,s \in K\}$ with
$(u,(r,s))(v,(k,\ell)) = (k\ell u + r^2 v,\ (rk, s\ell))$, and

$M = \{(u,(r,s) \mid u,r,s \in K\}$ with
$(u,(r,s))(v,(k,\ell)) = (\ell u + rv,\ (rk, s\ell))$.

Define $\varphi : M' \longrightarrow M$ by $\varphi(u,(r,s)) = (ru,(r,s))$. One checks that φ is a birational θ-morphism of degree one. Furthermore, φ induces an isomrophism

$$\varphi : \overline{T'} \underset{\cong}{\longrightarrow} \overline{T}\ .$$

We now compute the center of each monoid. Clearly,

$$Z(M) = \{(0,(r,r)) | r \in K\}$$

since M is the algebra of two by two upper triangular matrices. As for M', one needs a little more calculation, and we obtain

$$Z(M') = \{(0,(r,r)) \mid r \in K\} \cup \{(0,(0,s)) \mid s \in K\}\ .$$

In particular, the inclusion $Z(M) \subset Z(M')$ is proper, so that M and M' have different block structure even though $\varphi : M' \longrightarrow M$ is a birational equivalence with $\overline{T}' \underset{\cong}{\longrightarrow} \overline{T}$.

Example 5.8 Define a polarizable solvable monoid N as follows:

$$\begin{array}{l} N = \{(u,(r,s)) \mid u,r,s \in K\} \quad \text{with} \\ (u,(r,s))(v,(k,\ell)) = (k^2\ell u + r^3 v,\ (rk, s\ell)) \ . \end{array}$$

One checks that $\overline{T} = \{(0,(\alpha,\beta))|\alpha,\beta \in K\}$ is the closure in N of the maximal torus $T = \{(0,(\alpha,\beta)) \mid \alpha\beta \neq 0\}$. So assume $(0,(r,s)) \in \overline{T}$ is central. Then we must have

$$(0,(r,s))(v,(k,\ell)) = (v,(k,\ell))(0,(r,s)) \qquad \text{for all } v,k,\ell \ .$$

Thus, $r^3 v = sr^2 v$ for all v and so $r^3 = sr^2$. By the remarks preceding 6.4.6

$$\begin{array}{rcl} K[Y] & = & K[U,R,S]/(U, R^3 - SR^2) \\ & \cong & K[R,S]/(R^3 - SR^2) \ . \end{array}$$

Let $x = \overline{R}$, $y = \overline{S} \in K[y]$. Then $f = x(x-y) \neq 0$, yet

$$f^2 = x^2(x-y)^2 = (x^3 - x^2 y)(x-y) = 0 \ .$$

So $K[Y]$ is not reduced. We can also calculate the number of blocks of N of each θ-degree, using this presentation of $K[Y]$. In fact,

$$\begin{array}{rcll} |B\ell_0(N)| & = & 1 & \\ |B\ell_1(N)| & = & 2 & \\ |B\ell_n(N)| & = & 3 & \text{if } n \geq 2 \ . \end{array}$$

It appears that there may be an important structural relationship between the blocks of solvable monoids, and the blocks of arbitrary irreducible monoids (at least in characteristic zero).

Example 5.9 We start with $M = G$, a semisimple, simply connected algebraic group of rank r in characteristic zero. Here we find that

$$Bl(G) \quad = \quad IR(G),$$

and this is a monoid under Cartan product. Let $T \subset B \subset G$ be a maximal torus of the Borel subgroup B of G. Let $\{\rho_1, \ldots, \rho_r\}$ be the set of fundamental, dominant representations of G, so that $\rho_i : G \longrightarrow Gl(V_i)$. Let $L_i \subset V_i$ be the unique, one-dimensional subspace stabilized by B. Let $D_n(K)$ be the monoid of diagonal $n \times n$ matrices. Define

$$\varphi \quad : \quad T \longrightarrow D_n(K) \tag{5.1}$$

by $\varphi(t) = (\rho_1(t)|L_1, \ldots, \rho_r(t)|L_r)$. φ is the restriction of a $U \times U^-$-equivariant morphism $\psi : G \longrightarrow D_r(K)$, which is defined on UTU^- by $\psi(utv) = \varphi(t)$. Somehow, the $U \times U^-$-morphism ψ might be thought of as a kind of "basic monoid" associated with G. In particular,

a) $D_r(K)$ is solvable.
b) $Bl(G) \cong Bl(D_r(K)$.
c) G and $D_n(K)$ are (somehow) Morita equivalent via ψ.

