THE ESSENTIAL DIMENSION OF STACKS OF PARABOLIC VECTOR BUNDLES OVER CURVES

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Abstract. We find upper bounds on the essential dimension of the moduli stack of parabolic vector bundles over a curve. When there is no parabolic structure, we improve the known upper bound on the essential dimension of the usual moduli stack. Our calculations also give lower bounds on the essential dimension of the semistable locus inside the moduli stack of rank r degree d bundles without parabolic structure.

1. Introduction

We fix a base field k of characteristic zero that may not be algebraically closed. We further fix a smooth projective geometrically connected curve X of genus g at least 2 over this field. In Section 3 below we will define what a parabolic point is. These are necessarily k-rational points on X. We will need to assume that X has at least 3 k-rational points that are not parabolic points. This is needed in the proof of Theorem 8.4.

We denote by $\mathrm{Bun}^{r,d}_{X,\mathbf{D}}$ the moduli stack of vector bundles of rank r and degree d on X with parabolic structure along some reduced divisor (see § 4). Our aim is to compute an upper bound on its essential dimension. When the divisor is empty, the question has been considered before in [DL09]. Our results improve the upper bound obtained there; this is explained in Remark 13.3. Further, by carefully choosing our parabolic structure we are able to find lower bounds on the essential dimension of the semistable locus of the usual moduli stack (see § 14). There was no previously known lower bound better than the trivial bound given by the dimension of the moduli space, see Theorem 2.1.

For stable parabolic bundles, bounds can be found by using the calculation in [BD] of Brauer group of the moduli space and some standard facts about twisted sheaves.

The key to improving the bounds in [DL09] is to replace the Jordan-Hölder filtration with the socle filtration (see § 8). Unlike the Jordan-Hölder filtration, the socle filtration is Galois invariant, so it exists over the base field. This sidesteps the major difficulty in [DL09].

The other ingredient is the correspondence set up in [Bis97] between parabolic bundles and orbifold bundles. This allows us to compute extensions of parabolic bundles in terms of orbifold bundles. Finally we need the orbifold Riemann-Roch theorem, originally proved in [Toe99], to bound the dimensions of these groups.

The key results that compute upper bounds are Theorem 12.1, Proposition 11.2, Proposition 13.1 and Proposition 13.2. The first two results bound the essential

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dimension in terms of an auxillary function. The last two tell us about the growth rate of this function. The lower bound is given in Theorem 14.1.

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2. Essential Dimension

Denote by Fields_k the category of field extensions of k. Let $F : \mathrm{Fields}_k \longrightarrow \mathrm{Sets}$ be a functor. We say that $a \in F(L)$ is *defined over a field* $K \subseteq L$ if there exists a $b \in F(K)$ so that $r(b) = a$ where r is the restriction

$$F(K) \longrightarrow F(L) .$$

The *essential dimension* of a is defined to be

$$\mathrm{ed}(a) \stackrel{\mathrm{def}}{=} \min_K \mathrm{tr.deg}_k K ,$$

where the minimum is taken over all fields of definition K of a.

The *essential dimension* of F is defined to be

$$\mathrm{ed}(F) = \sup_a \mathrm{ed}(a) ,$$

where the supremum is taken over all $a \in F(K)$ and K varies over all objects of Fields_k.

For an algebraic stack $\mathfrak{X} \longrightarrow \mathrm{Aff}_k$ we obtain a functor

$$\mathrm{Fields}_k \longrightarrow \mathrm{Sets}$$

which sends K to the set of isomorphism classes of objects in $\mathfrak{X}(K)$. We define *the essential dimension of* $\mathfrak{X}$ to be the essential dimension of this functor, and denote this number by $\mathrm{ed}_k(\mathfrak{X})$.

We now recall some theorems from [BFRV] that will be needed in the future. We assume for the remainder of this section that $\mathfrak{X}/k$ is a Deligne-Mumford stack, of finite type, with finite inertia. By [KM97], such a stack has a coarse moduli space M. The first result that we shall need is the following theorem proved in [BFRV].

Theorem 2.1. *Recall that our base field has characteristic zero. Suppose $\mathfrak{X}$ is smooth and connected. Let K be the field of rational functions of M and let $\mathfrak{X}_K = \mathrm{Spec}(K) \times_M \mathfrak{X}$ be the base change. Then*

$$\mathrm{ed}_k(\mathfrak{X}) = \dim M + \mathrm{ed}_K(\mathfrak{X}_K) .$$

Proof. See [BFRV, Theorem 6.1]. □

The stack $\mathfrak{X}_K/K$ is called the generic gerbe. In the case where this gerbe is banded by μ_n, more can be said about $\mathrm{ed}_K(\mathfrak{X}_K)$.

Let $\mathfrak{G}$ be a gerbe over our field k banded by μ_n. There is an associated $\mathbb{G}_m$-gerbe over K, denoted by $\mathfrak{F}$, coming from the canonical inclusion $\mu_n \hookrightarrow \mathbb{G}_m$. It gives a torsion class in the Brauer group $\mathrm{Br}(K)$. The index of this class is called the *index* of the gerbe and is denoted by $\mathrm{ind}(\mathfrak{G}) = d$. There is a Brauer-Severi variety P/k of dimension $d-1$ whose class maps to the class of $\mathfrak{G}$ via the connecting homomorphism

$$\mathrm{H}^1(X, \mathrm{PGL}_d) \longrightarrow \mathrm{H}^2(X, \mathbb{G}_m) .$$

Let V be a smooth and proper variety over k. The set $V(k(V))$ is the collection of rational endomorphisms of V defined over k. Define

$$\mathrm{cd}_k(V) = \inf\{\dim \overline{\mathrm{im}(\phi)} \mid \phi \in V(k(V))\}\,.$$

The number $\mathrm{cd}_k(V)$ is called the *canonical dimension* of V. We recall another theorem from [BFRV].

Theorem 2.2. *In the above situation, if* $d > 1$, *then*

$$\mathrm{ed}(\mathfrak{G}) = \mathrm{cd}_K(P) + 1\,,$$

and

$$\mathrm{ed}(\mathfrak{F}) = \mathrm{cd}_K(P)\,.$$

See [BFRV, Theorem 4.1] for a proof.

Corollary 2.3. *In the above situation, if* $\mathrm{ind}(P) = p^r$ *is a prime power, we have*

$$\mathrm{ed}(\mathfrak{G}) = \mathrm{ind}(P)$$

and

$$\mathrm{ed}(\mathfrak{F}) = \mathrm{ind}(P) - 1$$

Proof. See [Kar00, Theorem 2.1] and [Mer03]. □

3. Parabolic Bundles

Definition 3.1. A *parabolic point* $\mathbf{x}$ on X consists of a triple

$$(x, \{k_i^x\}_{i=1}^{n(x)}, \{\alpha_i^x\}_{i=1}^{n(x)}),$$

where x is a k-point of X, the k_i^x are positive integers called the *multiplicities* and the α_i^x are rational numbers, called the *weights*. The weights are required to satisfy the following condition :

$$0 \leq \alpha_1^x < \alpha_2^x < \ldots < \alpha_{n(x)}^x < 1.$$

Definition 3.2. A *parabolic datum* $\mathbf{D}$ on X consists of a finite collection of parabolic points $\mathbf{x}_j = (x_j, \{k_i^{x_j}\}_{i=1}^{n(x_j)}, \{\alpha_i^{x_j}\}_{i=1}^{n(x_j)})$, so

$$\mathbf{D} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$$

We require the points to be pairwise distinct, that is $x_j \neq x_i$ for $j \neq i$.

The *support* of the datum is defined to be the reduced divisor $x_1 + \ldots + x_s$. We denote this divisor by $|\mathbf{D}|$.

Definition 3.3. Fix a parabolic datum $\mathbf{D}$ on X. If S is a scheme then a *family of parabolic bundles* $\mathcal{F}_*$ on X parameterised by S with parabolic datum $\mathbf{D} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ consists of a vector bundle $\mathcal{F}$ on $X \times S$ together with filtrations by vector subbundles

$$\mathcal{F}|_{\{x_j\}\times S} = F_1^{x_j}(\mathcal{F}) \supset F_2^{x_j}(\mathcal{F}) \supset \ldots \supset F_{n(x_j)}^{x_j}(\mathcal{F}) \supset F_{n(x_j)+1}^{x_j}(\mathcal{F}) = 0$$

with $F_i^{x_j}(\mathcal{F})$ locally free of rank

$$k_i^{x_j} + k_{i+1}^{x_j} + \ldots + k_{n(x_j)}^{x_j}.$$

The weight α_i^x is associated with $F_i^x(\mathcal{F})$. This definition forces $\mathrm{rk}(\mathcal{F}) = \sum_{i=1}^{n(x)} k_i^x$ for each $x \in \mathrm{supp}(|\mathbf{D}|)$.

When S is reduced to a point we call $\mathcal{F}_*$ a parabolic bundle.

Definition 3.4. Suppose that $\mathcal{F}_*$ is a parabolic bundle with datum $\mathbf{D}$. A *parabolic subbundle* $\mathcal{F}'_* = (\mathcal{F}', \{F_i^x(\mathcal{F}') : i = 1, \dots, n'(x), x \in \mathrm{supp}(|\mathbf{D}|)\})$ of $\mathcal{F}_*$ is a parabolic bundle with datum $\mathbf{D'}$ such that

1. $|\mathbf{D}| = |\mathbf{D}'|$
2. $\mathcal{F}'$ is a subbundle of $\mathcal{F}$
3. for each point x in the support, the weights $\{\alpha'^x_i\}_{i=1}^{n(x)}$ are a subset of the weights $\{\alpha_i^x\}_{i=1}^{n'(x)}$
4. if m is maximal so that $F_i^x(\mathcal{F}') \subseteq F_m^x(\mathcal{F})$ then $\alpha'^x_i = \alpha^x_m$.

Given a parabolic bundle $\mathcal{F}_*$ on X with parabolic datum

$$\mathbf{D} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_s\},$$

we define the *parabolic degree* of $\mathcal{F}_*$ to be the rational number

$$\mathrm{pardeg}(\mathcal{F}_*) = \deg(\mathcal{F}) + \sum_{j=1}^{s} \sum_{i=1}^{n(x_j)} k_i^{x_j} \alpha_i^{x_j}.$$

The *parabolic slope* is defined to be $\mathrm{par}\mu(\mathcal{F}_*) = \mathrm{pardeg}(\mathcal{F}_*)/\mathrm{rk}(\mathcal{F})$.

