

**UPPER BOUNDS FOR THE ESSENTIAL DIMENSION
OF THE MODULI STACK OF SL_n -BUNDLES
OVER A CURVE**

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Abstract. We find upper bounds for the essential dimension of various moduli stacks of SL_n -bundles over a curve. When n is a prime power, our calculation computes the essential dimension of the stack of stable bundles exactly and the essential dimension is not equal to the dimension in this case.

1. Introduction

We work over a field \mathbb{k} of characteristic 0 and fix a smooth projective geometrically connected curve X of genus $g \geq 2$ over \mathbb{k} . We assume that X has a point over \mathbb{k} . Our purpose in this paper is to study the essential dimension of various moduli stacks of SL_n -bundles on our curve. In order to use inductive arguments on the rank it will be convenient to slightly generalize the question. Let ξ be a line bundle on our curve. We will study the essential dimension of the stacks

$$\mathrm{Bun}_{SL_n}^{s,\xi} \quad \mathrm{Bun}_{SL_n}^{ss,\xi} \quad \mathrm{Bun}_{SL_n}^{\xi}$$

of (resp. stable, semistable, full) bundles with an identification of the top exterior power with ξ .

If our stacks possessed fine moduli spaces the essential dimension would just be the dimension of the moduli space. As no such space exists the question is open. At least when $\gcd(n, \xi)$ is a prime power, it seems that the essential dimension does not agree with the dimension of the moduli stack in the stable case.

For the stable case we compare the stack with its moduli space and use some theorems in [BRV] to study the essential dimension of the moduli stack. To pass from stable to semistable we use the Jordan–Hölder filtration. Some care is needed here as when considering essential dimension, one is forced into a position of having to consider nonalgebraically closed fields even if the base field \mathbb{k} is algebraically closed. For a semistable bundle its Jordan–Hölder filtration may not be defined

DOI: 10.1007/s00031-

*Supported by NSERC.

**Supported by NSERC.

Received December 16, 2008. Accepted August 3, 2009.

over the base field if it is not algebraically closed. To pass to the full moduli stack, we use the Harder–Narasimhan filtration.

An outline of the paper follows. Section 2 contains a review of the notion of essential dimension and pertinent results. Section 3 contains a review of the notions of stable and semistable with a view towards curves over nonalgebraically closed fields. Section 4 proves some elementary properties of our moduli stacks that will be needed later. Section 5 reviews twisted sheaves on gerbes and their relationship with period and index. Section 6 lists results regarding the Brauer group of the moduli space of vector bundles. A key invariant that is needed in our computations is the generic index of the gerbe

$$\mathrm{Bun}_{\mathrm{SL}_n}^{s,\xi} \rightarrow \mathrm{SU}^s(X, n, \xi),$$

where $\mathrm{SU}^s(X, n, \xi)$ is the coarse moduli space. In [BBGN07] and [DN89] the period of this gerbe is studied. In Section 7 we observe that the existence of some natural twisted sheaves implies that period equals index for this gerbe. The bound for the essential dimension of the stack $\mathrm{Bun}_{\mathrm{SL}_n}^{s,\xi}$ is obtained in Section 8. When $\gcd(\deg \xi, n)$ is a prime power this bound is an equality. The remaining sections contain results on bounds for the essential dimension of the full moduli stack and the semistable locus.

To describe the final result, we introduce a function $h_g : \mathbb{N} \rightarrow \mathbb{N}$ defined recursively by

$$\begin{aligned} h_g(1) &= 1, \\ h_g(n) - h_g(n-1) &= (n^3 - n^2) + \frac{n^2}{4}(g-1) + \frac{n}{2} + \frac{n^2 g^2}{4} + \frac{1}{4}. \end{aligned}$$

The final result (Theorem 11.1) that we obtain is the following.

Theorem. *We have*

$$\mathrm{ed}(\mathrm{Bun}_{\mathrm{SL}_n}^\xi) \leq [h_g(n)] + 1.$$

Notation and conventions

- \mathbb{k} our base field of characteristic 0.
- X a smooth geometrically connected curve of genus ≥ 2 defined over \mathbb{k} and having a point over \mathbb{k} .
- $\mathrm{Bun}_{\mathrm{SL}_n}^\xi$ the moduli stack of bundles over our curve with a fixed isomorphism of the top exterior power with ξ .
- $\mathrm{Bun}_{\mathrm{GL}_n}^\xi$ the moduli stack of bundles over our curve having determinant ξ .
- $\mathrm{Bun}_{\mathrm{GL}_n}^{\xi,s}$, $\mathrm{Bun}_{\mathrm{SL}_n}^{\xi,s}$ the open substacks of stable bundles.
- $\mathrm{Bun}_{\mathrm{GL}_n}^{\xi,ss}$, $\mathrm{Bun}_{\mathrm{SL}_n}^{\xi,ss}$ the open substacks of semistable bundles.
- $\mathrm{SU}^s(X, n, \xi)$ the moduli space of stable vector bundles with determinant ξ .

Acknowledgments. The authors would like to thank Patrick Brosnan, Daniel Krashen and Zinovy Reichstein for useful discussions and ideas. We would also like to thank the referees for numerous corrections to our original manuscript.

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2. Essential dimension

We denote by $\text{Fields}_{\mathbb{k}}$ the category of field extensions of \mathbb{k} . Let $F : \text{Fields}_{\mathbb{k}} \rightarrow \text{Sets}$ be a functor. We say that $a \in F(L)$ is *defined over a field* $K \subseteq L$ if there exists a $b \in F(K)$ so that $r(b) = a$ where r is the restriction

$$F(K) \rightarrow F(L).$$

The *essential dimension* of a is defined to be

$$\text{ed}(a) \stackrel{\text{def}}{=} \min_K \text{tr.deg}_{\mathbb{k}} K,$$

where the minimum is taken over all fields of definition K of a .

The *essential dimension* of F is defined to be

$$\text{ed}(F) = \sup_a \text{ed}(a),$$

where the supremum is taken over all $a \in F(K)$ and K varies over all objects of $\text{Fields}_{\mathbb{k}}$.

For an algebraic stack $\mathfrak{X} \rightarrow \text{Aff}_{\mathbb{k}}$ we obtain a functor

$$\text{Fields}_{\mathbb{k}} \rightarrow \text{Sets},$$

which sends K to the set of isomorphism classes of objects in $\mathfrak{X}(K)$. We define *the essential dimension of \mathfrak{X}* to be the essential dimension of this functor, and denote this number by $\text{ed}_{\mathbb{k}}(\mathfrak{X})$.

We now recall some theorems from [BRV] that will be needed in the future. We assume for the remainder of this section that \mathfrak{X}/\mathbb{k} is a Deligne–Mumford stack, locally of finite type, with finite inertia. By [KM97] such a stack has a coarse moduli space M . The first result that we shall need is

Theorem 2.1. *Suppose that $\text{char}(\mathbb{k}) = 0$ and \mathfrak{X} is also smooth and connected. Let K be the field of rational functions on M and let $\mathfrak{X}_K = \text{Spec}(K) \times_{\text{Spec}(\mathbb{k})} \mathfrak{X}$ be the base change. Then*

$$\text{ed}_{\mathbb{k}}(\mathfrak{X}) = \dim M + \text{ed}_K(\mathfrak{X}_K).$$

Proof. See [BRV, Theorem 6.1]. \square

The stack \mathfrak{X}_K/K is called the generic gerbe. In the case where this gerbe is banded by μ_n , more can be said about $\text{ed}_K(\mathfrak{X}_K)$.