Notice that even though G is a group, this "basic" object $D(K)$ is a monoid.

The above example leads us to an interesting conjecture about blocks.

Conjecture 5.10 *Let M be an irreducible, algebraic monoid. There exists an irreducible, algebraic monoid $B(M)$, and a certain dominant morphism $\psi : M \longrightarrow B(M)$ of algebraic varieties, such that*

a) *$B(M)$ is solvable.*
b) *M and $B(M)$ are (somehow) Morita equivalent via ψ*
c) *In particular, $Bl(M) \cong Bl(B(M))$.*

The reader might wonder what it means for algebraic monoids M and N to be *Morita equivalent*. Unfortunately, this has yet to be formulated precisely. Obviously, it will involve a bijection between $Bl(M)$ and $Bl(N)$, as well as a diagram of Morita equivalences between the corresponding (block) algebras one obtains from the "coordinate coalgebras" of M and N. In any case, there should be enough clues in the above example to find the "correct" definition, at least in the case of polarized monoids.

6 Modular representations of finite reductive monoids and the generating function of the adjoint quotient

In this section we consider reductive algebraic monoids M defined over a finite field $k = \mathbb{F}_q$. We readily obtain finite reductive monoids M_r, $r \geq 1$, as follows:

Let $M_r = M(\mathbb{F}_{q^r})$ be the finite monoid of $\mathbb{F}_{q^r}$-rational points of M. By standard facts about finite fields and Galois groups (Chapter II, Section 4 of [13]) there exists an $\mathbb{F}_q$-automorphism $\sigma : M \longrightarrow M$ of algebraic monoids such that

$$M_r = \{x \in M \mid \sigma^r(x) = x\} .$$

Thus, M_r is a monoid of Lie type in the sense of [17]

In this section we describe a useful formula for the number $|IR(M_r)|$ of irreducible, modular representations of M_r. The basic problem here is to consider the fomulas of the form

$$|IR(M_r)| = (q^r - 1)\sum_{i=1}^{n} a_i q^{ri}$$

where $a_i \in \mathbb{Z}$ and is independent of r. Whenever this can be done, it is particularly interesting to interpret the a_i's

We now state the main result of [13]. This is the main reason we are able to obtain so much information about irreducible modular representations of finite monoids of Lie type. Let M be a finite monoid of Lie type with unit group G of characteristic p.

Theorem 6.1 *Suppose $\rho : M \to End(V)$ is an irreducible representation of M over $\mathbb{F}_p$. Then $\rho|G$ is irreducible.*

The reader is reminded here that there is rarely such a direct and appealing relationship between the irreducible representations of a monoid and those of its unit group.

Let M be a finite monoid of Lie type of characteristic p, and let $\rho : M \to End(V)$ be an irreducible representation of M defined over $\mathbb{F}_p$. Let $\mathcal{U}(M)$ be the set of regular $\mathcal{J}$-classes of M. By standard results from [15], $\mathcal{U}$ is the set of $G \times G$-orbits on M. By the theory of Munn and Ponizovskii [2], ρ determines an *apex*, $Apex(\rho) \in \mathcal{U}(M)$. $Apex(\rho)$ is the unique smallest $\mathcal{J}$-class J of M such that $\rho(J) \neq 0$. But on the other hand, $\rho|G$ is irreducible by 6.1 above, and so by the theory of Richen [4], $\rho|G$ is determined by its *weight* $(I(\rho), \chi(\rho))$. In any case, $\rho : M \to End(V)$ determines the following data:

(i) $J = Apex(\rho) \in \mathcal{U}(M)$
(ii) $I = I(\rho) \in 2^S$
(iii) $\chi = \chi(\rho) : P_I \to \mathbb{F}_p^*$.

We consider the following two questions:

(a) Is ρ uniquely determined up to equivalence of representations by (J, I, χ)?
(b) What are the conditions on a triple (J, I, χ) with $\chi : P_I \to \mathbb{F}_p^*$ and $J \in \mathcal{U}(M)$, so that there exists an irreducible representation ρ of M with
  (i) $\rho|G$ of type (I, χ)
  (ii) $Apex(\rho) = J$?

To answer these two questions we need some further notions about monoids of Lie type. The reader should consult [16] for a detailed account of this theory (notice however that in [16], monoids of Lie type are referred to as *regular split monoids*).

