# FIRST HOMEWORK

*THE JOY OF NUMBER THEORY*

## Mathematics 3150a – Elementary Number Theory

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**FIRST HOMEWORK --- (Due by Wednesday, November 2, 2011)**

***Please show all of your work and justify your solutions carefully***

***Calculators are permitted, but all calculations should be carefully explained***

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**Exercise 1.** Show that for each $n \in \mathbb{N}$,

a) $6/n(n+1)(n+2)$, and

b) $6/n(n+1)(2n+1)$.

*10 marks*

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**Exercise 2.**

a) Find the greatest common divisor $d = (252, 198)$ of the numbers 252 and 198.

b) Find $\alpha, \beta \in \mathbb{Z}$ such that $\alpha.252 + \beta.198 = d$, where $d = (252, 198)$.

c) Find the greatest common divisors $e = (666, 1414)$.

d) Find $\gamma, \delta \in \mathbb{Z}$ such that $\gamma\ 666 + \delta\ 1414 = e$, where $e = (666, 1414)$.

*8 marks*

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**Exercise 3.** Let $\pi(n) = \#\{P = \text{prime} \mid P \leq n\}$.

Thus

$$\pi(2) = 1,\ \pi(3) = 2,\ \pi(4) = 2,\ \pi(5) = 3, \ldots$$

Find

$$\pi(505) - \pi(500).$$

Explain your result carefully.

*10 marks*

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**Exercise 4.** Determine the number of primes in the sequence:

$$\{102! + 2, 102! + 3, \ldots, (102)! + 102\} \text{ where } 102! = 1.2 \ldots 102.$$

*8 marks*

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**Exercise 5.** (Challenging!) Show that for no $n \in \mathbb{N}$, $n > 1$,

$S(n) := \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is an integer.

(For example $S(2) = \frac{1}{2} \times \frac{1}{3} = \frac{5}{6}$ is not an integer.)

*2 marks*

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**Exercise 6.** (Challenging!) Show that if $n = 2^k - 1$ for some $k$, then all binomial coefficients $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$ are odd.

For example if $k = 2$ then $n = 3$ and $\binom{3}{1} = 3$, $\binom{3}{2} = 3$, both are odd integers.

$\binom{n}{i} := \frac{n(n-1)\ldots(n-i+1)}{i(i-1)\ldots 1}$ is the formula for the binomial coefficient $\binom{n}{i}$.

*2 marks*

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**Exercise 7.** Factor into primes:

a) 100

b) 121

c) 1024

d) 2011

Please explain your calculations.

*8 marks*

**Exercise 8.**

Let $\sigma(n) = \sum_{d/n} d$ and $\mu(n)$ be the value of Möbius's function at $n$, and $\varphi(n)$ be the value of Euler's function at $n, n \in \mathbb{N}$.

Determine the values $\sigma(n)$, $\mu(n)$, $\varphi(n)$ for $n = 100, 101, 102, 103$.

*8 marks*

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**Exercise 9.** a) Find a value of $1+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}$ as a simple expression depending on $n$ which has minimum possible terms.

b) Suppose that $n$ is even. Find a simple expression for the value

$1+\binom{n}{2}+\binom{n}{4}+\cdots+\binom{n}{n}$.

(Here again $\binom{n}{i}$ is the binomial coefficient.)

*10 marks*

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**Exercise 10.** Find the number of integers between 100 and 500 which are divisible by 7.

*10 marks*

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**Exercise 11.** Show that the product of two integers of the form $4k+1, k \in \mathbb{N}$, is again an integer of this form.

*10 marks*

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**Exercise 12.** Show that there are infinitely many primes of the form $4k + 3$.

*4 marks*

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**Exercise 13.** Let $a, b$ be two odd integers such that $a - b = 2^n$ for some $n \in \mathbb{N}$. Show that $(a, b) = 1$. (This means that $a$ and $b$ are coprime with each other.)

*6 marks*

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**Exercise 14.** Find all prime numbers $P$ such that $2P^2 + 1$ is also prime.

*4 marks*