Denote by $\bar{k}$ an algebraic closure of the ground field k.

We say the $\mathcal{F}_*$ is *semistable* (respectively, *stable*) if for every parabolic subbundle $\mathcal{E}_*$ of $(\mathcal{F}_{\bar{k}})_*$ we have

$$\mathrm{par}\mu(\mathcal{E}_*) \le \mathrm{par}\mu((\mathcal{F})_*) \quad (\text{respectively, } \mathrm{par}\mu(\mathcal{E}_*) < \mathrm{par}\mu((\mathcal{F}_{\bar{k}})_*)\,,$$

where $(\mathcal{F}_{\bar{k}})_*$ is the base change of $\mathcal{F}_*$.

The usual arguments show that an arbitrary parabolic bundle has a unique maximal destabilising parabolic subbundle $\mathcal{E}_* \subseteq (\mathcal{F}_{\bar{k}})_*$ of maximal parabolic slope. The uniqueness implies that all the Galois conjugates $\sigma^*(\mathcal{E}_*) \subset (\mathcal{F}_{\bar{k}})_*$ coincide. Hence this subbundle is defined over the ground field k so that a base extension is not required in the definition of semistable parabolic bundles.

Construction 3.5. Let $\mathcal{F}_*$ be a parabolic bundle with datum $\mathbf{D}$. We wish to construct a bundle $\mathcal{F}_t$ for each $t \in \mathbb{R}$. Set $D = |\mathbf{D}|$. For each $t \in \mathbb{R}$ with $0 \le t < 1$, we construct a coherent sheaf $V_t(\mathcal{F}_*)$ supported on D by letting the component of $V_t(\mathcal{F}_*)$ on x_j be the subspace $F_i^{x_j}(\mathcal{F})$ of $\mathcal{F}|_{x_j}$, where $\alpha_{i-1}^{x_j} < t \le \alpha_i^{x_j}$; if $t > \alpha_{n(x_j)}^{x_j}$, then the component of $V_t(\mathcal{F}_*)$ on x_j is defined to be zero. Taking preimages of $V_t(\mathcal{F})$ gives a sheaf $\mathcal{F}_t$ with $\mathcal{F} \supseteq \mathcal{F}_t \supseteq \mathcal{F}(-D)$. We can extend this construction to $t \in \mathbb{R}$ by defining $\mathcal{F}_t = F_s(-\lfloor t \rfloor D)$ where $s = t - \lfloor t \rfloor$.

This collection is decreasing; has jumps at rational numbers only; is periodic; satisfies $\mathcal{F}_t(-D) = \mathcal{F}_{t+1}$ and is left continuous. It also uniquely determines the parabolic bundle.

Fix a reduced effective divisor $D = \sum_{i=1}^{s} x_i$ on X where the x_i are k-rational points.

Denote by $\mathrm{PVect}(X, D, N)$ the category of parabolic bundles with parabolic datum only inside the support of D and parabolic weights integer multiples of $\frac{1}{N}$. The morphisms in this category are given by the following definition.

Definition 3.6. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are parabolic bundles with parabolic bundles with parabolic data $\mathbf{D}$ and $\mathbf{D}'$. Suppose that $|\mathbf{D}| = |\mathbf{D}'|$. A *morphism of parabolic* *bundles* $f_* : \mathcal{F}_* \longrightarrow \mathcal{F}'_*$ is a morphism $f : \mathcal{F} \longrightarrow \mathcal{F}'$ of underlying bundles such that for every parabolic point $\mathbf{x}$ we have

$$f_x(F_i^x(\mathcal{F})) \subseteq F_j^x(\mathcal{F}')$$

whenever $\alpha_i^x > \alpha'^x_j$.

This category is in fact an abelian category. To construct the kernel $\ker(f_*)_*$ of a morphism $f_* : \mathcal{F}_* \longrightarrow \mathcal{F}'_*$ we take as the underlying bundle the kernel of the morphism of

$$f : \mathcal{F} \longrightarrow \mathcal{F}'.$$

The filtration on parabolic points is obtained by intersecting the filtration of $\mathcal{F}$ with $\ker(f)$. The weights are chosen so that if i is maximal with

$$F_j^x(\ker(f)) = F_i^x(\mathcal{F})$$

then $\alpha_j^{\ker(f),x} = \alpha_i^x$. This last condition is forced upon us by the requirement that the kernel be a subbundle.

One can also define notions such as exact sequence of parabolic bundles and describe cokernels. We refer the reader to [MS80] for further discussion on these matters.

4. Our Stacks

Let $\mathbf{D}$ be a parabolic datum on X. We will denote by $\mathrm{Bun}_{X,\mathbf{D}}^{r,d}$ the moduli stack of rank r degree d parabolic bundles with datum $\mathbf{D}$. Note that the weights only play a role when defining stability and semistability. Hence this stack is just a fibred product of flag varieties over the usual moduli stack.

Fix an ordinary line bundle ξ on X. We denote by $\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi}$ the moduli stack of parabolic bundles with fixed identification of the top exterior power with ξ. Precisely, there is a cartesian square

$$\begin{array}{ccc} \mathrm{Bun}_{X,\mathbf{D}}^{r,\xi} & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{X,\mathbf{D}}^{r,d} & \longrightarrow & \mathrm{Bun}_X^{1,d}. \end{array}$$

Here $\mathrm{Bun}_X^{1,d}$ is the moduli stack of line bundles of degree d, the right vertical arrow corresponds to the line bundle ξ and the bottom horizontal arrow is the determinant. As stability and semistability are open conditions, see [MS80, page 226-228], there are various open substacks, $\mathrm{Bun}_{X,\mathbf{D}}^{r,d,s}, \mathrm{Bun}_{X,\mathbf{D}}^{r,d,ss}, \mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s}$ and $\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,ss}$

We explain explicitly what $\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi}$ is. The objects of the category fibred in groupoids over a scheme S consist of pairs $(\mathcal{F}, \phi)$ where $\mathcal{F}$ is a family of parabolic bundles of rank r on $X \times S$ and ϕ is an isomorphism

$$\phi : \wedge^r \mathcal{F} \xrightarrow{\sim} \xi$$

The isomorphisms in the groupoid over S are isomorphisms of parabolic bundles compatible with the trivialisations.

The stack $\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi}$ is somewhat unnatural as ξ is not a parabolic line bundle. However it is a natural stepping stone to understanding the essential dimension of the stack $\mathrm{Bun}_{X,\mathbf{D}}^{r,d}$. Below we will see that $\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s}$ is a smooth Deligne-Mumford stack with finite inertia so that Theorem 2.1 applies. We will be able to compute the period and index of its generic gerbe and apply Theorem 2.2 to understand its essential dimension.

Proposition 4.1. *The stack* $\mathrm{Bun}^{r,\xi}_{X,\boldsymbol{D}}$ *is smooth.*

Proof. First recall that the moduli stack $\mathrm{Bun}^{r,\xi}_{X}$ of vector bundles is smooth. If $\mathbf{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_s\}$ then set

$$\mathbf{D}' = \{\mathbf{x}_1, \ldots, \mathbf{x}_{s-1}\}\,.$$

The morphism forgetting one parabolic point

$$\mathrm{Bun}^{r,\xi}_{X,\mathbf{D}} \longrightarrow \mathrm{Bun}^{r,\xi}_{X,\mathbf{D}'}$$

is representable with fibres flag varieties. Hence the above morphism is smooth and the result follows by induction. □

Remark 4.2. The stack $\mathrm{Bun}^{r,d,s}_{X,\mathbf{D}}$ is in fact a global quotient stack. For simplicity, in this remark we will assume that $\mathbf{D} = \{\mathbf{x}\}$ with multiplicities $k_1, \ldots, k_n$.

From [MS80, page 226] the family of stable parabolic bundles of rank r and degree d is a bounded family. We may find an integer N so that for every $n \geq N$ we have that $\mathrm{H}^1(X, \mathcal{F}(n)) = 0$ and $\mathcal{F}(n)$ is generated by global sections for every bundle underlying a stable parabolic bundle. Let Q be the corresponding quot scheme. Let $\mathcal{W}$ be the universal bundle on $Q \times X$. There is a flag variety F over Q parameterising flags of $\mathcal{W}_x$ of type $k_1, \ldots, k_n$. To give an S-point of F is the same as giving a quotient :

$$\pi_X^* \mathcal{O}_X(-N)^m \twoheadrightarrow \mathcal{F}$$

on $S \times X$ and a flag of $\mathcal{F}|_{S\times\{x\}}$. There is an open subset Ω parameterising quotients that are stable as parabolic bundles. We have

$$[R/\mathrm{GL}_m] = \mathrm{Bun}^{r,d,s}_{X,\mathbf{D}}.$$

The stack $\mathrm{Bun}^{r,\xi,s}_{X,\mathbf{D}}$ is also a global quotient stack. There is a $\mathbb{G}_m$-torsor I over R parameterising isomorphisms of the determinant with ξ (see [LMB00, page 29]). Then

$$[I/\mathrm{GL}_m] = \mathrm{Bun}^{r,\xi,s}_{X,\mathbf{D}}$$

Proposition 4.3. *Let $\mathcal{F}$ be a family of stable parabolic bundles on* $\mathrm{Spec}(R) \times X$*. Then all parabolic endomorphisms of $\mathcal{F}$ are scalar multiplication by elements of R.*

Proof. This is well known when R is a field. We will explain how to deduce this result from the case of a field.