Let M be a finite monoid of Lie type with unit group G. Let S be the Coxeter-Dynkin diagram of G. Let $\mathcal{U} = \mathcal{U}(M)$ be the set of regular $\mathcal{J}$-classes of M. It turns out that $\mathcal{U}$ is the set of two-sided G-orbits of M. Define $GaG \geq GbG$ if $b \in MaM$. In this way, $\mathcal{U}$ becomes a lattice. There is a cross-section of idempotents $\Lambda = \{e_J | J \in \mathcal{U}\} \subseteq E(M)$, such that $J = Ge_JG$, and for all $J_1, J_2 \in \mathcal{U}, e_{J_1}e_{J_2} = e_{J_2}e_{J_1} = e_{J_1 \wedge J_2}$. Λ is called a *cross-section lattice*. Furthermore, $E(J) = \{ge_Jg^{-1} | g \in G\}$.

Definition 6.2 The *type map*

$$\lambda : \mathcal{U} \to 2^S$$

is defined so that for all $J \in \mathcal{U}$

$$P(e_J) = P_{\lambda(J)}.$$

Here $P(e) = C_G^r(e) =: \{G \in G | ge = ege\}$, and $P_I \subseteq G$ denotes the *parabolic subgroup of type I* (see Section 2.5 of [1]).

The type map is the key ingredient in understanding how the orbits of a reductive monoid fit together to make things work.

We now introduce a new invariant. Let Λ be a cross-section lattice and let $e \in \Lambda$. Then $\{e\} = \Lambda \cap J$ and $C_G^r(e) = P_{\lambda(J)}$, where $\lambda(J) \subseteq S$. If $B \subseteq C_G^r(e)$ is a Borel subgroup and $H = \{g \in G | ge = e\}$ then BH is a parabolic subgroup containing B.

Definition 6.3

$$\nu(J) \in 2^S \text{ via } BH = P_{\nu(J)}.$$

Notice also, that if we let $K_J = \{g \in G | ge = eg = e\}$, then $P_{\nu(J)} = BK_J$. We can now state the main theorem (in particular, answering questions a) and b) above).

Theorem 6.4 *Let $I \in 2^S$ and $J \in \mathcal{U}(M)$. Assume that $\nu(J) \subseteq I \subseteq \lambda(J)$. Define*

$$\alpha_{I,J} = \{\chi : L_I \to \mathbb{F}_q^* | \chi(g) = \chi(h) \text{ if } e_Jg = e_Jh\}.$$

Then there is a one to one correspondence between the irreducible representations of M and the set

$$\bigsqcup_{\substack{I \in 2^S, J \in \mathcal{U}(M) \\ \nu(J) \subseteq I \subseteq \lambda(J)}} \alpha_{I,J}.$$

Under this correspondence $\chi \in \alpha_{I,J}$ corresponds to the unique irreducible representation $\rho : M \to End(M)$ such that

(i) $Apex(\rho) = J$
(ii) *There is a line* $Y \subseteq V$ *such that* $\{g \in G | \rho(g)Y = Y\} = P_I$.
(iii) *If* $g \in P_I$ *and* $y \in Y$ *then* $\rho(g)(y) = \chi(g)y$.

Proof Let $\rho : M \to End(V)$ be irreducible with apex $J \in \mathcal{U}(M)$. So by Munn-Ponizovskii, V is also an irreducible J^0-module; and by Theorem 6.1 above, for $e \in E(J)$, $e(V)$ is an irreducible H_e-module, where H_e is the unit group of eMe. But $H_e = eC_G(e)$, and so $e(V)$ is also an irreducible $C_G(e)$-module. Now $C_G(e) \subseteq G$ is the Levi factor of $P_{\lambda(J)} = C^r_G(e)$, and so it is also a finite group of Lie type. Hence, Richen's theory applies to the $C_G(e)$-module $e(V)$. Thus, for any Borel subgroup $B_0 \subseteq C_G(e)$ there exists a unique line $Y \subseteq e(V)$ such that $\rho(B_0)Y = Y$.

Let $H = \{g \in G | \rho(g)Y = Y\}$. One checks, as in [27], that $H \subseteq C^r_G(e)$ and H contains a Borel subgroup of G.