Let \mathfrak{G} be a gerbe over our field \mathbb{k} banded by μ_n . Such a gerbe gives a torsion class in the Brauer group $\text{Br}(K)$. The index of this class is called the *index* of the gerbe and denoted by $\text{ind}(\mathfrak{G}) = d$. There is a Brauer-Severi variety P/\mathbb{k} of dimension $d - 1$ whose class maps to the class of \mathfrak{G} via the connecting homomorphism

$$H^1(X, \text{PGL}_d) \rightarrow H^2(X, \mathbb{G}_m).$$

Let X be a smooth and proper variety over \mathbb{k} . The set $X(\mathbb{k}(X))$ is the collection of rational endomorphisms of X defined over \mathbb{k} . Define

$$e_{\mathbb{k}}(X) = \inf\{\dim \overline{\text{im}(\phi)} \mid \phi \in X(\mathbb{k}(X))\}.$$

The number $e_{\mathbb{k}}(X)$ is called the *canonical dimension* of X .

Theorem 2.2. *In the above situation*

$$\mathrm{ed}(\mathfrak{G}) = e_K(P) + 1.$$

Proof. See [BRV, Theorem 4.1]. \square

Corollary 2.3. *In the above situation if $n = p^r$ is a prime power we have*

$$\mathrm{ed}(\mathfrak{G}) = \mathrm{ind}(P) + 1.$$

Proof. See [Kar00, Theorem 2.1] and [Mer03]. \square

In this paper we will be interested in studying the essential dimension of the stack $\mathrm{Bun}_{\mathrm{SL}_n}^\xi$. Let us recall what it is precisely.

Fix a line bundle ξ on our curve X and denote by $\mathrm{Bun}_{\mathrm{SL}_n}^\xi$ the moduli stack of SL_n -vector bundles on X with determinant ξ . For a \mathbb{k} -scheme U the objects in the groupoid over U are pairs (\mathcal{E}, ϕ) where \mathcal{E} is a rank n bundle on $X \times_{\mathbb{k}} U$ and ϕ is an isomorphism

$$\phi : \bigwedge^n \mathcal{E} \xrightarrow{\sim} \xi.$$

A morphism $(\mathcal{E}, \phi) \rightarrow (\mathcal{E}', \phi')$ is an isomorphism of vector bundles $\alpha : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ such that the following diagram commutes:

$$\begin{array}{ccc} \bigwedge^n \mathcal{E} & \xrightarrow{\phi} & \xi \\ \wedge \alpha \downarrow & & \parallel \\ \bigwedge^n \mathcal{E}' & \xrightarrow{\phi'} & \xi. \end{array}$$

In the case where ξ is the trivial bundle this is just the moduli stack of SL_n -torsors.

In order to study the essential dimension of this stack it will be useful to introduce another auxiliary stack $\mathrm{Bun}_{\mathrm{GL}_n}^\xi$. For a \mathbb{k} -scheme U the objects in the groupoid over U are rank n vector bundles on $X \times_{\mathbb{k}} U$ with $\det \mathcal{E} \otimes \mathrm{pr}_X^* \xi^\vee$ isomorphic to $\mathrm{pr}_U^* \eta$ where η is a line bundle on U . The morphisms of the groupoid are just isomorphisms of vector bundles. It follows from the generalized seesaw theorem [Mum70, p. 89] that this is in fact a closed substack of the moduli stack of vector bundles on X .

3. Stability and semistability for bundles

Notation 3.1. Let \mathcal{E} be a vector bundle on X_K . We denote by \mathcal{E}_L the pullback of \mathcal{E} under the natural projection $X_L \rightarrow X_K$ where $K \hookrightarrow L$ is a field extension.

Let \mathcal{E} be a vector bundle on our curve X . The *slope* of \mathcal{E} is defined to be

$$\mu(\mathcal{E}) \stackrel{\mathrm{defn}}{=} \frac{\mathrm{deg}(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}.$$

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A vector bundle \mathcal{E} is said to be semistable (resp. stable) if

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}_L) \quad (\text{resp. } \mu(\mathcal{F}) < \mu(\mathcal{E}_L))$$

for every subsheaf \mathcal{F} of \mathcal{E}_L as L varies over all algebraic field extensions of \mathbb{k} . An SL_n -bundle is said to be semistable (resp. stable) if its associated vector bundle is so.

Given a vector bundle \mathcal{E} , set

$$\mu = \sup\{\mu(\mathcal{E}') \mid \mathcal{E}' \subseteq \mathcal{E}\}.$$

One can show that there exists a unique subsheaf \mathcal{E}_k of \mathcal{E} such that $\mu = \mu(\mathcal{E}_k)$ and \mathcal{E}_k is maximal with respect to inclusion amongst subsheaves of slope μ , see [Pot97, Prop. 5.4.2]. Such a sheaf is called a *maximal destabilizing subsheaf*. Induction yields a unique filtration

$$0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_k \subseteq \mathcal{E} = \mathcal{E}_{k+1}$$

such that:

- (i) The associated graded objects $\mathcal{E}_i/\mathcal{E}_{i-1}$ are semistable.
- (ii) The slopes $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$ are decreasing.

This is the *Harder–Narasimhan* filtration.

Proposition 3.2. *Let \mathcal{E} be a semistable bundle. There exists an increasing filtration, defined over a finite Galois extension L/\mathbb{k} ,*

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_n = \mathcal{E}_L,$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1} = \text{gr}_i(\mathcal{E}_\bullet)$ is stable. Moreover, any two filtrations \mathcal{E}_i and \mathcal{E}'_i have the same length and there exists $\sigma \in S_n$ so that $\text{gr}_i(\mathcal{E}_\bullet) \cong \text{gr}_{\sigma(i)}(\mathcal{E}'_\bullet)$.

Proof. This is [Pot97, Prop. 5.3.7]. \square

Remark 3.3. It follows from the uniqueness of the Harder–Narasimhan filtration that the field extension is not needed in the definition of semistable. In other words, semistable may be defined in the following way, a bundle \mathcal{E} is semistable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for all subbundles \mathcal{F} of \mathcal{E} .

This is not true for the notion of stable. For example, consider a curve X/\mathbb{k} of genus at least 1, and a quadratic extension L/\mathbb{k} . We can arrange things so that there is a point $p \in X(L)$ such that its Galois conjugate p^σ is different from itself. The rank 2 bundle

$$\mathcal{E} = \mathcal{O}(p) \oplus \mathcal{O}(p^\sigma)$$

has a Galois action and descends to a bundle on X . However, its Jordan–Hölder filtration exists only over the curve X_L .

Two bundles \mathcal{E} and \mathcal{F} are said to be *S-equivalent* if the $\bigoplus \text{gr}_i(\mathcal{E})$ and $\bigoplus \text{gr}_i(\mathcal{F})$ are isomorphic.

We summarize below some basic properties of stable and semistable bundles.

Theorem 3.4.

- (i) *Let \mathcal{F} be a stable bundle on X . Then $H^0(X, \text{End}(\mathcal{F}))$ is one dimensional.*
- (ii) *More generally, let R be a ring and let \mathcal{F} be a family of stable bundles on X_R parametrized by R , i.e. for every closed point x of $\text{Spec}(R)$, the restriction of the family to x is stable. Then $H^0(X, \text{End}(\mathcal{F})) = R$.*
- (iii) *Being stable and semistable are open conditions.*
- (iv) *Fix a line bundle ξ on X . There exists a moduli space $\text{SU}^{\text{ss}}(X, n, \xi)$ of semistable bundles of rank n and determinant ξ on X . Its closed points correspond to S -equivalence classes of semistable bundles. There is an open substack $\text{SU}^s(X, n, \xi)$ parametrizing stable bundles.*

Proof. This is essentially carried out in Part I of [Pot97] when \mathbb{k} is algebraically closed. For our slightly more general setting choose an algebraic closure $\bar{\mathbb{k}} \subseteq \mathbb{k}$.

(i) Suppose that we have an endomorphism $\phi : \mathcal{E} \rightarrow \mathcal{E}$ of a stable bundle. We know that its base extension $\phi_{\bar{\mathbb{k}}}$ is multiplication by a scalar λ . The scalar λ must come from \mathbb{k} as ϕ is defined over \mathbb{k} .

