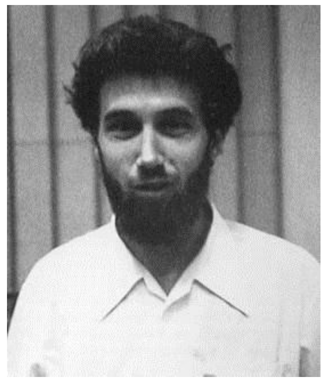
EXERCISES

THE JOY OF NUMBER THEORY

Mathematics 3150a – Elementary Number Theory

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EXERCISES --- December 5, 2011

This homework is just additional 'self-work' to be discussed at parties, student circles, and office hours. It should be helpful preparation for the Final Exam. This homework will not be graded.

Exercise 1. Let P be an odd prime ≥ 5.

Find the smallest positive $s \in \{1, \ldots, P-1\}$ such that $(P-3)! \equiv s \bmod P$.

Exercise 2. Let a given P be a prime ≥ 3. Set

$$S(P) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{P-1} = \frac{N}{(P-1)!}, \text{ where } n \in \mathbb{N}.$$

Show that P/N. (For example $S(3) = 1 + \frac{1}{2} = \frac{3}{2!}$ and indeed $3/3$.)

Exercise 3. Show that no prime of the form $P = 4k + 3, k \in \mathbb{N}$, is a sum of two squares.

Exercise 4. Show that for each prime $P = 4k + 1, k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $P/(n^2 + 1)$.

Exercise 5. Show that $(\frac{-1}{P}) = (-1)^{\frac{P-1}{2}}$.

Exercise 6. Determine for which P, $(\frac{-3}{P}) = 1$.

Exercise 7. Show that there are

a.) Infinitely many primes of the form $6k + 5$, $k \in \mathbb{N}$, and

b.) Infinitely many primes of the form $6k + 1$.

Exercise 8. Find all primes of the form $P = 4n^4 + 1, n \in \mathbb{N}$.

Exercise 9. Determine the sum

$$T(P) := \sum_{j=0}^{P-2} \left(\frac{j}{P}\right).$$

Exercise 10. Determine all primes P such that $\left(\frac{-2}{P}\right) = 1$.

Exercise 11. Determine all primes P such that $P/(a^2+2b^2)$ for some $a, b \in \mathbb{Z}$, $(a, b) = 1$.

Exercise 12. Show that if $P > 5$, then $(P-1)/(P-2)!$

Exercise 13. Show that $(P-1)^2/(P-1)!$

Exercise 14. Show that $(P-1)^2/(1+l\,(P-1)-P^l)$ for any $l \in \mathbb{N}$.

Exercise 15. Show that if $P > 5$, then $(P-1)! + 1 \neq P^l$, for any $l \in \mathbb{N}$.

(Hint: Use Exercises 12, 13 and 14.)

Exercise 16. (A comment related to Exercise 15.)

What happens when $P = 3$ and 5? Why does your solution of Exercise 15 fail for $P = 3$ and 5?

Exercise 17. Calculate all Legendre symbols $\left(\frac{j}{19}\right)$, $j = 1, 2, \ldots, 19$.

Exercise 18. Find all primes when

a.) $\left(\frac{3}{P}\right) = 1$,

b.) $\left(\frac{3}{P}\right) = -1$,

c.) $\left(\frac{3}{P}\right) = 0$,

d.) $\left(\frac{7}{P}\right) = 1$.

Exercise 19. Show that for any $n \in \mathbb{N}$ and any $a \in \mathbb{N}$, such that $(a, n) = 1$, we have $a^{\varphi(n)} \equiv 1 \pmod{n}$.

Exercise 20. Show that $\left(\frac{2}{P}\right) = (-1)^{\frac{P^2-1}{8}}$.