There exists a natural inclusion

$$\epsilon : R \hookrightarrow \mathrm{H}^0(X_R, \mathrm{End}(\mathcal{F}))$$

that we wish to show is an isomorphism. By flat base change, we may assume $R = (R, \mathfrak{m})$ is local. Via Nakayama's Lemma we need to show that

$$\bar{\epsilon} : R/\mathfrak{m} \hookrightarrow \mathrm{H}^0(X_R, \mathrm{End}(\mathcal{F})) \otimes_R R/\mathfrak{m}$$

is surjective. But by the field case, the composition

$$R/\mathfrak{m} \longrightarrow \mathrm{H}^0(X, \mathrm{End}(\mathcal{F})) \otimes_R R/\mathfrak{m} \longrightarrow \mathrm{H}^0(X_{R/\mathfrak{m}}, \mathrm{End}(\mathcal{F}_{R/\mathfrak{m}}))$$

is surjective. The result follows from the base change theorem, [Har77, Ch. III, Theorem 12.11] □

Theorem 4.4. *The stack* $\mathrm{Bun}_{X,\boldsymbol{D}}^{r,\xi,s}$ *is a smooth Deligne-Mumford stack with finite inertia.*

Proof. This stack is of finite type due to the fact that stable parabolic bundles form a bounded family, see [MS80]. We need to show that the diagonal is formally unramified. Consider an extension of Artinian local k-algebras

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0\,.$$

An A'-point of $\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s} \times \mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s}$ amounts to two families $(\mathcal{F}_1, \phi_1)$ and $(\mathcal{F}_2, \phi_2)$ of stable bundles with identifications of their top exterior powers with ξ parametrised by A'. Completing this to a diagram of the form

$$\begin{array}{ccc} \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(A') \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s} & \longrightarrow & \mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s} \times \mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s} \end{array}$$

amounts to an isomorphism $\alpha : \mathcal{F}_1|_A \cong \mathcal{F}_2|_A$ compatible with the identifications of the top exterior powers. We need to show that any extension of the isomorphism α to A' is unique. In view of Proposition 4.3, this follows from the following claim:

Claim 4.5. *Let* $(B, \mathfrak{m})$ *be a local* k*-algebra. Suppose that* $y_i \in B$ *and* $y_1^r = y_2^r = 1$*. Further assume that* y_i *have the same images under the projection*

$$q : B \longrightarrow B/\mathfrak{m}\,.$$

Then $y_1 = y_2$*.*

Proof of claim. We may write $y_2 = y_1 + x$ where $x \in \mathfrak{m}$. As we are in characteristic 0, we have

$$1 = (y_2)^r + x(\text{another unit in } B)\,.$$

Since $y_2^r = 1$ we must have $x = 0$.

The fact about finite inertia is easily verified using Proposition 4.3. □

It follows that $\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s}$ has a coarse moduli space that we shall denote by $M(X, \mathbf{D}, r, \xi)^s$. By Proposition 4.3, this makes the stack into a gerbe banded by the rth roots of unity over the moduli space.

5. Period and Index

For a parabolic datum $\mathbf{D} = (D, \{k_i^x, \alpha_i^x : x \in \mathrm{supp}(D), i = 1, \ldots, n(x)\})$ we define an integer

$$l(\mathbf{D}) = \gcd(\deg(\xi), r, \{k_i^x : x \in \mathrm{supp}(D), i = 1 \ldots, n(x)\})$$

where the greatest common divisor is taken over the rank and all multiplicities of parabolic points in the parabolic datum.

Theorem 5.1. *Assume the base field is the complex numbers. The period of the gerbe*

$$\mathrm{Bun}_{X,\boldsymbol{D}}^{r,\xi,s} \longrightarrow M(X, \boldsymbol{D}, r, \xi)^s$$

is $l(\boldsymbol{D})$*.*

Proof. This follows from [BD]. Note that the gerbe of splittings of the Severi-Brauer variety in the cited paper is easily identified with the moduli stack. □

We will see below that the above result is true over any field of characteristic zero.

A useful tool for understanding the difference between the period and the index is the notion of a twisted sheaf. A *twisted sheaf* on a $\mathbb{G}_m$-gerbe $\mathfrak{G} \longrightarrow X$ is a coherent sheaf $\mathcal{F}$ on $\mathfrak{G}$ such that inertial action of $\mathbb{G}_m$ on $\mathcal{F}$ coincides with natural module action of $\mathbb{G}_m$ on $\mathcal{F}$. We spell out the meaning of this statement in the next paragraph.

Suppose that we have a T-point $T \longrightarrow X$ and an object a of $\mathfrak{G}$ above this point. Part of the data of the coherent sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}_a$ on T. These sheaves are required to satisfy compatibility conditions on pullbacks for morphisms in the category $\mathfrak{G}$. In particular, every object a of the gerbe $\mathfrak{G}$ has an action of $\mathbb{G}_m$ and hence there is an action of $\mathbb{G}_m$ on $\mathcal{F}$. The above definition says that the action of $\mathbb{G}_m$ on $\mathcal{F}$ should be the same as the $\mathbb{G}_m$-action coming from the fact that $\mathcal{F}$ is an $\mathcal{O}_{\mathfrak{G}}$-module.

Example 5.2. We have a μ_r-gerbe

$$\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s} \longrightarrow M(X, \mathbf{D}, r, \xi)^s\ .$$

It gives rise to a $\mathbb{G}_m$-gerbe via the natural group homormorphism $\mu_r \longrightarrow \mathbb{G}_m$. We denote this gerbe by $\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s,\mathbb{G}_m}$. One can describe this stack explicitly. The objects over a scheme S are families $\mathcal{F}_*$ of stable parabolic bundles on $S \times X$ such that $\wedge^r \mathcal{F}$ is isomorphic to

$$\pi_S^* \mathcal{L} \otimes \pi_X^* \xi$$

for some line bundle $\mathcal{L}$ on S. There is a universal stable parabolic bundle $\mathcal{W}_*$ on

$$\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s,\mathbb{G}_m} \times X.$$

The data that makes up $\mathcal{W}_*$ consists of a vector bundle $\mathcal{W}$ of rank r on

$$\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s,\mathbb{G}_m} \times X.$$

and, for each parabolic point $\mathbf{x}$ in the datum $\mathbf{D}$, a universal flag

$$F_1^x(\mathcal{W}) \supseteq \ldots \supseteq F_{n(x)}^x(\mathcal{W})$$

on

$$\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s,\mathbb{G}_m} \times \{x\}.$$

As the automorphism group of a stable parabolic bundle is multiplication by a scalar we see that each of these bundles produces a twisted sheaf. Fix a k-rational point $y \in X$. We have on

$$\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s,\mathbb{G}_m}$$

the following twisted sheaves

(1) $\mathcal{W}_{\mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s,\mathbb{G}_m}} \times \{y\}$ of rank r
(2) $F_i^x(\mathcal{W})$ of rank $k_i^x + \ldots + k_{n(x)}^x$ for each $\mathbf{x} \in \mathbf{D}$, $1 \le i \le n(x)$.

There is one more twisted sheaf that will be of importance to us below. We have a projection

$$p : \mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s,\mathbb{G}_m} \times X \longrightarrow \mathrm{Bun}_{X,\mathbf{D}}^{r,\xi,s,\mathbb{G}_m}\ .$$

The sheaf

$$\pi_* \mathcal{W} \otimes \mathcal{O}_X(Ny)$$

is locally free for large N and gives us another twisted sheaf on the stack. Using Riemann-Roch, it has rank

$$\deg(\xi) + Nr + r(1-g).$$

We will need the following :

Proposition 5.3. *Let $\mathfrak{G} \longrightarrow \mathrm{Spec}(K)$ be a $\mathbb{G}_m$-gerbe over a field. Then the index of $\mathfrak{G}$ divides m if and only if there is a locally free rank m twisted sheaf on $\mathfrak{G}$.*

Proof. See [Lie08, Proposition 3.1.2.1]. □

Theorem 5.4. *The index equals the period for the gerbe*

$$\mathrm{Bun}^{r,\xi,s}_{X,\boldsymbol{D}} \longrightarrow M(X,\boldsymbol{D},r,\xi)^s .$$

Proof. Over the complex numbers the result follows from Theorem 5.1, Example 5.2 and Proposition 5.3. For a non algebraically closed field we proceed as follows. Set $e = \gcd(\deg(\xi), r, \{k^x_j : x \in \mathrm{supp}(|\mathbf{D}|), j = 1,\ldots,n(x)\})$. From Example 5.2 it follows that the index divides e. After base change to an algebraically closed field we find that e is the period, using a Lefschetz principle. This means the period of the original gerbe was larger than e. However the period always divides the index. □

6. The Essential Dimension of the Stable Locus

Let $\mathbf{x}$ be a point in the datum $\mathbf{D}$. We denote by $\mathrm{Flag}_{\mathbf{x}}(\mathbf{D})$ or $\mathrm{Flag}(k^x_1, k^x_2, \ldots, k^x_{n(x)})$ the flag variety determined by the multiplicities of $\mathbf{x}$. Explicitly, if the multiplicities at $\mathbf{x}$ are $k^x_1, k^x_2, \ldots, k^x_{n(x)}$ then $\mathrm{Flag}_{\mathbf{x}}(\mathbf{D})$ is the flag variety parameterising flags

$$V_1 \supseteq V_2 \supseteq \ldots \supseteq V_{n(x)}$$

with $\dim V_1 = \sum_{i=1}^{n(x)} k^x_i$ and $\dim V_i/V_{i+1} = k^x_i$.

Let g be the genus of X.

Theorem 6.1. *Set*

$$l(\boldsymbol{D}) = \gcd(\deg\xi, r, k^x_i),$$

where the gcd ranges over all possible multiplicities of all parabolic points in the datum $\boldsymbol{D}$. If $l(\boldsymbol{D}) > 1$ then the essential dimension of $\mathrm{Bun}^{r,\xi,s}_{X,\boldsymbol{D}}$ is bounded by

$$l(\boldsymbol{D}) + \dim M(X,\boldsymbol{D},\xi)^s = l(\boldsymbol{D}) + (r^2-1)(g-1) + \sum_{\boldsymbol{x}\in\boldsymbol{D}} \dim \mathrm{Flag}_{\boldsymbol{x}}(\boldsymbol{D}).$$

For any $l(\boldsymbol{D})$ we have

$$\mathrm{ed}(\mathrm{Bun}^{r,\xi,s,\mathbb{G}_m}_{X,\boldsymbol{D}}) \le l(\boldsymbol{D}) - 1 + (r^2-1)(g-1) + \sum_{\boldsymbol{x}\in\boldsymbol{D}} \dim \mathrm{Flag}_{\boldsymbol{x}}(\boldsymbol{D}).$$

This upper bound is an equality when $l(\boldsymbol{D})$ is a prime power.