(ii) There exists a natural inclusion

$$\epsilon : R \hookrightarrow H^0(X_R, \text{End}(\mathcal{F}))$$

that we wish to show is an isomorphism. By flat base change, we may assume $R = (R, \mathfrak{m})$ is local. Via Nakayama's lemma we need to show that

$$\bar{\epsilon} : R/\mathfrak{m} \hookrightarrow H^0(X_R, \text{End}(\mathcal{F})) \otimes_R R/\mathfrak{m}$$

is surjective. But by (i), the composition

$$R/\mathfrak{m} \rightarrow H^0(X, \text{End}(\mathcal{F})) \otimes_R R/\mathfrak{m} \rightarrow H^0(X_{R/\mathfrak{m}}, \text{End}(\mathcal{F}_{R/\mathfrak{m}}))$$

is surjective. The result follows from the base change theorem [Har77, III, Theorem 12.11].

(iii) One may just adapt the proofs from [Pot97] to our situation or use the fact that for every scheme S/\mathbb{k} , the projection

$$S_{\bar{\mathbb{k}}} \rightarrow S$$

is an open morphism.

(iv) We wish to construct a moduli space for the stacks $\text{Bun}_{\text{SL}_n}^{s, \xi}$ and $\text{Bun}_{\text{GL}_n}^{s, \xi}$. These are the open substacks of $\text{Bun}_{\text{SL}_n}^{\xi}$ and $\text{Bun}_{\text{GL}_n}^{\xi}$ parametrizing stable bundles. The result actually follows from a theorem of Keel and Mori [KM97], once we know that $\text{Bun}_{\text{SL}_n}^{s, \xi}$ is a Deligne–Mumford stack with finite inertia and $\text{Bun}_{\text{SL}_n}^{s, \xi}$ and $\text{Bun}_{\text{GL}_n}^{s, \xi}$ have the same moduli space. However we will need an explicit description of the moduli space below. The family of stable bundles of given rank and determinant is a bounded family. This can be proved by passing to $\bar{\mathbb{k}}$ and applying the result there. Hence there exists an integer N such that

$$H^1(X, \mathcal{E}(n)) = 0 \quad \text{and} \quad \mathcal{E}(n) \quad \text{is generated by global sections}$$

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for every stable bundle \mathcal{E} of given rank and determinant and for every $n \geq N$. Recall that we have assumed that our curve has a point p over \mathbb{k} so we define $\mathcal{E}(n) = \mathcal{E} \otimes \mathcal{O}_X(np)$. Let $h = \dim H^0(X, \mathcal{E}(N))$ for a stable bundle of given rank and determinant. Consider the quot-scheme parametrizing quotients

$$\mathcal{O}_X(-N)^h \rightarrow \mathcal{E}$$

with rank $\mathcal{E} = n$ and $\deg \mathcal{E} = \deg \xi$. There is a locally closed subset Ω parametrizing quotients (use (ii)) with \mathcal{E} stable and $\det \mathcal{E} = \xi$. Using part (i) we have

$$\mathrm{Bun}_{SL_n}^{s,\xi} = [\Omega/SL_h] \quad \text{and} \quad \mathrm{Bun}_{GL_n}^{s,\xi} = [\Omega/GL_h].$$

The center of GL_h acts trivially on Ω , and is in fact the stabilizer of a point by (i). It follows that we can identify the coarse moduli space with the quotient

$$\mathrm{Bun}_{GL_n}^{s,\xi} \rightarrow \Omega/GL_h = \mathrm{SU}^s(X, n, \xi). \quad \square$$

Note that a consequence of the above is that a family \mathcal{F} of stable bundles on $S \times X$ with $\det \mathcal{F} \otimes \mathrm{pr}_X^* \xi^\vee$ being the pullback of a line bundle on S , determines a morphism

$$\phi_{\mathcal{F}} : S \rightarrow \mathrm{SU}^s(X, n, \xi).$$

We would like to record a kind of partial converse to the above.

Proposition 3.5.

- (i) *Given an S -point $\phi : S \rightarrow \mathrm{SU}^s(X, n, \xi)$ there is an étale cover $e : T \rightarrow S$ such that $\phi \circ e$ is equal to $\phi_{\mathcal{F}}$ for some family on $T \times X$.*
- (ii) *Suppose that we have two families \mathcal{F}_1 and \mathcal{F}_2 on $S \times X$ that determine the same S -point of $\mathrm{SU}^s(X, n, \xi)$. Then there is an étale cover $T \rightarrow S$ such that pullbacks $\mathcal{F}_{1,T}$ and $\mathcal{F}_{2,T}$ are isomorphic.*

Proof. Using the notation of the proof of the preceding proposition we note that $\Omega \rightarrow \Omega/\mathrm{PGL}_h = \mathrm{SU}^s(X, n, \xi)$ is a PGL_h -principal bundle. To see this, we note that the question is local in the étale topology, so we may pass to an algebraically closed field and use the known result there. To finish off the proof, recall that for any PGL_h -principal bundle $P \rightarrow B$, an S -point $S \rightarrow B$ lifts to P upon passing to an étale cover of S . \square

4. Basic properties of our moduli stacks

Recall that we denote by $\mathrm{Bun}_{SL_n}^{s,\xi}$ and $\mathrm{Bun}_{GL_n}^{s,\xi}$ the open substacks of $\mathrm{Bun}_{SL_n}^\xi$ and $\mathrm{Bun}_{GL_n}^\xi$ parametrizing stable bundles.

There is an obvious morphism of stacks

$$\mathrm{Bun}_{SL_n}^{s,\xi} \rightarrow \mathrm{Bun}_{GL_n}^{s,\xi}$$

that will give a u -morphism of gerbes below.

Lemma 4.1. *The natural map $\text{Bun}_{\text{GL}_n}^{s,\xi} \rightarrow \text{SU}^s(X, n, \xi)$ makes the stack into a gerbe banded by \mathbb{G}_m over the moduli space.*

Proof. The previous proposition is saying that it is a gerbe. The band is computed in Theorem 3.4. \square

Similarly we have

Lemma 4.2. *The natural map $\text{Bun}_{\text{SL}_n}^{s,\xi} \rightarrow \text{SU}^s(X, n, \xi)$ makes the stack into a gerbe banded by μ_n over the moduli space.*

We need to show that $\text{Bun}_{\text{SL}_n}^{s,\xi}$ is a Deligne–Mumford stack with finite inertia.

Proposition 4.3. *The stack $\text{Bun}_{\text{SL}_n}^{s,\xi}$ is a Deligne–Mumford stack.*

Proof. The stack is of finite type as the collection of stable bundles forms a bounded family, see [Pot97, Chap. 7]. We need to show that the diagonal is formally unramified. Consider an extension of Artinian local k -algebras

$$0 \rightarrow I \rightarrow A' \rightarrow A \rightarrow 0.$$

An A' -point of $\text{Bun}_{\text{SL}_n}^{s,\xi} \times \text{Bun}_{\text{SL}_n}^{s,\xi}$ amounts to two families (\mathcal{F}_1, ϕ_1) and (\mathcal{F}_2, ϕ_2) of stable bundles with identifications of their top exterior powers with ξ parametrized by A' . Completing this to a diagram of the form

$$\begin{array}{ccc} \text{Spec}(A) & \longrightarrow & \text{Spec}(A') \\ \downarrow & & \downarrow \\ \text{Bun}_{\text{SL}_n}^{s,\xi} & \longrightarrow & \text{Bun}_{\text{SL}_n}^{s,\xi} \times \text{Bun}_{\text{SL}_n}^{s,\xi} \end{array}$$

amounts to an isomorphism $\alpha : \mathcal{F}_1|_A \cong \mathcal{F}_2|_A$ compatible with the identifications of the top exterior powers. We need to show that any extension of the isomorphism α to A' is unique. In view of Theorem 3.4 this follows from the following claim.