Observe that $K_J \subseteq H$, and so $P_{\nu(J)} = BK_J \subseteq H \subseteq P_{\lambda(J)} =: C^r_G(e)$. So we can now summarize the relevant properties of an irreducible representation $\rho : M \to End(V)$ with apex $J \in \mathcal{U}(M)$.

(a) Let $H = \{g \in G | \rho(g)Y = Y\}$. Then $H = P_I$ is parabolic and $\nu(J) \subseteq I \subseteq \lambda(J)$.
(b) If $g \in K_J$ then $\rho(g)y = y$ for all $y \in Y$.
(c) $\rho|G$ is the irreducible representation of type (I, χ), where χ is defined via $\rho(g)(y) = \chi(g)y$ for $g \in L_I$.

On the other hand, suppose $\rho' : M \to End(V')$ is an irreducible representation with apex J, and parabolic subgroup $H' = P_I$ with character χ. Then by Richen's results, $(\rho'|C_G(e), \rho'(e)(V)) \cong (\rho|C_G(e), \rho(e)(V))$ since they have the same (I, χ). But then by Munn-Ponizovskii, ρ and ρ' are equivalent because they come from the same irreducible representation of eMe. Thus, the correspondence

$$\rho \rightarrowtail (I, J, \chi)$$

is injective. To complete the proof, it remains only to be shown that all possible invariants (I, J, χ) actually arise from irreducible representations of M. But this is now a counting problem. It is easy to check, using Richen's results, that the number of irreducible representations of H_e is

$$\sum_{\nu(J) \subseteq I \subseteq \lambda(J)} |\alpha_{I,J}|$$

Thus, by Munn-Ponizovskii, there are exactly

$$\sum_{J \in \mathcal{U}(M)} \sum_{\nu(J) \subseteq I \subseteq \lambda(J)} |\alpha_{I,J}|$$

irreducible representations of M. This concludes the proof. □

Let $x \in M_r$. We say x is *semisimple* if

i) x is a unit in the monoid eM_re for some idempotent e of M_r.
ii) $x^k = e$ for some k with $(k, q) = 1$.

Let $M_r^{ss} = \{x \in M_r^{ss} \mid x \text{ is semisimple}\}$ and let $M_r^{ss}/\sim$ denote the set of conjugacy classes of semisimple elements of M_r.

Lemma 6.5 $|IR(M_r)| = |M_r^{ss}/\sim|$.

Proof By the theory of Munn and Ponizovskii $|IR(M_r)| = \sum_{e\in\Lambda}|IR(H(e))|$ where $H_r(e)$ is the unit group of eM_re. But from Theorem 42 of [28], $|IR(H_r(e))| = H_r(e)^{ss}/\sim$. Hence, it suffices to see that two elements of $H_r(e)$ are $H_r(e)$-conjugate if and only if they are G_r-conjugate. But this is straightforward. □

Let $\pi : M \longrightarrow X$ be the adjoint quotient, as in 2.4. We need some further assumptions to relate $M_r^{ss}/\sim$ with $X(\mathbb{F}_{q^r})$.

Definition 6.6 Let M be reductive. We say that *locally simply connected* (lsc) if $H(e)$ is a simply connected group for each $e \in \Lambda$.

From Remark 2.1.3 of [33] this means that $H'(e) = (H(e), H(e))$ is a simply connected semisimple group for each $e \in \Lambda$, The Levi subgroups of any simply connected groups are also simply connected (Lemma 2.1.7 of [33]).

We say that M is *split* over $\mathbb{F}_q$ if its unit group G is split over $\mathbb{F}$ in the usual sense [31]. Recall that any M, defined over $\mathbb{F}_q$, is split over $\mathbb{F}_{q^r}$ for some $r > 0$.

Proposition 6.7 *Let M be a lsc, reductive monoid defined over $\mathbb{F}_q$.*

a) *The canonical map $M_r^{ss}/\sim\longrightarrow X(\mathbb{F}_{q^r})$ is bijective for each $r > 0$.*
b) $|IR(M_r)| = \sum_{e\in\lambda} q^{r(e)}|H(e)_{ab}(\mathbb{F}_{q^r})|$ *where $r(e)$ is the semisimple rank of $H(e)$ and $H(e)_{ab}$ is the abelianization of $H(e)$.*
c) *If M is split over $\mathbb{F}_q$ then $|H(e)_{ab}(\mathbb{F}_{q^r})| = (q^r - 1)^a$ for some $a \geq 0$.*

Proof From 2.4, X parametrizes G-conjugacy classes of semisimple elements of M. But now we can apply Theorem 10.3 of [30]. This says that for each $e \in \Lambda$, $(H^{ss}(\mathbb{F}_q)/\sim) \longrightarrow X(\mathbb{F}_q)$ is bijective. Combined with 6.1 we obtain our result.