Proof. If $l(\mathbf{D}) > 1$ this follows from Theorems 2.2, 2.1 and 5.4. To deduce the result for $l(\mathbf{D}) = 1$ observe that the gerbe

$$\mathrm{Bun}^{r,\xi,s,\mathbb{G}_m}_{X,\mathbf{D}} \longrightarrow M(X,\mathbf{D},r,\xi)^s$$

is neutral as the period is one. Hence there is a universal bundle on the moduli space and the result follows. □

Corollary 6.2. *The essential dimension of* $\mathrm{Bun}^{r,d,s}_{X,\boldsymbol{D}}$ *(degree d, and not fixed determinant) is bounded by*

$$l(\boldsymbol{D}) - 1 + \dim M(X, \boldsymbol{D}, r, \xi)^s = l(\boldsymbol{D}) - 1 + (r^2 - 1)(g - 1) + \sum_{\boldsymbol{x}\in\boldsymbol{D}} \dim \mathrm{Flag}_{\boldsymbol{x}}(\boldsymbol{D}) + g.$$

Proof. Suppose that we have a family $\mathcal{F}$ over a field K. By adjoining at most g parameters to the base field we may assume that $\det \mathcal{F}$ is defined over the base field and then apply the above result. □

7. Some Linear Algebra

Let V be a finite dimensional vector space of dimension r. We equip V with two full flags

$$V = F^x_1 \supset F^x_2 \supset \ldots F^x_r \supset F^x_{r+1} = 0$$

and

$$V = F^y_1 \supset F^y_2 \supset \ldots F^y_r \supset F^y_{r+1} = 0$$

with $\dim F^x_i = r - i + 1$. We say that the flags are *generic* if

$$\dim(F^x_i + F^y_j) = \min(r, \dim(F^x_i) + \dim(F^y_j)).$$

It is clear that generic flags exist. Fix a one dimensional subspace $l \subseteq V$ with

$$\dim(l + F^x_i + F^y_j) = \min(1 + \dim(F^x_i) + \dim(F^y_j), r).$$

We say that the tuple (V, F_*, F'_*, l) is *generic* if the flags are generic and the subspace l satisfies the above condition. It is easy to see that generic triples exist and any generic pair of flags can be completed to a generic triple.

For a subspace $W \subseteq V$ define the *degree* of W to be

$$\begin{aligned}\deg_V(W) &= \sum_{i=1}^{r}((i-1)(\dim(W \cap F^x_i) - \dim(W \cap F^x_{i+1})) \\ &\quad + \sum_{i=1}^{r}((i-1)(\dim(W \cap F^y_i) - \dim(W \cap F^y_{i+1})) \\ &= \sum_{i=2}^{r} \dim(W \cap F^x_i) + \sum_{i=2}^{r} \dim(W \cap F^y_i).\end{aligned}$$

We also need the notation

$$\deg W = \deg_V W + (r-1)\dim(l \cap W).$$

Note that $\deg_V W$ only depends on the first two flags and not the line.

Let's set $d^x_i(W) = \dim(W \cap F^x_i)$ and $d^y_i(W) = \dim(W \cap F^y_i)$

We define the *slope* of W to be $\mu(W) = \deg W / \dim W$.

Lemma 7.1. *We have*

$$(W \cap F^x_i) \oplus (W \cap F^y_{r+2-i}) \subseteq W$$

so that $d^x_i(W) + d^y_{r+2-i}(W) \le \dim(W)$ *for all* $2 \le i \le r$.

In the case $l \subseteq W$*, we have*

$$l \oplus (W \cap F^x_i) \oplus (W \cap F^y_{r+3-i}) \subseteq W$$

so that $d^x_i(W) + d^y_{r+3-i}(W) \le \dim(W) - 1$ *for all* $2 \le i \le r+1$. *In particular,* $d^x_2(W) \le \dim(W) - 1$ *and* $d^y_2(W) \le \dim(W) - 1$.

Proof. This follows from the definition of generic. □

Proposition 7.2. *We have*

$$\mu(V) > \mu(W)$$

for every proper subspace W.

Proof. Note that $\deg(V) = 2\sum_{i=1}^{r-1} i + (r-1) = r^2 - 1$ so that $\mu(V) = r - \frac{1}{r}$.

First consider the case that $W \cap l = \{0\}$. Then by Lemma 7.1, we have

$$\begin{aligned} \deg(W) &= \deg_V W \\ &= \sum_{i=2}^{r} (d_i^x(W) + d_{r+2-i}^y(W)) \\ &\leq (r-1)\dim W. \end{aligned}$$

So $\mu(W) \leq r - 1 < \mu(V)$.

Then consider the case in which $l \subseteq W$. Then by Lemma 7.1, we have

$$\begin{aligned} \deg(W) &= \deg_V W + (r-1) \\ &= (\sum_{i=3}^{r} (d_i^x(W) + d_{r+3-i}^y(W))) + (d_2^x(W) + d_2^y(W)) + (r-1) \\ &\leq r(\dim(W) - 1) + (r-1). \end{aligned}$$

So $\mu(W) \leq r - \frac{1}{\dim(W)} < \mu(V)$. □

8. The Socle

Definition 8.1. Let K a field containing k. We say that a parabolic bundle $\mathcal{F}$ on X_K is *polystable* if $\mathcal{F} \otimes_K \bar{K}$ is a direct sum of stable parabolic bundles of the same parabolic slope.

Proposition 8.2. *Let $\mathcal{F}_*$ be a semistable parabolic bundle on $X_{\bar{K}}$ with parabolic slope μ. Then there exists a unique maximal polystable subbundle with parabolic slope μ. We call this bundle the socle of $\mathcal{F}_*$ and write* $\mathrm{Soc}(\mathcal{F}_*)$. *If $\mathcal{F}_*$ is defined over K then so is* $\mathrm{Soc}(\mathcal{F}_*)$.

Proof. If $\mathcal{F}_*$ is stable then $\mathrm{Soc}(\mathcal{F}_*) = \mathcal{F}_*$ and we are done. Otherwise we can find a proper stable subbundle. Suppose that we are given two polystable subbundles of $\mathcal{F}_*$, say

$$\mathcal{P} = \oplus_{i=1}^{l} \mathcal{F}_i \quad \text{and} \quad \mathcal{P}' = \oplus_{i=1}^{l'} \mathcal{F}'_i.$$

The bundle $\mathcal{P} \cap \mathcal{P}'$ has parabolic slope μ and hence is polystable. After reindexing, we can write it as

$$\mathcal{P} \cap \mathcal{P}' = \oplus_{i=1}^{m} \mathcal{F}_i = \oplus_{i=1}^{m'} \mathcal{F}'_i.$$

After further reindexing, we have $m = m'$ and $\mathcal{F}_i = \mathcal{F}'_i$. If $m \neq l$ then we contradict maximality as

$$(\mathcal{P} \cap \mathcal{P}') \oplus (\oplus_{i=m}^{l} \mathcal{F}_i) \oplus (\oplus_{i=m}^{l'} \mathcal{F}'_i)$$

is bigger.

To see the assertion about ground fields notice that $\mathrm{Soc}(\mathcal{F}_*)$ will always be defined over some finite Galois extension L/K with Galois group G. But uniqueness forces the subbundles $\gamma^*\mathrm{Soc}(\mathcal{F}_*)$, $\gamma \in G$, to be the same. □

Consider the functor

$$\mathbf{F} = \mathbf{F}_{X,\mathbf{D}}^{\mathrm{poly},r,d} : \mathrm{Fields}_k \longrightarrow \mathrm{Sets}$$

with

$$\mathbf{F}(K) = \{\text{families of polystable bundles on } X_K \text{ of degree } d \text{ and rank } r\}/\sim,$$

where $\sim$ is the equivalence relation given by isomorphism of families. We will need to say something about the essential dimension of this functor.

Let $\mathcal{F} \in \mathbf{F}(K)$. Let V be the common fibre of the underlying vector bundle of $\mathcal{F}$. Then V is an r dimensional vector space. Choose three k-points $x, y, z \in X$ that are not parabolic points. We turn these points into parabolic points by defining full flags at x and y to be

$$V = F_1^x \supset F_2^x \supset \dots F_r^x \supset F_{r+1}^x = \{0\}$$

and

$$V = F_1^y \supset F_2^y \supset \dots F_r^y \supset F_{r+1}^y = \{0\}$$

so that $k_i^x = k_i^y = 1$ for all $i = 1, \dots, r$. Choose the flag at z to be

$$V = F_1^z \supset F_2^z = l \supset F_3^z = 0$$

where l is a line in V so that $k_1^z = r - 1$ and $k_2^z = 1$. The weights for x and y are chosen to be $\alpha_i^x = (i-1)\epsilon$, $i = 1, \dots, r$ and the weights for z are chosen to be $\alpha_1^z = 0$ and $\alpha_2^z = (r-1)\epsilon$ where ϵ is chosen so that the largest weight is smaller than 1. The corresponding parabolic points are denoted $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$.

Let

$$E = \mathrm{Flag}_{\mathbf{x}}\mathcal{F} \times_K \mathrm{Flag}_{\mathbf{y}}(\mathcal{F}) \times_K \mathrm{Flag}_{\mathbf{z}}(\mathcal{F}).$$

On $E \times X$ there is a universal extension of the (quasi) parabolic structure of $\mathcal{F}$ to the three new points. That is, there is a parabolic bundle $\mathcal{E}_*$ on $E \times X$ with datum $\mathbf{D'} = \mathbf{D} \cup \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$. Note by construction, the parabolic slope of the new parabolic bundle $\mathcal{E}_*$ is

$$\mathrm{par}\mu(\mathcal{E}_*) = \mathrm{par}\mu(\mathcal{F}_*) + \sum_{i=1}^{r} k_i^x \alpha_i^x + \sum_{i=1}^{r} k_i^y \alpha_i^y + \sum_{i=1}^{2} k_i^z \alpha_i^z = \mu + \mu(V)\epsilon$$

where $\mu(V)$ is defined in Section 7 . Any parabolic subbundle $\mathcal{E}'_*$ of $\mathcal{E}_*$ has parabolic slope

$$\mathrm{par}\mu(\mathcal{E}'_*) = \mu + \mu(W)\epsilon$$

where W is the common fibre of the subbundle $\mathcal{E}'_*$ and hence is a vector subspace of V. Then by Proposition 7.2, we have that $\mathcal{E}_*$ is a stable parabolic bundle. There is an open subscheme $E^s \subseteq E$ where this bundle is stable.