Claim 4.4. *Let (B, \mathfrak{m}) be a local \mathbb{k} -algebra. Suppose that $y_i \in B$ and $y_1^n = y_2^n = 1$. Further assume that y_i have the same images under the projection*

$$q : B \rightarrow B/\mathfrak{m}.$$

Then $y_1 = y_2$.

Proof of claim. We may write $y_2 = y_1 + x$ where $x \in \mathfrak{m}$. As we are in characteristic 0, we have

$$1 = (y_2)^n + x(\text{another unit in } B).$$

Since $y_2^n = 1$ we must have $x = 0$. \square

In order to make use of the work in [BRV] we need to see that the morphism

$$\mathcal{I}(\text{Bun}_{\text{SL}_n}^{s,\xi}) \rightarrow \text{Bun}_{\text{SL}_n}^{s,\xi},$$

where $\mathcal{I}(\mathcal{X})$ means inertia stack, is a finite morphism.

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Proposition 4.5. *The stack $\text{Bun}_{SL_n}^{s,\xi}$ has finite inertia.*

Proof. Using Theorem 3.4, one identifies the inertia stack with $\text{Bun}_{SL_n}^{s,\xi} \times_{\mathbb{k}} \mu_n$. Hence the projection

$$\text{Bun}_{SL_n}^{s,\xi} \times_{\mathbb{k}} \mu_n \rightarrow \text{Bun}_{SL_n}^{s,\xi}$$

is a finite morphism. \square

5. Twisted sheaves and the Brauer group

This section collects some general results about the Brauer group and twisted sheaves. Let X/\mathbb{k} be a scheme. A gerbe $\mathfrak{G} \rightarrow X$ banded by μ_n gives a class $[\mathfrak{G}]$ in $H^2(X_{\text{ét}}, \mu_n)$ and hence a torsion class in $H^2(X_{\text{ét}}, \mathbb{G}_m)$. Recall that the *period* of \mathfrak{G} is defined to be the order of this class. If $X = \text{Spec}(K)$ for a field K we define the *index* of $[\mathfrak{G}]$ to be the greatest common divisor of the degrees of splitting fields of $[\mathfrak{G}]$.

The following is well known.

Proposition 5.1. *When $X = \text{Spec}(K)$ in the above situation the period divides the index.*

Proof. This is well known, for example, see [FD, Prop. 4.16]. \square

A useful tool for understanding the difference between the period and the index is the notion of a twisted sheaf. A *twisted sheaf* on a \mathbb{G}_m -gerbe $\mathfrak{G} \rightarrow X$ is a coherent sheaf \mathcal{F} on \mathfrak{G} such that inertial action of \mathbb{G}_m on \mathcal{F} coincides with natural module action of \mathbb{G}_m on \mathcal{F} . We spell out the meaning of this statement in the next paragraph.

Suppose that we have a T -point $T \rightarrow X$ and an object a of \mathfrak{G} above this point. Part of the data of the coherent sheaf \mathcal{F} is a sheaf \mathcal{F}_a on T . These sheaves are required to satisfy compatibility conditions on pullbacks for morphisms in the category \mathfrak{G} . In particular, every object a of the gerbe \mathfrak{G} has an action of \mathbb{G}_m and hence there is an action of \mathbb{G}_m on \mathcal{F} . The above definition says that action of \mathbb{G}_m on \mathcal{F} should be the same as the \mathbb{G}_m -action coming from the fact that \mathcal{F} is an $\mathcal{O}_{\mathfrak{G}}$ -module.

Example 5.2. We have a μ_n -gerbe

$$\text{Bun}_{SL_n}^{s,\xi} = [\Omega/SL_h] \rightarrow [\Omega/PGL_h] = \text{SU}^s(X, n, \xi).$$

It gives rise to a \mathbb{G}_m -gerbe

$$\text{Bun}_{GL_n}^{s,\xi} = [\Omega/GL_h] \rightarrow [\Omega/PGL_h] = \text{SU}^s(X, n, \xi),$$

where $\text{Bun}_{SL_n}^{s,\xi}$ is the moduli stack of bundles with determinant ξ but the isomorphisms do not induce the identity on the determinant. The universal bundle on $\text{Bun}_{GL_n}^{s,\xi} \times X$ is a twisted sheaf since the only automorphisms of a stable bundle are given by multiplication by a scalar.

We will need the following.

Proposition 5.3. *Let $\mathfrak{G} \rightarrow \text{Spec}(K)$ be a \mathbb{G}_m -gerbe over a field. Then the index of \mathfrak{G} divides m if and only if there is a locally free rank m twisted sheaf on \mathfrak{G} .*

Proof. See [Lie08, Prop. 3.1.2.1]. \square

Corollary 5.4. *The index of the gerbe $\text{Bun}_{\text{GL}_n}^{s,\xi} \rightarrow \text{SU}^s(X, n, \xi)$ over the generic point of $\text{SU}^s(X, n, \xi)$ divides n .*

6. The Brauer group of $\text{SU}^s(X, n, \xi)$

In this section we recall the results of [BBGN07] and we present some minor modifications of these results for our own context.

There is a natural Severi–Brauer variety over $\text{SU}^s(X, n, \xi) \times X$. To construct it, using the notation of §4, notice that the PGL_h action on Ω lifts to the projectivisation of the universal bundle on the quot-scheme. Let \mathbb{P} be the quotient Severi–Brauer variety. Each closed point $x \in X$ gives an inclusion

$$\text{SU}^s(X, n, \xi) \hookrightarrow \text{SU}^s(X, n, \xi) \times X.$$

Denote by \mathbb{P}_x the pullback of \mathbb{P} via this inclusion.

Proposition 6.1. *When working over $\mathbb{k} = \mathbb{C}$ we have:*

- (i) *The Brauer group $\text{Br}(\text{SU}^s(X, n, \xi))$ is cyclic of order $\text{gcd}(n, \text{deg}(\xi))$.*
- (ii) *The Brauer group is generated by the class of the gerbe*

$$\text{Bun}_{\text{GL}_n}^{s,\xi} \rightarrow \text{SU}^s(X, n, \xi).$$

- (iii) *The class of the Brauer-Severi variety \mathbb{P}_x in $\text{H}^2(\text{SU}^s(X, n, \xi), \mathbb{G}_m)$ coincides with the class of the gerbe*

$$\text{Bun}_{\text{GL}_n}^{s,\xi} \rightarrow \text{SU}^s(X, n, \xi).$$

This class does not depend on x .

Proof. This is [BBGN07, Theorem 1.8] and the discussion immediately before it. \square

In our setting we are not working over the complex numbers but we do not need the full power of the result above. We can prove the following which is sufficient for our question on essential dimension.

Proposition 6.2. *The period of the gerbe*

$$\text{Bun}_{\text{GL}_n}^{s,\xi} \rightarrow \text{SU}^s(X, n, \xi)$$

is $\text{gcd}(n, \text{deg}(\xi))$.

To prove this we need to recall some constructions from [DN89]. Recall that the moduli spaces $\text{SU}^s(X, \xi, n)$ and $\text{SU}^{\text{ss}}(X, \xi, n)$ can be constructed as geometric invariant theory quotients

$$\text{SU}^s(X, \xi, n) = \Omega/\text{GL}_h, \quad \text{SU}^{\text{ss}}(X, \xi, n) = \Omega^{\text{ss}}//\text{GL}_h,$$

where Ω^* is an appropriate open subset of the quot-scheme as in Section 4. We write Ω^* to mean one of Ω or Ω^{ss} . Let L be a GL_h -line bundle on Ω^* . There is an integer $e(L)$ such that the center of GL_h acts on L with weight $e(L)$.

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Proposition 6.3. *Let k be an integer. There exists a GL_h line bundle on Ω with $e(L) = k$ if and only if k is divisible by $\gcd(\deg(\xi), n)$*

Proof. This is precisely Proposition 5.1 of [DN89]. There it was proved over the complex numbers but the proof goes through in our case. We briefly outline it here for the convenience of the reader.