To prove b) one needs a careful calculation combining a) above, Richen's theory [4], Munn-Ponizovskii theory [2], and some basic results from [29]. See Theorem 4.2 of [27] for more details.

For c), first notice that $|H(e)_{ab}(\mathbb{F}_q r)|$ is a factor of $det(\sigma^* - 1)$ using 6.1(d) of [25]. But M is split so that $det(\sigma^* - 1) = (q^r - 1)^m$, where m is the rank of G. □

We can now obtain very precise information relating $\{IR(M_r)\}$ and X, for lsc monoids.

First, we recall a key definition from [18].

Definition 6.8 A reductive monoid M is $\mathcal{J}-$ *irreducible* if $\mathcal{U}(M)$ contains exactly one, minimal, nonzero $\mathcal{J}$-class.

In [18] the authors give a recipe for computing $\mathcal{U}(M)$ and the type map $\lambda : \Lambda \to 2^S$, in case M is $\mathcal{J}$-irreducible.

Theorem 6.9 *Let M be lsc and split over $\mathbb{F}_q$ with adjoint quotient $\pi : M \longrightarrow X$. Then*

a) $|IR(M_r)| = 1 = (q^r - 1)\sum_{i\geq 0} b_i q^{ri}$ *for some integers b_i independent of $r > 0$.*
b) *If M is $\mathcal{J}$-irreducible then*

$$|IR(M_r)| - 1 = (q^r - 1)\sum_{i\geq 0} a_i(q^r - 1)^{ri}$$

where

$$a_i = \left|\left\{(I, e) \in 2^S \times \Lambda \,\middle|\, \begin{array}{c}\nu(e) \subseteq I \subseteq \lambda(e)\\ |\lambda(e)\backslash I| = i\end{array}\right\}\right| .$$

c) *If M is $\mathcal{J}$-irreducible then $\mathbb{P}(X) := (X\backslash 0)/K^* = \bigsqcup_{e\in\Lambda\backslash\{0\}} C_e$ where $C_e \cong K^{b_i}$. In particular, b_i is the $2i-th$ Betti number of $\mathbb{P}(X)$.*

Proof For a) use 6.7 b) and c). For b) one needs Theorem 3.1 of [27] which calculates $|IR(M)|$ in terms of $\{(I,e)\in 2^s\times\Lambda \mid \nu(e)\subseteq I\subseteq\lambda(e)\}$. For c) notice that $X = \bigsqcup_{e\in\Lambda} H^{ss}(e)/\sim$. But from Theorem 1.6 of [25], $(H^{ss}(e)/\sim) = K^{b_e}\times K^*$. $\square$

Example 6.10 Let $M = M_n(K)$ where $K = \overline{\mathbb{F}_q}$. So $M_r = M_n(\mathbb{F}_{q^r})$. Then, by Munn-Ponizovskii [2],

$$|IR(M_r)| = \sum_{m=0}^{n} |IR(G\ell_m(\mathbb{F}_{q^r}))|$$

while, by Richen [4],

$$|IR(G\ell_m(\mathbb{F}_{q^r}))| = (q^r-1)q^{r(m-1)} .$$

Thus,

$$|IR(M_r)| = (q^r-1)\sum_{i=0}^{n-1} q^{ri} .$$

So

$$b_i = \begin{cases} 1 & , \quad i = 0,\ldots,n-1 \\ 0 & , \quad \text{otherwise} \ . \end{cases}$$

Plainly, b_i is the 2i-th Betti number of $\mathbb{P}(X) = \mathbb{P}^{n-1}$.

But we can also write

$$|IR(G\ell_m(\mathbb{F}_{q^r}))| = \sum_{i=0}^{m-1} \binom{m-1}{i}(q^r-1)^{i+1} .$$

So

$$\begin{aligned} |IR(M_r)| - 1 &= \sum_{m=1}^{n} |IR(G\ell_m(\mathbb{F}_{q^r}))| \\ &= \sum_{i=0}^{n} \binom{n}{i+1}(q^r-1)^{i+1} . \end{aligned}$$

By 6.4b) we obtain the curious combinatorial formula

$$\binom{n}{i+1} = \left|\left\{(I,e)\in 2^S\times\Lambda \;\middle|\; \begin{array}{c} \nu(e)\subseteq I\subseteq\lambda(e) \\ |\lambda(e)\backslash I| = i \end{array}\right\}\right| .$$

Example 6.11 In [19] the author and M. Putcha construct for each finite group G of Lie type, a certain canonical monoid $M(G)$ having the following properties:

(a) G is the unit group of $M(G)$.