Lemma 8.3. *The open subscheme F^s is not empty.*

Proof. It suffices to show that $F^s \otimes_K \bar{K}$ is not empty so we may assume that

$$\mathcal{F} = \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_l$$

with the $\mathcal{F}_i$ non isomorphic stable bundles of the same slope μ. We may find an open U of X_K which contains x, y and z such that $\mathcal{F}|_U$ is trivial. Then we apply the argument above to obtain the result. $\square$

Theorem 8.4. *We have*

$$\begin{aligned}\mathrm{ed}(\boldsymbol{F}) &\leq (r^2-1)(g-1)+\sum_{\boldsymbol{x}\in\boldsymbol{D}}\dim\mathrm{Flag}_{\boldsymbol{x}}(\boldsymbol{D})+g+r^2-1\\ &= r^2g+\sum_{\boldsymbol{x}\in\boldsymbol{D}}\dim\mathrm{Flag}_{\boldsymbol{x}}(\boldsymbol{D})\end{aligned}$$

Proof. As above, we may add parabolic structure at x, y and z to obtain a stable parabolic bundle. Since $l(\mathbf{D}')=1$ and

$$\dim\mathrm{Flag}_{\mathbf{x}}(\mathbf{D}')=\dim\mathrm{Flag}_{\mathbf{y}}(\mathbf{D}')=r(r-1)/2, \dim\mathrm{Flag}_{\mathbf{z}}(\mathbf{D}')=r-1$$

we may apply Corollary 6.2 to obtain the result. □

9. Orbifold Bundles and Orbifold Riemann-Roch

Let Y be a smooth projective curve with an action of the finite group Γ defined over k. If $\mathcal{E}$ is a Γ bundle on Y the cohomology groups $\mathrm{H}^*(Y,\mathcal{E})$ are naturally representations of the group Γ. We define $\chi(\Gamma,Y,\mathcal{E})$ to be the equivariant Euler characteristic. Precisely, it is the class

$$\chi(\Gamma,Y,\mathcal{E})=[\mathrm{H}^0(Y,\mathcal{E})]-[\mathrm{H}^1(Y,\mathcal{E})]$$

in the K-ring of representations of Γ. The orbifold Riemman–Roch theorem is a formula for this class.

Theorem 9.1. *Suppose that $k=\bar{k}$. Consider the projection*

$$\pi: Y\longrightarrow X=Y/\Gamma.$$

For each $y\in Y$ write e_y for the ramification index of π at y and Γ_y for the isotropy group at y. We have a character

$$\chi_y:\Gamma_y\longrightarrow\mathbb{G}_m$$

coming from the action of Γ_y on the cotangent space $\mathfrak{m}_y/\mathfrak{m}_y^2$.

We have

$$\begin{aligned}|\Gamma|\chi(\Gamma,Y,\mathcal{E}) &= (|\Gamma|(1-g)\mathrm{rk}(\mathcal{E})+\deg(\mathcal{E}))[k[\Gamma]]-\\ &\quad \textstyle\sum_{y\in Y}\sum_{d=0}^{e_y-1}d[\mathrm{Ind}_{\Gamma_y}^{\Gamma}(\mathcal{E}|_y\otimes\chi_y^d)].\end{aligned}$$

Proof. See [Köc05]. This formula can also be deduced from [Toe99, Theorem 4.11] by considering the morphism of quotient stacks

$$[Y/\Gamma]\longrightarrow B\Gamma.$$

□

We now recall the main result of [Bis97] in the case of curves. Consider a reduced divisor $D=\sum_{i=1}^s x_i$ on X where x_i are k-rational points.

Consider a curve Y with an action of a finite group Γ such that $Y/\Gamma=X$. There is a projection $\pi: Y\longrightarrow X$. We further assume that for each $x\in\mathrm{supp}(D)$, we have $\pi^*(x)=kN(\pi^*x)_{\mathrm{red}}$, for some positive integer k. Denote by $\mathrm{Vect}_\Gamma^D(Y,N)$ the full subcategory of Γ-bundles on Y with $W\in\mathrm{Vect}_\Gamma^D(Y)$ if and only if

- for all geometric points y in Y with $y\in\mathrm{supp}((\pi^*D)_{\mathrm{red}})$ and for each $\gamma\in\Gamma_y$ in the isotropy group, γ^N acts on the fibre W_y trivially;
- for all geometric points y in Y with $y\notin\mathrm{supp}((\pi^*D)_{\mathrm{red}})$, the isotropy group Γ_y acts on the fibre W_y trivially.

Note : We have not asserted that such a Y exists over our base field k. If such a curve Y with Γ action exists then we will say that it *splits* the parabolic structure on X.

The category $\mathrm{Vect}(X, D, N)$ is a tensor category. To define the tensor product, it is convenient to think of parabolic bundles as being an appropriate family $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$, as described in Construction 3.5. Then $(\mathcal{F} \otimes \mathcal{F}')_t$ is the subsheaf of $i_* i^* \mathcal{F} \otimes \mathcal{F}'$ generated by

$$\mathcal{F}_a \otimes \mathcal{F}'_b \quad a + b \geq t.$$

Here we denote the inclusion $X \setminus D \hookrightarrow X$ by i. One checks that the resulting collection $(\mathcal{F}_* \otimes \mathcal{F}'_*)_{t\in\mathbb{R}}$ gives a bundle with parabolic datum $\mathbf{D}$.

We can also define an internal hom object. Set

$$\mathcal{H}om(\mathcal{F}, \mathcal{F}')_t = \mathcal{H}om(\mathcal{F}, \mathcal{F}'[t]).$$

Here $\mathcal{F}'[t]$ is the parabolic bundle with $\mathcal{F}'[t]_s = \mathcal{F}'_{s+t}$.

With these definitions $\mathrm{PVect}(X, D, N)$ becomes a rigid tensor category. The unit U for the tensor product is the bundle with $U_0 = \mathcal{O}_X$ and $U_t = \mathcal{O}_X(-D)$ for $0 < t < 1$. The proof that this is indeed a rigid tensor category can be found in [Yok95, § 3].

Theorem 9.2. *There is an additive equivalence of rigid tensor categories between* $\mathrm{PVect}(X, D, N)$ *and* $\mathrm{Vect}^D_\Gamma(Y, N)$.

Proof. This can be found in [BBN01, page 344], [Bis97]. Also see below for a description of one of the functors in this equivalence. □

We will denote the Γ-bundle associated to a parabolic bundle $\mathcal{F}_*$ by $\mathcal{F}^Y_*$.

There is a usual notion of exact sequence in the category $\mathrm{Vect}^D_\Gamma(Y)$. There is also a notion of exact sequence in the category $\mathrm{PVect}(X, D, N)$ that was described in Section 3.

Let us remark.

Proposition 9.3. *The equivalence in Theorem 9.2 preserves exact sequences.*

Proof. We use the notation set up before Theorem 9.2. Write $D = \sum_{i=1}^s x_i$ and $y_i = \pi^*(x_i)_{\mathrm{red}}$. Set $\pi^*(x_i) = n_i y_i$.

It will be convenient to think of parabolic bundles in terms of the construction in 3.5. From [Bis97], the functor $M : \mathrm{Vect}^D_\Gamma(Y, N) \longrightarrow \mathrm{PVect}(X, D, N)$ is given by the formula $M(W) = \mathcal{E}_t$ where

$$\mathcal{E}_t = (\pi_*(W \otimes \mathcal{O}_Y(\lceil -t n_i \rceil y_i))^\Gamma.$$

The functor is clearly additive. It suffices to remark that an equivalence of abelian categories by additive functors must preserve exact sequences. □

10. A Universal Construction

Let $\mathcal{E}_*$ and $\mathcal{E}'_*$ be parabolic bundles with parabolic datum $\mathbf{D}$ and $\mathbf{D}'$. We choose an integer N so that all the weights are integer multiples of $\frac{1}{N}$. We denote by $\mathrm{Ext}_{\mathrm{par}}(\mathcal{E}'_*, \mathcal{E}_*)$ the Yoneda Ext group in the category $\mathrm{PVect}(X, D, N)$ where D is chosen large enough to contain all parabolic points. It is a k-vector space and we view it as a variety. We would like to construct a universal extension on it. (A quick check shows that exact sequences are preserved by Baer sum and scalar multiplication.)

After some finite base extension L/k, there exists a group Γ, a smooth curve Y; and an action of Γ on Y defined over L such that $Y/\Gamma = X$ and Y splits the parabolic structures of $\mathcal{E}_*$ and $\mathcal{E}'_*$. That is, there is an equivalence of categories between $\mathrm{PVect}(X, D, N)$ and $\mathrm{Vect}_\Gamma^D(Y, N)$. By further extension of L we may assume all representations of Γ are defined over L.

Proposition 10.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be Γ bundles on Y. There exists an L-vector space $\mathrm{Ext}^1_\Gamma(\mathcal{F}, \mathcal{G})$, which we view as an L-variety, and an extension of Γ sheaves on $\mathrm{Ext}^1_\Gamma(\mathcal{F}, \mathcal{G}) \times Y$,*

$$0 \longrightarrow \pi^*\mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \pi^*\mathcal{F} \longrightarrow 0 \qquad (E)$$

with the following universal property: Given a scheme $f : V \longrightarrow \mathrm{Spec} L$ and an extension

$$0 \longrightarrow f^*\mathcal{G} \longrightarrow \mathcal{E}' \longrightarrow f^*\mathcal{F} \longrightarrow 0 \qquad (E')$$

of Γ sheaves on $V \times Y$ there exists a unique L-morphism $t : V \longrightarrow \mathrm{Ext}^1_\Gamma(\mathcal{F}, \mathcal{G})$ with $t^*(E) \cong (E')$.