First consider the reverse implication. We have a universal bundle U on $\Omega \times X$. The result follows by considering the weight of central torus actions on the line bundles

$$\det(i^*(U)) \quad \text{and} \quad \det(\pi_*(U \otimes \mathcal{O}_X(m))).$$

Here $i : \Omega \hookrightarrow \Omega \times X$ is the inclusion at some point of X and $\pi : \Omega \times X \rightarrow \Omega$ the projection.

For the other direction one can simply observe that $e(L)$ doesn't change under a base extension

$$\text{Spec}(K) \rightarrow \text{Spec}(\mathbb{k}).$$

So one may base change to an algebraically closed field and use a Lefschetz principle. \square

Proof (of Proposition 6.2). With the above lemma the proof can be now copied from [BBGN07]. \square

7. The period index problem for our gerbe

Let K be the function field of $SU^s(X, n, \xi)$. We have a gerbe over K defined by the 2-Cartesian square:

$$\begin{array}{ccc} \mathfrak{G} & \longrightarrow & \text{Bun}_{GL_n}^{s, \xi} \\ \downarrow & & \downarrow \\ \text{Spec}(K) & \longrightarrow & SU^s(X, n, \xi). \end{array}$$

Set $d = \gcd(n, \deg(\xi))$. We know that the period of $\text{Bun}_{GL_n}^{s, \xi}$ is d . Let us remark that the period of \mathfrak{G} is also d . This follows from the following two facts.

Proposition 7.1. *Let X be a regular scheme with function field K . The pullback map*

$$Br(X) \rightarrow Br(K)$$

is injective.

Proof. See [Mil, IV, Cor. 2.6]. \square

Proposition 7.2. *The moduli space $SU^s(X, n, \xi)$ is a smooth algebraic variety.*

Proof. By [GIT, Theorem 1.1] geometric invariant theory quotients are uniform. So

$$SU^s(X, n, \xi)_{\bar{\mathbb{k}}} = SU^s(X_{\bar{\mathbb{k}}}, n, \xi).$$

Hence one may base change to an algebraically closed field and apply the result there, see [Pot97, Chap. 8]. \square

By Propositions 5.1 and 6.1 and the above discussion we know that d divides the index of \mathfrak{G} .

In fact we have

Proposition 7.3. *We have $d = \text{ind}(\mathfrak{G})$ so that period equals the index for this gerbe.*

Proof. It suffices to show that $\text{ind}(\mathfrak{G})$ divides n and $\text{deg}(\xi)$. It follows from Example 5.2 and Proposition 5.3 that the index divides n .

Recall that X has a point. Taking $\mathcal{L} = \mathcal{O}_X(d)$ for d large we may assume that

$$R^1\pi_*(\mathcal{F}^{\text{univ}} \otimes \pi_X^*\mathcal{L}) = 0,$$

where $\mathcal{F}^{\text{univ}}$ is the universal bundle on

$$\text{Bun}_{\text{GL}_n}^{s,\xi} \times X$$

and π the projection onto $\text{Bun}_{\text{GL}_n}^{s,\xi}$. As we are over the stable locus, the bundle $\pi_*(\mathcal{F}^{\text{univ}})$ is a twisted sheaf of rank

$$\chi = \text{deg}(\xi) + n(1 - g).$$

Applying Proposition 5.3 again the result follows. \square

8. The stable locus

Proposition 8.1. *Let α be the class of $\text{Bun}_{\text{SL}_n}^{s,\xi}$ inside $\text{H}^2(\text{SU}^s(X, \xi, n), \mu_n)$. The image of α under the natural map*

$$\text{H}^2(\text{SU}^s(X, \xi, n), \mu_n) \rightarrow \text{H}^2(\text{SU}^s(X, \xi, n), \mathbb{G}_m)$$

is the class of $\text{Bun}_{\text{GL}_n}^{s,\xi}$.

Proof. We have a natural inclusion $u : \mu_n \hookrightarrow \mathbb{G}_m$ and a diagram

$$\begin{array}{ccc} \text{Bun}_{\text{SL}_n}^{s,\xi} & \xrightarrow{\phi} & \text{Bun}_{\text{GL}_n}^{s,\xi} \\ & \searrow & \swarrow \\ & \text{SU}^s(X, n, \xi) & \end{array}$$

The map ϕ is a u -morphism in the sense of [Gir71, Chap. IV, 2.1.5]. The result now follows from [Gir71, Chap. IV, 3.1.5]. \square

Theorem 8.2. *Suppose that $\text{char}(\mathbb{k}) = 0$. We have a bound*

$$\text{ed}(\text{Bun}_{\text{GL}_n}^{s,\xi}) \leq (n^2 - 1)(g - 1) + d,$$

where $d = \text{gcd}(n, \text{deg}(\xi))$. This inequality is an equality when $d = p^r$ is a prime power.

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Proof. Let K be the function field of $SU^s(X, n, \xi)$ and $\mathfrak{G} \rightarrow \text{Spec}(K)$ the generic gerbe defined by the Cartesian diagram:

$$\begin{array}{ccc} \mathfrak{G} & \longrightarrow & \text{Bun}_{SL_n}^{s, \xi} \\ \downarrow & & \downarrow \\ \text{Spec}(K) & \longrightarrow & SU^s(X, n, \xi). \end{array}$$

By Theorem 2.1 we have

$$\text{ed}(\text{Bun}_{SL_n}^{s, \xi}) = \dim SU^s(X, n, \xi) + \text{ed}(\mathfrak{G}/K),$$

and $\dim SU^s(X, n, \xi) = (n^2 - 1)(g - 1)$, see [Pot97, Theorem 8.3.2].

It remains to understand the essential dimension of the generic gerbe. By Theorem 2.2 we have

$$\text{ed}(\mathfrak{G}/K) = e(SB) + 1,$$

where SB is a Severi–Brauer variety of dimension $\text{ind}(\mathfrak{G}) - 1$. The index of the generic gerbe is computed in Proposition 7.3. Recall $e(X)$ is the minimum element of the set

$$\{\dim \overline{\text{Im}(\phi)} \mid \phi \text{ a rational endomorphism of } X\}.$$

It follows that $e(SB) \leq \dim_K(SB) \leq n - 1$.

For the equality one applies the corollary to Theorem 2.2 which states that

$$\text{ed}(\mathfrak{G}/K) = \text{index of } \mathfrak{G}/K = n,$$

when n is a prime power. \square

9. The Galois theory of stable bundles

Fix a Galois extension L/K with Galois group G . Let \mathcal{E} be a semistable bundle on X_K with slope μ . We shall abuse notation and write \mathcal{E} for the pullback to X_L . Note that there are canonical identifications $h^*\mathcal{E} \cong \mathcal{E}$ for every $h \in G$.

Let V be a stable bundle on X_L of slope μ and suppose that $\text{Hom}(\mathcal{E}, V)$ is nonzero. Let $q = \dim(\text{Hom}(\mathcal{E}, V))$. Choose an ordered basis $\phi_1, \phi_2, \dots, \phi_q$ for $\text{Hom}(\mathcal{E}, V)$. We will need the fact that the induced map

$$\mathcal{E} \rightarrow V^q$$

is surjective. This follows from

Proposition 9.1. *Let \mathcal{E} be a semistable bundle and let V be a stable bundle of the same slope. Suppose*

$$\psi_1, \dots, \psi_k \in \text{Hom}(\mathcal{E}, V)$$

are linearly independent. Then the induced map

$$\mathcal{E} \rightarrow V^k$$

is surjective.

