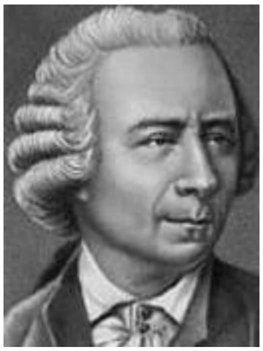
SECOND HOMEWORK

THE JOY OF DISCOVERY

Mathematics 3150a – Elementary Number Theory



L. Euler

A. Wiles

B. Mazur

R. Taylor

GOOD LUCK!

HAVE FUN!

SECOND HOMEWORK --- (Due by Monday, December 5, 2011)

Please show all of your work and justify your solutions carefully

Calculators are permitted, but all calculations should be carefully explained

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Exercise 1. Determine the smallest possible positive C_1, C_2, C_3 and C_4 such that

A) $2^{50} \equiv C_1 \pmod{17}$,

B) $3^{1094} \equiv C_2 \pmod{1093}$,

C) $2^{15} \equiv C_3 \pmod{17}$,

D) $50! \equiv C_4 \pmod{53}$.

Please explain how you arrived at your answer.

12 marks

Exercise 2. Prove that $(1+2)^{1093} \equiv 1 + 2^{1093} \pmod{1093}$.

10 marks

Exercise 3. Let $\varphi(n) = \#\{k \mid (k,n) = 1, 1 \leq k \leq n\}$ be an Euler function.

Find all $a \in \mathbb{N}$ such that $\varphi(2a) = \varphi(a)$.

8 marks

Exercise 4. Let $[s]$ be the integer part of s for each $s \in \mathbb{R}$. Thus

$$[s] \leq s < [s] + 1 \text{ and } [s] \in \mathbb{Z}.$$

Determine and carefully justify your solutions:

a) $\left[\frac{1092!}{1093}\right] - \frac{1092!}{1093}$.

b) $\frac{4124!}{4125} - \left[\frac{4124!}{4125}\right]$.

4 marks

Exercise 5. Show that for all primes p and all $n \in \mathbb{N}$, we have:

$$2^{n(p-1)+1} \equiv 2 \pmod{p}$$

16 marks

Exercise 6. Show that if P_1 and P_2 are two distinct prime numbers, then

$$P_1^{P_2-1} + P_2^{P_1-1} \equiv 1 \pmod{P_1\, P_2}.$$

4 marks

Exercise 7. Show that for odd primes P, the congruence below is solvable

$x^2 + 2x + c \equiv 0 \pmod{P}$, if and only if

$$(1-c)^{\frac{P-1}{2}} \equiv 1 \pmod{P}.$$

4 marks

Exercise 8. Find all quadratic residues modulo 23.

Please explain your method of finding them.

20 marks

Exercise 9. Calculate Legendre's symbols and please explain your calculations:

A) $\left(\frac{17}{53}\right)$

B) $\left(\frac{35}{17}\right)$.

12 marks

Exercise 10. Show that for all primes P the congruence

$$(x^2 - 13)(x^2 - 17)(x^2 - 221) \equiv 0 \pmod{P} \text{ is solvable.}$$

10 marks