Proof. There exists a universal extension on $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G})$. This follows via base change for cohomology. To obtain a universal Γ-extension, just restrict this extension to

$$\mathrm{Ext}^1_\Gamma(\mathcal{F}, \mathcal{G}) \overset{\mathrm{defn}}{=} \mathrm{Ext}^1(\mathcal{F}, \mathcal{G})_{\mathrm{triv}}.$$

This proves the proposition. □

Proposition 10.2. *There exists a universal extension of parabolic bundles on*

$$\mathrm{Ext}_{\mathrm{par}}(\mathcal{E}'_*, \mathcal{E}_*)\,.$$

Proof. We may assume that L/k is Galois with group G. Using the equivalence in Theorem 9.2 we see that there is a universal extension on the base extension $\mathrm{Ext}_{\mathrm{par}}(\mathcal{E}'_*, \mathcal{E}_*) \otimes L$. However, the universal extension inherits a Galois action, in view of its universal property, and hence descends. □

We need to bound the dimension of $\mathrm{Ext}_{\mathrm{par}}(\mathcal{E}'_*, \mathcal{E}_*)$. The following lemma will be useful.

Lemma 10.3. *Let Γ be a finite group and Γ_y a cyclic subgroup with generator γ. Let V be a finite dimensional representation of Γ_y on which γ^N acts trivially and T a one dimensional representation of Γ whose restriction to Γ_y is faithful. Then*

$$\sum_{d=0}^{|\Gamma_y|-1} d.\dim(\mathrm{Ind}_{\Gamma_y}^{\Gamma} V \otimes T^d)_{\mathrm{triv}}$$

is bounded by

$$(\dim V)|\Gamma_y|(1 - \frac{1}{N})$$

(here $W_{\mathrm{triv}} = W^{\Gamma_y}$ is the fixed part).

Proof. By base change we may assume that the ground field contains all roots of unity. Let ζ be a primitive $|\Gamma_y|$th root of unity. Write $V = \oplus_{i=0}^{N-1} V_{(\frac{i}{N}|\Gamma_y|)}$ where the generator γ acts by scalar multiplication by ζ^j on V_j. Note that only these weight spaces can occur as γ^N acts trivially on V. Then

$$\sum_{d=0}^{|\Gamma_y|-1} d.\dim(\mathrm{Ind}_{\Gamma_y}^{\Gamma} V \otimes T^d)_{\mathrm{triv}}$$

$$= \sum_{d=0}^{|\Gamma_y|-1} \sum_{i=0}^{N-1} d.\dim((V_{\frac{i}{N}|\Gamma_y|} \otimes T^d)_{\mathrm{triv}})$$

by Frobenius reciprocity. But γ acts by multiplication by the scalar ζ^{s+d} on $V_s \otimes T^d$. We see that for fixed i, the set of invariants $(V_{\frac{i}{N}|\Gamma_y|} \otimes T^d)_{\mathrm{triv}}$ is non-zero if and only if $(d,i) = (0,0)$ or $i > 0$ and $d = |\Gamma_y|(1 - \frac{i}{N})$. When $(V_{\frac{i}{N}|\Gamma_y|} \otimes T^d)_{\mathrm{triv}}$ is non-zero, $\dim((V_{\frac{i}{N}|\Gamma_y|} \otimes T^d)_{\mathrm{triv}}) = \dim(V_{\frac{i}{N}|\Gamma_y|})$. So the above sum becomes

$$\sum_{i=1}^{N-1} |\Gamma_y|(1 - \frac{i}{N}) \dim(V_{\frac{i}{N}|\Gamma_y|}) \leq |\Gamma_y|(1 - \frac{1}{N}) \dim(V)$$

since $V = \oplus_{i=0}^{N-1} V_{\frac{i}{N}|\Gamma_y|}$. □

Recall from [Bis97] that a Γ-bundle is semistable if and only if the underlying bundle is semistable. This fact follows from the uniqueness of the Harder-Narasimhan filtration. We also need the following lemma.

Lemma 10.4. *Let $\mathcal{E}$ be a semistable Γ-bundle on Y. Then*

$$\dim \mathrm{H}^0(Y, \mathcal{E})_{\mathrm{triv}} \leq \begin{cases} 0 & \text{if } \deg(\mathcal{E}) < 0 \\ \mathrm{rk}(\mathcal{E}) + \frac{\deg(\mathcal{E})}{|\Gamma|} & \text{otherwise} \end{cases}$$

Proof. The assertion is obvious when the degree is negative. We induct on the degree. By extending L, we may find a point $y \in Y$ on which Γ acts freely. We let D be the divisor $\mathrm{orb}(y)$. The result follows from the exact sequence

$$0 \longrightarrow \mathcal{E}(-D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_D \longrightarrow 0 \,.$$

Note that the twist $\mathcal{E}(-D)$ is indeed semistable. □

Proposition 10.5. *Let $\mathcal{F}_*$ and $\mathcal{G}_*$ be parabolic bundles such that $\mathcal{F}_*^\vee \otimes \mathcal{G}_*$ is parabolic semistable and* $\mathrm{pardeg}(\mathcal{F}_*^\vee \otimes \mathcal{G}_*) \geq 0$. *Then*

$$\dim \mathrm{Ext}_{\mathrm{par}}(\mathcal{F}_*, \mathcal{G}_*) \leq \mathrm{rk}(\mathcal{F}_*)\mathrm{rk}(\mathcal{G}_*)g + (\deg |\boldsymbol{D}|)(1 - \frac{1}{N})\mathrm{rk}(\mathcal{F}_*)\mathrm{rk}(\mathcal{G}_*) \,,$$

where g is the genus of X. (Recall the notation $|\boldsymbol{D}|$ from Section 3.)

Proof. We may pass to a field extension L/k so that there is a Γ cover $Y \longrightarrow X$ as in Theorem 9.2. Write $\mathcal{W}_*^Y$ for the Γ-bundle associated to a parabolic bundle $\mathcal{W}_*$ under this equivalence of categories. We need to compute the dimension of $\mathrm{Ext}^1_\Gamma(\mathcal{F}_*^Y, \mathcal{G}_*^Y)$. Note that the fact that

$$\dim_L \mathrm{Ext}^1_\Gamma(\mathcal{F}_*^Y, \mathcal{G}_*^Y) = \dim_{\bar{L}} \mathrm{Ext}^1_\Gamma(\mathcal{F}_*^Y \otimes \bar{L}, \mathcal{G}_*^Y \otimes \bar{L}).$$

allows us to pass to an algebraic closure and apply Theorem 9.1 to the Γ bundle $(\mathcal{F}_*^Y)^\vee \otimes \mathcal{G}_*^Y$. Then

$$\begin{aligned}\dim \mathrm{Ext}_{\mathrm{par}}(\mathcal{F}_*, \mathcal{G}_*) &= h^1((\mathcal{F}_*^Y)^\vee \otimes \mathcal{G}_*^Y)_{\mathrm{triv}} \\ &= h^0((\mathcal{F}_*^Y)^\vee \otimes \mathcal{G}_*^Y)_{\mathrm{triv}} - \chi(\Gamma, Y, (\mathcal{F}_*^Y)^\vee \otimes \mathcal{G}_*^Y)_{\mathrm{triv}} \\ &\leq \mathrm{rk}((\mathcal{F}_*^Y)^\vee \otimes \mathcal{G}_*^Y))g + \\ & \frac{1}{|\Gamma|}\sum_{y\in Y}\sum_{d=0}^{e_y-1} d\dim[\mathrm{Ind}_{\Gamma_y}^{\Gamma}((\mathcal{F}_*^Y)^\vee \otimes G_*^Y)|_y \otimes \chi_y^d)_{\mathrm{triv}}]\end{aligned}$$

Here we applied the last lemma under the hypothesis that the Γ-bundle $(\mathcal{F}_*^Y)^\vee \otimes \mathcal{G}_*^Y$ is semistable of non-negative degree. This is true since, by hypothesis, the corresponding parabolic bundle $\mathcal{F}^\vee \otimes \mathcal{G}$ is parabolic semistable with non-negative parabolic degree. We also applied Theorem 9.1 with the observation that the trivial part of the regular representation $k[\Gamma]$ has dimension one. Since $\mathrm{rk}((\mathcal{F}_*^Y)^\vee \otimes \mathcal{G}_*^Y)) = \mathrm{rk}(\mathcal{F})\mathrm{rk}(\mathcal{G})$, we need only bound the second term. If $y \notin \mathrm{supp}((\pi^*(D))_{\mathrm{red}})$, then Lemma 10.3 shows that the sum corresponding to y vanishes as the isotropy group acts trivially on $(\mathcal{F}^\vee \otimes \mathcal{G})|_y$. If N is an integer so that all weights are integer multiples of $\frac{1}{N}$ and $y \in \mathrm{supp}((\pi^*(D))_{\mathrm{red}})$, then Lemma 10.3 shows that the sum corresponding to y is bounded by $(1-\frac{1}{N})(\mathrm{rk}(\mathcal{F})\mathrm{rk}(\mathcal{G}))$ since in this case the kernel of Γ_y on $(\mathcal{F}^\vee \otimes \mathcal{G})|_p$ has order dividing N. Since

$$\frac{1}{|\Gamma|}|\{y \in Y : y \in \mathrm{supp}((\pi^*(D))_{\mathrm{red}})\}| \leq \deg(|\mathbf{D}|)\,,$$

the proof is complete. □

Let $\mathbf{D}$ be a parabolic datum on X. We denote by $N(\mathbf{D})$ the smallest positive integer so that the weights in $\mathbf{D}$ are scalar multiples of $\frac{1}{N(\mathbf{D})}$. Set

$$(1-\frac{1}{N(\mathbf{D})})\deg(|\mathbf{D}|) = M(\mathbf{D}).$$

Corollary 10.6. *Let $\mathcal{E}_*$ be a non-stable parabolic bundle of rank r with parabolic data $\boldsymbol{D}$. Let*

$$0 \subset (\mathcal{E}_1)_* \subset (\mathcal{E}_2)_* \subset \cdots \subset (\mathcal{E}_m)_* = \mathcal{E}_*$$

be the Harder-Narasimhan filtration of $\mathcal{E}_*$. Define $(\mathcal{E}')_* := (\mathcal{E}_{m-1})_*$. Then

$$\dim(\mathrm{Ext}^1_{\mathrm{par}}((\mathcal{E}/\mathcal{E}')_*, (\mathcal{E}')_*)) \leq r'(r-r')(g + M(\boldsymbol{D}))$$