Proof. One inducts on k . In the case $k = 1$, since \mathcal{E} is semistable, V is stable and both have slope μ , we see that

$$\mu = \mu(\mathcal{E}) \leq \mu(\psi_1(\mathcal{E})) \leq \mu(V) = \mu$$

and so $\mu(\psi_1(\mathcal{E})) = \mu(V) = \mu$ which implies, from the stability of V , that $\psi_1(\mathcal{E}) = V$. So ψ_1 is surjective.

In general, let \mathcal{K} be the kernel of ψ_k . By the previous argument, ψ_k is surjective. As we have an exact sequence

$$0 \rightarrow \text{Hom}(V, V) \rightarrow \text{Hom}(\mathcal{E}, V) \rightarrow \text{Hom}(\mathcal{K}, V),$$

$\psi_1, \psi_2, \dots, \psi_{k-1}$ restrict to linearly independent homomorphisms from \mathcal{K} to V . Then one applies the induction hypothesis to \mathcal{K} . \square

We write $\Phi : \mathcal{E} \rightarrow V^q$ for the surjection induced by the basis $\phi_1, \phi_2, \dots, \phi_q$ and \mathcal{K} for its kernel.

We have for each $h \in G$ a composition of surjective maps

$$\mathcal{E} \xrightarrow{\sim} h^*\mathcal{E} \xrightarrow{h^*\Phi} (h^*V)^q.$$

We abuse notation and write $h^*\Phi$ for the composition of these two maps. Note that $h^*(\mathcal{K}) = \text{Ker}(h^*\Phi)$ so that, for each $g \in G$, we have a short exact sequence

$$0 \rightarrow h^*(\mathcal{K}) \rightarrow \mathcal{E} \xrightarrow{h^*\Phi} (h^*V)^q \rightarrow 0.$$

Proposition 9.2. *Suppose that we are given a different basis*

$$\psi_1, \psi_2, \dots, \psi_q \in \text{Hom}(\mathcal{E}, V).$$

Then there exists a unique automorphism

$$\alpha_h^{\Phi, \Psi} = \alpha_h : (h^*V)^q \xrightarrow{\sim} (h^*V)^q$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{h^*\Phi} & (h^*V)^q \longrightarrow 0 \\ \parallel & & \downarrow \alpha_h \\ \mathcal{E} & \xrightarrow{h^*\Psi} & (h^*V)^q \longrightarrow 0. \end{array}$$

These isomorphisms are functorial with respect to h , that is, $h^\alpha_{h'} = \alpha_{hh'}$.*

Proof. The uniqueness is clear. We first construct α_{id} . In this case there is an $\alpha \in \text{GL}(\text{Hom}(\mathcal{E}, V))$ that sends the basis $\Phi = \{\phi_j\}$ to the basis $\Psi = \{\psi_j\}$. Then we take α_{id} to be the induced automorphism of the polystable bundle V^q . One defines α_h to be $h^*\alpha_{\text{id}}$. \square

Let S be the stabilizer of the G -action on V , that is,

$$S = \{g \in G \mid g^*V \cong V\}.$$

Also let $\text{id} = h_1, h_2, \dots, h_l$ be coset representatives for G/S .

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Proposition 9.3. *Let $h \in G$ and suppose that $hS = h_i S$ for some $i = 1, \dots, l$. Choose an isomorphism $\beta : h^*V \xrightarrow{\cong} h_i^*V$. Then there exists a unique isomorphism $(h^*(V))^q \cong (h_i^*(V))^q$ such that the following diagram commutes:*

$$\begin{array}{ccccc} h^*\mathcal{E} & \longrightarrow & (h^*V)^q & \longrightarrow & 0 \\ \parallel & & \downarrow & & \\ h_i^*\mathcal{E} & \longrightarrow & (h_i^*V)^q & \longrightarrow & 0. \end{array}$$

Proof. The isomorphism is the composite $\beta^q \circ \alpha_h$ where α_h is the isomorphism from the last proposition. \square

Let us recall a definition.

Definition 9.4. Suppose a finite group H acts on a scheme Y . Let \mathcal{F} be a sheaf on Y . We say \mathcal{F} (really (\mathcal{F}, α_g)) is an H -sheaf if there are isomorphisms

$$\alpha_g : g^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

for each $g \in G$ subject to the conditions:

- (1) $\alpha_1 = \text{identity}$.
- (2) For every $g, h \in G$ the following diagram commutes:

$$\begin{array}{ccc} g^*h^*\mathcal{F} & \xrightarrow{g^*\alpha_h} & g^*\mathcal{F} \\ & \searrow \alpha_{hg} & \downarrow \alpha_g \\ & & \mathcal{F}. \end{array}$$

Corollary 9.5. *The coherent sheaf V is an S -sheaf, that is, there is an action of the group S on V compatible with the action of S on X_L .*

Proof. This is because \mathcal{E} is an S -sheaf and the uniqueness part of Proposition 9.3. \square

Proposition 9.6. *Let V be a stable bundle with the same slope as the semistable bundle \mathcal{E} . Set $q = \dim \text{Hom}(\mathcal{E}, V)$. If the associated graded bundles of the Jordan–Hölder filtration of \mathcal{E} are $\mathcal{G}_1, \dots, \mathcal{G}_\alpha$, then at least q of the \mathcal{G}_i are isomorphic to V .*

Proof. Let $\phi_1, \phi_2, \dots, \phi_q$ be a basis for $\text{Hom}(\mathcal{E}, V)$. Using the discussion at the start of this section we obtain an exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow V^q \rightarrow 0.$$

Then we can obtain a Jordan–Hölder filtration of \mathcal{E} by extending such a filtration of \mathcal{K} . Precisely, if $\mathcal{E}_1, \dots, \mathcal{E}_k$ form a Jordan Hölder filtration of \mathcal{K} then for each $j = 1, \dots, q$, define \mathcal{E}_{k+j} to be the kernel of the surjective map

$$\mathcal{E} \rightarrow V^{q-j}$$

given by $\phi_1, \dots, \phi_{q-j}$. The result now follows. \square

Proposition 9.7. *We have*

$$|G/S| \leq \frac{\text{rank}(\mathcal{E})}{q \cdot \text{rank}(V)},$$

where $q = \dim \text{Hom}(\mathcal{E}, V)$ and S is the stabilizer subgroup of V in G .

Proof. Let $\{\mathcal{G}_i\}$ be a set of associated graded bundles of the Jordan–Hölder filtration of \mathcal{E} . By Proposition 9.6 applied to the semistable bundle \mathcal{E} and the stable bundle h_i^*V for each coset representative h_i of $S = \text{Stab}_G(V)$ in G , we see that we have q of the \mathcal{G}_i isomorphic to h_i^*V for each $i = 1, \dots, [G : S]$. Then $\text{rank}(\mathcal{E}) \geq q \cdot [G : S] \text{rank}(V)$. The result follows. \square

For the following corollary we will make use of Galois descent. An introduction to this subject can be found in [BOU, p. 60] and [KO, Chap. 2]. As stated, the theorems in these two references are not quite general enough for our purposes. A very general version of this theorem is written down in [Mil, p. 19]. The relationship of this last theorem to Galois descent is established by realizing that for a Galois cover $S \rightarrow T$ with group H we have $S \times_T S \cong S \times H$.

Corollary 9.8. *There is a field extension L'/K of degree at most*

$$\frac{\text{rank}(\mathcal{E})}{q \cdot \text{rank}(V)},$$

over which V is defined. Furthermore, there is a surjection

$$\mathcal{E} \rightarrow V^q \rightarrow 0$$

defined over L' .

Proof. One takes $L' = L^S$ and applies Corollary 9.5 above to see that V descends to L' . Note that

$$\text{Hom}_{X_{L'}}(\mathcal{E}_{L'}, V_{L'}) \otimes L = \text{Hom}_{X_L}(\mathcal{E}_L, V_L)$$

so one takes a new basis defined over L' and applies Proposition 9.1. Note that we are not asserting that the original surjection descends to L' . \square

Remark 9.9. In order to obtain a bound on the essential dimension we replace L with the Galois closure of L'/K . So G is some subgroup of the symmetric group S_p with $p = \dim L'/K$ and hence by the corollary is a subgroup of $S_{\text{rank}(\mathcal{E})}$.