(b) The type map $\lambda : \mathcal{U}(M(G)) \to 2^S$ of $M(G)$ satisfies

  (i) $\lambda : \mathcal{U}(M(G))\backslash\{0\} \to 2^S$ is bijective

  (ii) $\lambda(J_e\wedge J_f) = \lambda(J_e)\cap\lambda(J_f)$, where $\mathcal{U}(M(G))$ has been identified with a cross-section lattice Λ of $M(G)$.

(c) For each $e\in\Lambda, \{g\in G | ge = eg = e\} = \{1\}$.

By the results of [16], $M(G)$ is determined up to isomorphism by these properties. This monoid also enjoys a number of other useful properties that were important in the proof of Theorem 2.1. In any case, if $J \in \mathcal{U}(M(G))$ then by (c), $v(J) = \phi$. Furthermore, $\alpha_{I,J} = Hom(L_I, \mathbb{F}_q^*)$ for any $I \subseteq \lambda(J)$. So $\alpha_{I,J}$ is independent of J if it is non-empty.

Define, for any finite monoid M of Lie type,

$$IR(M) = \{\rho : M \to End(V) | \rho \text{ is irreducible}\} / \sim$$

where "$\sim$" denotes equivalence of representations. Thus, by Theorem 3.1

$$\begin{aligned} |IR(M(G))| &= \sum_{I \subseteq \lambda(J)} |\alpha_{I,J}| \\ &= \sum_{I \subseteq S} \sum_{J \in \mathcal{U}(S)} |\alpha_{I,J}| \\ &= \sum_{I \subseteq S} 2^{|S \setminus I|} \alpha(I) \end{aligned}$$

where $\alpha(I)$ is the common value of $|\alpha_{I,J}|$ for $I \subseteq \lambda(J)$. This agrees with the formula (1) of Theorem 2.2 of [19].

If $G = S\ell_{n+1}(\mathbb{F}_q)$ then $|S| = n$, and for $|I| = i, \alpha(I) = (q-1)^{n-i}$. Thus

$$\begin{aligned} |IR(M(G))| &= \sum_{I \subseteq S} 2^{|S \setminus I|} \alpha(I) \\ &= \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} (q-1)^{n-i} \\ &= \sum_{i=0}^{n} \binom{n}{i} (2q-2)^{n-i} \\ &= (2q-1)^n \end{aligned}$$

Notice that this corrects the calculation error of Example 2.3 of [19].

The reader might wonder if, for any reductive monoid M, there is a finite, dominant morphism

$$\varphi : M' \longrightarrow M$$

of reductive monoids with M' a lsc monoid. In general this seems to be a delicate problem. However, we do have some positive results.

If M is reductive and normal we denote by $C\ell(M)$ the *divisor class group* of M. See [9] for the whole story on class groups.

Theorem 6.12 *Let M be reductive and normal.*

a) *If $C\ell(M) = (0)$ then M is lsc.*
b) *If $M \backslash G$ is irreducible then $C\ell(M)$ is finite, and there exists $\pi : M' \longrightarrow M$ finite and dominant such that $C\ell(M') = (0)$.*

Proof Note first that if $C\ell(X) = (0)$ and $U \subseteq X$ is open, then $C\ell(U) = (0)$. On the other hand, an algebraic group G is simply connected if and only if $C\ell(G) = (0)$. Thus to prove a) it suffices to show that $C\ell(eMe) = (0)$ for any $e \in \Lambda$. But this follows from Theorem 10.6 of [9].

The proof of b) requires three steps. First, we may assume M has a zero element. Second, we may assume $C\ell(G) = (0)$ by taking integral closure along the universal cover of G. Similarly, we can assume $G = G_0 \times K^*$ where G_0 is semisimple. The final step amounts to showing that if we take integral closure along $G_0 \times K^* \longrightarrow G_0 \times K^*$, $(g, \alpha) \longmapsto (g, \alpha^n)$, for some n, the resulting monoid M' satisfies b). $\square$

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