Proof. The Harder-Narasimhan filtration of $(\mathcal{E}' \otimes (\mathcal{E}/\mathcal{E}')^\vee)_*$ as a parabolic bundle is

$$\begin{gathered}0 \subset (\mathcal{E}_1)_* \otimes (\mathcal{E}/\mathcal{E}_{m-1})_*^\vee \subset (\mathcal{E}_2)_* \otimes (\mathcal{E}/\mathcal{E}_{m-1})_*^\vee \subset \cdots \subset \\ (\mathcal{E}_{m-1})_* \otimes (\mathcal{E}/\mathcal{E}_{m-1})_*^\vee = \mathcal{E}'_* \otimes (\mathcal{E}/\mathcal{E}_{m-1})_*^\vee\end{gathered}$$

Note that

$$\begin{aligned}\mathrm{par}\mu(((\mathcal{E}/\mathcal{E}_{m-1})_*^\vee \otimes (\mathcal{E}_i/\mathcal{E}_{i-1})_*) &= \frac{(-r_m d_i + d_m r_i)}{r_i r_m} \\ &= \mathrm{par}\mu((\mathcal{E}_i/\mathcal{E}_{i-1})_*)) - \mathrm{par}\mu((\mathcal{E}_m/\mathcal{E}_{m-1})_*) \\ &> 0\end{aligned}$$

where d_i is the parabolic degree of $(\mathcal{E}_i/\mathcal{E}_{i-1})_*)$ and r_i is its rank. So the previous proposition applies to each $((\mathcal{E}/\mathcal{E}_{m-1})^\vee_* \otimes (\mathcal{E}_i/\mathcal{E}_{i-1})_*)$ since this is a semistable parabolic bundle with positive parabolic slope. We find that

$$\dim(\mathrm{Ext}^1_{\mathrm{par}}((\mathcal{E}/\mathcal{E}_{m-1})_*, (\mathcal{E}_i/\mathcal{E}_{i-1})_*)) \leq (r_i)(r-r')(g+M(\mathbf{D})).$$

The result follows by a simple induction. □

11. The Semistable Locus

We define a function recursively by $F_{g,\mathbf{D}} : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$F_{g,\mathbf{D}}(r) = \max_{\substack{s+t=r \\ s\geq 0,\ t>0 \\ s,t \text{ integers}}} F_{g,\mathbf{D}}(s) + t^2 g + st(g+M(\mathbf{D}))$$

and $F_{g,\mathbf{D}}(1) = g$ and $F_{g,\mathbf{D}}(0) = 0$.

Let us record the following :

Lemma 11.1. *Consider positive integers k_i with partitions $s_i + t_i = k_i$. Here s_i and t_i are non negative. Then*

$$\dim \mathrm{Flag}(s_1, \ldots, s_n) + \dim \mathrm{Flag}(t_1, \ldots, t_n) \leq \dim \mathrm{Flag}(k_1, \ldots, k_n)\,.$$

Proof. Recall that

$$\dim \mathrm{Flag}(k_1, \ldots, k_n) = \sum_{i=1}^{n-1} k_i(k_{i+1} + \ldots + k_n).$$

The result follows. □

Proposition 11.2. *We have*

$$\mathrm{ed}(\mathrm{Bun}^{r,ss,d}_{X,\boldsymbol{D}}) \leq F_{g,\boldsymbol{D}}(r) + \sum_{\boldsymbol{x}\in\boldsymbol{D}} \dim(\mathrm{Flag}_{\boldsymbol{x}}(\boldsymbol{D}))\,.$$

Proof. We proceed by induction on the rank with the case of rank one being trivial. Consider a parabolic bundle $\mathcal{E}_*$. If $\mathcal{E}_*$ is stable we are done by Corollary 6.2. Suppose next that

$$\mathrm{Soc}(\mathcal{E}_*) = \mathcal{E}_*.$$

Then by Theorem 8.4, the bundle is defined over a field of transcendence degree at most

$$r^2 g + \sum_{\mathbf{x}\in\mathbf{D}} \dim(\mathrm{Flag}_{\mathbf{x}}(\mathbf{D})).$$

In the remaining case there is an exact sequence

$$0 \longrightarrow \mathrm{Soc}(\mathcal{E}_*) \longrightarrow \mathcal{E}_* \longrightarrow \mathcal{F}_* \longrightarrow 0\,.$$

Suppose that $\mathbf{D}_1$ and $\mathbf{D}_2$ are the parabolic structures for $\mathrm{Soc}(\mathcal{E}_*)$ and $\mathcal{F}_*$ respectively. The previous lemma shows that for every parabolic point $\mathbf{x}$ we have

$$\dim \mathrm{Flag}_{\mathbf{x}}(\mathbf{D}_1) + \dim \mathrm{Flag}_{\mathbf{x}}(\mathbf{D}_2) \leq \dim \mathrm{Flag}_{\mathbf{x}}(\mathbf{D})\,.$$

Let the ranks of $\mathrm{Soc}(\mathcal{E}_*)$ and $\mathcal{F}_*$ be t and s respectively. By Theorem 8.4, we know that $\mathrm{Soc}(\mathcal{E}_*)$ is defined over a field of transcendence degree at most $t^2 g + \sum_{\mathbf{x}\in\mathbf{D}_1} \dim(\mathrm{Flag}_{\mathbf{x}}(\mathbf{D}_1))$. By induction the parabolic bundle $\mathcal{F}_*$ is defined over a field of transcendence degree $F_g(s) + \sum_{\mathbf{x}\in\mathbf{D}_2} \dim(\mathrm{Flag}_{\mathbf{x}}(\mathbf{D}_2))$. Let the compositum of these two fields be K. We have

$$\dim \mathrm{Ext}_{\mathrm{par}}(\mathcal{F}_*, \mathrm{Soc}(\mathcal{E}_*))_{\mathrm{triv}} \leq st(g+M(\mathbf{D})))$$

by Proposition 10.5. Note that this result applies as

$$\mathcal{F}_*^\vee \otimes \mathrm{Soc}(\mathcal{E}_*)$$

is semistable of parabolic degree 0. To obtain the result we apply Proposition 10.2. □

12. The Full Moduli Stack

We form a function

$$G_{g,\mathbf{D}}(r) = \max_{\substack{s+t=r \\ s\geq 0,\ t>0 \\ s,t \text{ integers}}} F_{g,\mathbf{D}}(t) + G_{g,\mathbf{D}}(s) + st(g + M(\mathbf{D}))$$

with $G_{g,\mathbf{D}}(1) = g$ and $G_{g,\mathbf{D}}(0) = 0$.

Theorem 12.1. *We have a bound*

$$\mathrm{ed}(\mathrm{Bun}^{r,d}_{X,\boldsymbol{D}}) \leq G_{g,\boldsymbol{D}}(r) + \sum_{\boldsymbol{x}\in\boldsymbol{D}} \dim(\mathrm{Flag}_{\boldsymbol{x}}(\boldsymbol{D})).$$

Note : The left handside in the inequality does not depend upon the weights in the parabolic datum. Hence the inequality is true for all possible choices of weights.

Proof. We prove this by induction on the rank r with the case of rank 1 being trivial. Consider a parabolic bundle $\mathcal{E}_*$ and the exact sequence

$$0 \longrightarrow (\mathcal{E}_1)_* \longrightarrow (\mathcal{E})_* \longrightarrow (\mathcal{E}_2)_* \longrightarrow 0\,.$$

where $(\mathcal{E}_1)_*$ is the (destabilising) parabolic proper subbundle of maximal rank in the Harder-Narasimhan of $\mathcal{E}_*$. Suppose that $\mathbf{D}_1$ and $\mathbf{D}_2$ are the parabolic structures for $(\mathcal{E}_1)_*$ and $(\mathcal{E}_2)_*$ respectively. Lemma 11.1 shows that for every parabolic point $\mathbf{x}$ we have

$$\dim(\mathrm{Flag}_{\mathbf{x}}(\mathbf{D}_1)) + \dim(\mathrm{Flag}_{\mathbf{x}}(\mathbf{D}_2)) \leq \dim(\mathrm{Flag}_{\mathbf{x}}(\mathbf{D}))\,.$$

Let the ranks of $(\mathcal{E}_1)_*$ and $(\mathcal{E}_2)_*$ be t and s respectively. By Proposition 11.2 we know that $(\mathcal{E}_2)_*$ is defined over a field of transcendence degree at most $F_{g,\mathbf{D}_1}(s) + \sum \dim \mathrm{Flag}_{\mathbf{x}}(\mathbf{D}_1)$. By induction the parabolic bundle $(\mathcal{E}_1)_*$ is defined over a field of transcendence degree $G_{g,\mathbf{D}_2}(t) + \sum_{\mathbf{x}\in\mathbf{D}_2} \dim \mathrm{Flag}_{\mathbf{x}}(\mathbf{D}_2)$. Let the compositum of these two fields be K. Note that $\deg(|\mathbf{D}_i|) \leq \deg(|\mathbf{D}|)$ and $N(\mathbf{D}_i) \leq N(\mathbf{D})$ for $i = 1, 2$ so that $M(\mathbf{D}_i) \leq M(\mathbf{D})$ for $i = 1, 2$ and hence $F_{g,\mathbf{D}_i} \leq F_{g,\mathbf{D}}$. Then

$$\mathrm{trdeg} K \leq F_{g,\mathbf{D}}(s) + G_{g,\mathbf{D}}(t) + \sum_{\mathbf{x}\in\mathbf{D}} \dim(\mathrm{Flag}_{\mathbf{x}}(\mathbf{D}))\,.$$