Let us record the following result.

Proposition 9.10. *Let \mathcal{F} be a semistable vector bundle of rank n and degree d over our curve X of genus g . Then $h^0(\mathcal{F}) \leq \max(d/n + 1, 0)n$. Furthermore, when \mathcal{F} has nonnegative slope we have $h^0(\mathcal{F}) \leq n + d$ and $h^1(\mathcal{F}) \leq ng$.*

Proof. The first part is Lemma 7.1.2 of [Pot97]. The second statement follows from the first via Riemann–Roch. \square

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Corollary 9.11. *Let \mathcal{E} be a nonstable vector bundle of rank n over X . Let \mathcal{E}' be a maximal destabilizing proper subbundle with $\mu(\mathcal{E}') > \mu(\mathcal{E})$ and rank $n' < n$. Then*

$$\dim(\mathrm{Ext}^1(\mathcal{E}/\mathcal{E}', \mathcal{E}')) \leq n'(n - n')g$$

Proof. Suppose that the Harder–Narasimhan filtration of \mathcal{E} is

$$0 \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \cdots \subseteq \mathcal{E}_k = \mathcal{E}$$

so that $\mathcal{E}_{k-1} = \mathcal{E}'$. The Harder–Narasimhan filtration of

$$\mathcal{E}' \otimes (\mathcal{E}/\mathcal{E}')^\vee$$

is then

$$0 \subseteq \mathcal{E}_1 \otimes (\mathcal{E}/\mathcal{E}_{k-1})^\vee \subseteq \mathcal{E}_2 \otimes (\mathcal{E}/\mathcal{E}_{k-1})^\vee \subseteq \cdots \subseteq \mathcal{E}_{k-1} \otimes (\mathcal{E}/\mathcal{E}_{k-1})^\vee = \mathcal{E}' \otimes (\mathcal{E}/\mathcal{E}_{k-1})^\vee.$$

Notice that

$$\mu((\mathcal{E}/\mathcal{E}_{k-1})^\vee \otimes \mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E}_i/\mathcal{E}_{i-1}) - \mu(\mathcal{E}/\mathcal{E}_{k-1}),$$

which is positive and the bundle $(\mathcal{E}/\mathcal{E}_{k-1})^\vee \otimes \mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable so that the proposition applies to it. A long exact sequence and simple induction completes the proof. \square

10. From stable to semistable

We begin with a couple of simple observations.

Lemma 10.1. *Let L/L_1 be a field extension and consider the morphism*

$$f : X_L \rightarrow X_{L_1}.$$

Let \mathcal{F}, \mathcal{G} be coherent sheaves on X_{L_1} and suppose we have two morphisms

$$\alpha_i : \mathcal{F} \rightarrow \mathcal{G}, \quad i = 1, 2.$$

If $f^\alpha_1 = f^*\alpha_2$ then $\alpha_1 = \alpha_2$.*

Proof. Note that the morphism f is flat. Hence

$$\mathrm{Hom}_{X_{L_1}}(\mathcal{F}, \mathcal{G}) \otimes L = \mathrm{Hom}_{X_L}(f^*\mathcal{F}, f^*\mathcal{G}).$$

\square

Proposition 10.2. *In the situation of the above lemma suppose that a finite group G acts on both L and L_1 . Suppose further that $f^*\mathcal{F}$ is a G -sheaf and we have associated isomorphisms*

$$\alpha_g : g^*\pi^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}.$$

Consider $\{g_1, g_2, \dots, g_k\}$ a generating set for G . Suppose that there exist isomorphisms

$$\beta_{g_i} : g_i^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

with $f^(\beta_{g_i}) = \alpha_{g_i}$ then there is a G -action on \mathcal{F} that pulls back to the G -action on $f^*\mathcal{F}$.*

Proof. For every $g \in G$, we fix an expression

$$g = g_{i_1} g_{i_2} \cdots g_{i_k}.$$

We define

$$\beta_g : g^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

as the composition of the following list of morphisms

$$\begin{array}{lcl} g_{i_k}^* \cdots g_{i_3}^* g_{i_2}^* (\beta_{g_1}) & : & g_{i_k}^* \cdots g_{i_2}^* g_{i_1}^* \mathcal{F} \rightarrow g_{i_k}^* \cdots g_{i_3}^* g_{i_2}^* \mathcal{F}, \\ \vdots & & \vdots \\ g_k^* \beta_{g_{k-1}} & : & g_k^* g_{k-1}^* \mathcal{F} \xrightarrow{\sim} g_{k-1}^* \mathcal{F}, \\ \beta_{g_k} & : & g_k^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}. \end{array}$$

We have $f^*(\beta_g) = \alpha_g$. To check that this is an action we need to see that the conditions of Definition 9.4 hold. As they hold for α_g and β_g pullbacks to α_g , they hold for β_g by the lemma. \square

To obtain the bound we will need to make use of the following construction.

Proposition 10.3. *Fix a projective scheme Y and another scheme Q over \mathbb{k} . Let \mathcal{F} and \mathcal{G} be families of coherent sheaves on $Y \times Q$. Consider the functor*

$$\mathbf{Isom}(\mathcal{F}, \mathcal{G}) : \mathbf{Schemes}/Q \rightarrow \mathbf{Sets}$$

whose value on $f : P \rightarrow Q$ is the set of isomorphisms

$$\mathbf{Isom}(\mathcal{F}, \mathcal{G})(f : P \rightarrow Q) = \{\alpha : (f \times 1)^*(\mathcal{F}) \xrightarrow{\sim} (f \times 1)^*(\mathcal{G})\}.$$

This functor is representable by a scheme $\mathbf{Isom}(\mathcal{F}, \mathcal{G}) \rightarrow Q$.

Proof. See [LMB, p. 29, Proof of Theorem 4.6.2.1]. \square

Remark 10.4. Suppose $Q = \mathrm{Spec}(K)$, where K is a field. Consider an isomorphism $\alpha : \mathcal{F} \xrightarrow{\sim} \mathcal{G}$. The Zariski tangent space to $\mathbf{Isom}(\mathcal{F}, \mathcal{G})$ at α can be identified with $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$. The reason is that a tangent vector is just a morphism

$$\mathrm{Spec}(K[\epsilon]) \rightarrow \mathbf{Isom}(\mathcal{F}, \mathcal{G})$$

lifting the K -point induced by α . (Here $K[\epsilon] = K[t]/t^2$.) This is just a matrix of maps

$$\begin{pmatrix} \alpha & d \\ 0 & \alpha \end{pmatrix} : \mathcal{F} \oplus \epsilon \mathcal{F} \xrightarrow{\sim} \mathcal{G} \oplus \epsilon \mathcal{G}.$$

The only unknown parameter is d which is just a homomorphism $d : \mathcal{F} \rightarrow \mathcal{G}$.