We have

$$\dim \mathrm{Ext}_{\mathrm{par}}((\mathcal{E}_2)_*, (\mathcal{E}_1)_*)_{\mathrm{triv}} \leq st(g + M(\mathbf{D}))$$

by Corollary 10.6. Let $W = \mathrm{Ext}((\mathcal{E}_2)_*, (\mathcal{E}_1)_*)_{\mathrm{triv}}$. The parabolic bundle $\mathcal{E}_*$ is defined over the function field K' of a subvariety of $\mathbb{P}(W)$. Then

$$\mathrm{trdeg} K' \leq F_{g,\mathbf{D}}(s) + G_{g,\mathbf{D}}(t) + \sum_{\mathbf{x}\in\mathbf{D}} \dim(\mathrm{Flag}_{\mathbf{x}}(\mathbf{D})) + st(g + M(\mathbf{D})) - 1$$

The result follows. □

13. Some Facts about $F_{g,\mathbf{D}}$ and $G_{g,\mathbf{D}}$

Let

$$H_{g,\mathbf{D},r}(t) = F_{g,\mathbf{D}}(t) + (r-t)^2 g + (g + M(\mathbf{D}))t(r-t)$$

so that

$$F_{g,\mathbf{D}}(r) = \max_{0\le t\le r-1} H_{g,\mathbf{D},r}(t)$$

Proposition 13.1. *If $g \le M(\boldsymbol{D})$, then for all $r \ge 0$, we have*

$$F_{g,\boldsymbol{D}}(r) = \frac{r(r+1)}{2} g + \frac{r(r-1)}{2} M(\boldsymbol{D})$$

If $g \ge M(\boldsymbol{D})$, then for all $r \ge 0$, we have

$$F_{g,\boldsymbol{D}}(r) = r^2 g$$

Proof. **Case 1:** $g \le M(\mathbf{D})$.

For $r = 0, 1$, this follows by definition of $F_{g,\mathbf{D}}$. Assume the result for $0 \le t < r$ by induction. Then by the inductive hypothesis, for all $0 \le t \le r-1$, we have

$$H_{g,\mathbf{D},r}(t) = (\frac{t(t+1)}{2})g + (\frac{t(t-1)}{2})M(\mathbf{D}) + (r-t)^2 g + (g + M(\mathbf{D}))t(r-t).$$

Simplifying this, we find that, for all $0 \le t \le r-1$, we have

$$H_{g,\mathbf{D},r}(t) = \left(\frac{g - M(\mathbf{D})}{2}\right) t^2 - (r - \frac{1}{2})(g - M(\mathbf{D}))t + r^2 g$$

Note that

$$H_{g,\mathbf{D},r}(r-1) = \frac{r(r+1)}{2} g + \frac{r(r-1)}{2} M(\mathbf{D}).$$

So it suffices to prove the claim that

$$\max_{0\le t\le r-1} H_{g,\mathbf{D},r}(t) = H_{g,\mathbf{D},r}(r-1).$$

If $g = M(\mathbf{D})$, it is clear that $H_{g,\mathbf{D},r}(t) = r^2 g$ for all $0 \le t \le r-1$ so the claim holds in this case.

Assume then that $g < M(\mathbf{D})$.

Consider the parabola that agrees with $H_{g,\mathbf{D},r}(t)$:

$$f(t) = \left(\frac{g - M(\mathbf{D})}{2}\right) t^2 - (r - \frac{1}{2})(g - M(\mathbf{D}))t + r^2 g$$

Since

$$f'(t) = (g - M(\mathbf{D}))(t - (r - 1/2)).$$

then $f'(t) \ge 0$ iff $t \le (r - 1/2)$ under the hypothesis $g - M(\mathbf{D}) < 0$. So in particular, $f(t)$ is increasing on the interval $0 \le t \le r-1$. But then since $H_{g,\mathbf{D},r}(t) = f(t)$, we have

$$F_{g,\mathbf{D}}(r) = \max_{0\le t\le r-1} H_{g,\mathbf{D},r}(t) = H_{g,\mathbf{D},r}(r-1).$$

as required.

Case 2: $g \ge M(\mathbf{D})$.

We will prove by induction on r that

$$F_{g,\mathbf{D}}(r) = H_{g,\mathbf{D},r}(0) = r^2 g$$

if $g \geq M(\mathbf{D})$. The statement is true for $r = 0, 1$ by definition and since we have more generally that

$$H_{g,\mathbf{D},r}(0) - H_{g,\mathbf{D},r}(1) = (r-1)(g - M(\mathbf{D})) \geq 0\,.$$

this shows that, in particular, we have

$$F_{g,\mathbf{D}}(2) = H_{g,\mathbf{D},2}(0) = 4g\,.$$

By the inductive hypothesis, we may assume that for $0 \leq t \leq r-1$, we have

$$H_{g,\mathbf{D},r}(t) = t^2 g + (r-t)^2 g + (g + M(\mathbf{D}))t(r-t)$$

Simplifying this, we obtain

$$H_{g,\mathbf{D},r}(t) = (g - M(\mathbf{D}))(t(t-r)) + r^2 g$$

by the above. Since $(g - M(\mathbf{D})) \geq 0$, we have $(g - M(\mathbf{D}))(t(t-r)) \leq 0$ if $0 \leq t \leq r-1$. So $H_{g,\mathbf{D},r}(t) \leq r^2 g = H_{g,\mathbf{D},r}(0)$ if $0 \leq t \leq r-1$. This implies that $F_{g,\mathbf{D}}(r) = r^2 g$.

Observe that

$$F_{g,\mathbf{D}}(r) = r^2 g = \frac{r(r+1)}{2} g + \frac{r(r-1)}{2} M(\mathbf{D})$$

if $g = M(\mathbf{D})$ so that the answers agree on the overlap. □

Recall that

$$G_{g,\mathbf{D}}(r) = \max_{\substack{s+t=r \\ s\geq 0,\ t>0 \\ s,t \text{ integers}}} F_{g,\mathbf{D}}(t) + G_{g,\mathbf{D}}(s) + st(g + M(\mathbf{D}))$$

and $G_{g,\mathbf{D}}(1) = g, G_{g,\mathbf{D}}(0) = 0$.

Proposition 13.2. *$F_{g,\boldsymbol{D}}(r) = G_{g,\boldsymbol{D}}(r)$ for all $r \geq 0$.*

Proof. The result is true by definition for $r = 0, 1$. It suffices to prove that for all $0 < s \leq t$, we have

$$F_{g,\mathbf{D}}(s+t) - F_{g,\mathbf{D}}(s) - F_{g,\mathbf{D}}(t) - (g + M(\mathbf{D}))st \geq 0$$

using the previous proposition.

Case 1: $g \leq M(\mathbf{D})$.

Since

$$F_{g,\mathbf{D}}(r) = \frac{r(r+1)}{2} g + \frac{r(r-1)}{2} M(\mathbf{D})\,,$$

we find that

$$\begin{aligned} &F_{g,\mathbf{D}}(s+t) - F_{g,\mathbf{D}}(s) - F_{g,\mathbf{D}}(t) - (g + M(\mathbf{D}))st \\ &= st(g + M(\mathbf{D})) - st(g + M(\mathbf{D})) = 0\,. \end{aligned}$$

Case 2: $g > M(\mathbf{D})$.

Since

$$F_{g,\mathbf{D}}(r) = r^2 g\,,$$

we find that

$$\begin{aligned} &F_{g,\mathbf{D}}(s+t) - F_{g,\mathbf{D}}(s) - F_{g,\mathbf{D}}(t) - (g + M(\mathbf{D}))st \\ &= 2stg - st(g + M(\mathbf{D})) = st(g - M(\mathbf{D})) \geq 0\,. \end{aligned}$$

□

Remark 13.3. The main result of [DL09] shows that

$$\mathrm{ed}(\mathrm{Bun}_X^{r,d}) \leq \lfloor h_g(r) \rfloor + g.$$

The function $h_g(r)$ is defined recursively by $h_g(1) = 1$ and

$$h_g(r) - h_g(r-1) = r^3 - r^2 + \frac{r^4}{4}(g-1) + \frac{r}{2} + \frac{r^2 g^2}{4} + \frac{1}{4}.$$

(Note : solving the recursion would produce a quartic.) Putting together Theorem 12.1, Proposition 13.1 and Proposition 13.2 for $\mathbf{D} = \emptyset$ and the original hypothesis $g \geq 2$ on the curve, we have

$$\mathrm{ed}(\mathrm{Bun}_X^{r,d}) \leq r^2 g,$$

which is a substantial improvement. The main reason for the improvement is the use of the socle filtration as opposed to the Jordan-Hölder filtration.

14. Lower Bounds

The issue of finding lower bounds in questions on essential dimension is more subtle. We fix a rank r and denote by $\mathrm{Bun}_X^{r,\xi,ss}$ the semistable locus of the moduli stack of vector bundles of rank r and determinant ξ. We would like to find a lower bound on its essential dimension.

Suppose that p^l divides r where p is a prime. Construct a parabolic datum $\mathbf{D} = \{\mathbf{x}\} = (x, \{p^l, r - p^l\}, \{\alpha_1, \alpha_2\})$ where α_i are chosen so small so that if a vector bundle is semistable for the datum $\mathbf{D}$ then it is semistable. It is easy to see one can do this using the definition of parabolic slope.

Theorem 14.1. *We have*

$$\mathrm{ed}(\mathrm{Bun}_X^{r,\xi,ss}) \geq (r^2-1)(g-1) + p^l.$$

Proof. Let $\mathbf{D} = \{\mathbf{x}\}$ be the datum constructed above. Now the greatest common divisor of the multiplicities is p^l. Hence by Theorem 6.1, as we are in the prime power case, there is a family of parabolic bundles $\mathcal{F}_*$ on X_L that cannot be compressed. Further,

$$\mathrm{tr.deg}_k(L) = p^l + (r^2-1)(g-1) + \dim \mathrm{Flag}_{\mathbf{x}}(\mathbf{D})\,.$$

The underlying vector bundle $\mathcal{F}$ is semistable. We claim that it does not descend to a field of transcendence degree less than

$$(r^2-1)(g-1) + p^l\,.$$

Suppose it did. Denote the field by K. Then the original parabolic bundle comes from a point of $\mathrm{Flag}_{\mathbf{x}}(\mathbf{D})$ over this field. This is a contradiction. □

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