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Construction 10.5. Let us recall the set-up of the previous section. Consider a semistable bundle \mathcal{E} of rank n on X_K . By passing to a Galois extension L/K with Galois group G we can find an exact sequence, defined over L ,

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow V^q \rightarrow 0. \quad (1)$$

Suppose that K and V descend to a subfield \tilde{N} of L . We may replace \tilde{N} by N its Galois closure in L . Set

$$W = \text{Ext}_{X_N}^1(V^q, \mathcal{K}).$$

There is a universal extension

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E}^{\text{univ}} \rightarrow V^q \rightarrow 0$$

on $W \times X$. (The universal extension amounts to constructing a universal cohomology class. Let R be an N -algebra. By [Har77, III, 12.11],

$$\text{Ext}_{X_R}^1(V^q, \mathcal{K}) = W \otimes R,$$

and hence a cohomology class gives a homomorphism $R \otimes \text{Sym}^\bullet W^\vee \rightarrow R$. The class is constructed by thinking of W as $\text{Spec}(\text{Sym}^\bullet W^\vee)$.) Using [Ram73], the universal extension descends to an extension on $\mathbb{P}(W) \times X$ of the form

$$0 \rightarrow \mathcal{K} \otimes \mathcal{O}(1) \rightarrow \mathcal{E}^{\text{univ}} \rightarrow V^q \rightarrow 0.$$

There is a morphism $\phi_E : \text{Spec}(L) \rightarrow \mathbb{P}(W)$ with $\phi_E^*(\mathcal{E}^{\text{univ}}) = \mathcal{E}$. The generic point of the image of ϕ_E is of the form $\text{Spec}(\tilde{M})$ for some subfield \tilde{M} of L . We let M be its Galois closure in L . We have a diagram of fields

$$\begin{array}{ccc} L & \longleftarrow & M \\ \uparrow & & \uparrow \\ K & \longleftarrow & M^G \end{array}$$

with \mathcal{E} defined over M . Finally, we need to construct an extension L_1 of M so that the Galois action on \mathcal{E} descends to L_1 . By Proposition 10.2 we only need to make the generators descend. By Remark 9.9 we can assume that G is a subgroup of S_n and hence by [Cam, Theorem 1.13] it can be generated by $n - 1$ group elements. Choose a generating set $\{g_1, g_2, \dots, g_{n-1}\}$ for G . Consider the scheme

$$\mathbf{Isom}(\mathcal{E} : g_1, \dots, g_{n-1}) \stackrel{\text{def}}{=} \mathbf{Isom}(\mathcal{E}, g_1^* \mathcal{E}) \times_M \cdots \times_M \mathbf{Isom}(\mathcal{E}, g_{n-1}^* \mathcal{E}).$$

The isomorphisms $g_i^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ defining the Galois action of \mathcal{E} are defined over the function field of some point of $\mathbf{Isom}(\mathcal{E} : g_1, \dots, g_{n-1})$ by Proposition 10.2.

Proposition 10.6. *In the above situation, suppose that V and \mathcal{K} descend to a field N with $\text{tr deg } N/\mathbb{k} = p$. Then there is a subfield L_1 of L , stable under G , to which \mathcal{E} with its Galois action descends. Furthermore, we have*

$$\text{tr deg } L_1/\mathbb{k} \leq q \cdot \text{rank}(V)(n - q \cdot \text{rank}(V))g - 1 + n^3 - n^2 + p.$$

Proof. In Construction 10.5 we constructed L_1 . So we just need to count transcendence degrees. The transcendence degree of M is bounded by $\dim W - 1 + p$. Noticing that V^q and \mathcal{K} are semistable of the same slope we have, by Proposition 9.10,

$$\mathrm{tr\,deg} M/\mathbb{k} \leq q \cdot \mathrm{rank}(V)(n - q \cdot \mathrm{rank}(V))g - 1 + p.$$

By construction 10.5 and by remark 10.4, we have

$$\begin{aligned} \mathrm{tr\,deg} L_1/M &\leq \dim(\mathbf{Isom}(\mathcal{E} : g_1, \dots, g_{n-1})) \\ &= \sum_{i=1}^{n-1} \dim(\mathbf{Isom}(\mathcal{E}, g_i^* \mathcal{E})) \\ &= \sum_{i=1}^{n-1} \dim \mathrm{Hom}(\mathcal{E}, g_i^* \mathcal{E}) \\ &\leq (n-1)n^2, \end{aligned}$$

where the last inequality follows from Proposition 9.10 since for each $i = 1, \dots, n-1$, $\mathrm{Hom}(\mathcal{E}, g_i^* \mathcal{E})$ is a semistable bundle of rank n^2 and degree 0. \square

Set

$$\Lambda(n) = (n^3 - n^2) + \frac{n^2}{4}(g-1) + \frac{n}{2} + \frac{n^2 g^2}{4} + \frac{1}{4}.$$

We define a function $h_g : \mathbb{N} \rightarrow \mathbb{N}$ recursively by the formula $h_g(1) = 0$ and

$$h_g(n) - h_g(n-1) = \Lambda(n).$$

Proposition 10.7. *If $n \geq 1$ then $h_g(n+1) \geq h_g(n)$.*

Theorem 10.8. *We have*

$$\mathrm{ed}(\mathrm{Bun}_{\mathrm{SL}_n}^{\xi, ss}) \leq [h_g(n)] + 1.$$

Proof. We induct on n . The result for $n = 1$ is by choice of the constant.

Let \mathcal{E} be a rank $n \geq 2$ semistable bundle defined over a field K/\mathbb{k} . Notice that

$$\begin{aligned} n^2 g^2 - 4(n^2 - 1)(g - 1) &= n^2(g - 2)^2 + 4g - 4 \\ &\geq 0 \end{aligned}$$

and

$$n^3 - n^2 = n^2(n - 1) \geq n^2.$$

The first inequality implies $n^2 g^2 / 4 \geq (n^2 - 1)(g - 1)$ so if \mathcal{E} is stable then by Theorem 8.2 we are done.

Otherwise we can find a Galois extension L/K with group G and an exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow V^q \rightarrow 0$$

defined over L . By induction and the above proposition, \mathcal{K} is defined over a field of transcendence degree at most $h_g(n-1)$. The stable bundle V is defined over

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a field of transcendence degree at most $(\text{rank}(V)^2 - 1)(g - 1) + \text{rank}(V)$. Writing $\alpha = \text{rank}(V)$ and applying Proposition 10.6 to our bundle \mathcal{E} , along with its Galois action, descends to a field of of transcendence degree

$$h_g(n - 1) + \alpha^2(g - 1) + \alpha - q^2\alpha^2g + \alpha qng + (n^3 - n^2).$$

It suffices to show that if

$$\lambda(\alpha, q) = \alpha^2(g - 1) + \alpha - q^2\alpha^2g + \alpha qng,$$

then for all pairs of integers (α, q) with $0 < q\alpha < n$ we have

$$(n^3 - n^2) + \lambda(\alpha, q) \leq \Lambda(n).$$

(The extra $+1$ in the statement of the theorem comes from the choice of trivialization $\det \mathcal{E} \cong \xi$.) To prove the above assertion we consider two cases.

Case I: $\alpha \leq n/2$.

In this case, some calculus shows that

$$-q^2\alpha^2g + \alpha qng \leq n^2g/4,$$

by considering $p(x) = xng - x^2g$. So we obtain

$$\begin{aligned} \lambda(\alpha, q) &\leq \frac{n^2}{4}(g - 1) + \frac{n}{2} + \frac{n^2g}{4} \\ &\leq \frac{n^2}{4}(g - 1) + \frac{n}{2} + \frac{n^2g^2}{4} + \frac{1}{4}. \end{aligned}$$

Case II: $\alpha > n/2$.

Now we must have $q = 1$. The bound is much easier as

$$\begin{aligned} \lambda(\alpha, q) &= -\alpha^2 + \alpha + \alpha ng \\ &\leq \frac{(1 + ng)^2}{4}. \end{aligned}$$

One obtains

$$\begin{aligned} 4\Lambda(n) - (1 + ng)^2 - 4n^3 + 4n^2 &= n^2(g - 1) + 2n - 2ng \\ &= (g - 1)n(n - 2). \end{aligned}$$

This quantity is nonnegative as $n \geq 2$. \square

11. The full moduli stack

Theorem 11.1. *Suppose that $\text{char}(\mathbb{k}) = 0$. We have a bound*

$$\text{ed}(\text{Bun}_{SL_n}^\xi) \leq [h_g(n)] + 1,$$

where $h_g(n)$ is as defined in the last section.

Proof. We prove this by induction on the rank n . Note that all rank 1 bundles are stable so that $\text{ed}(\text{Bun}_{\text{SL}_n}^\xi) = \text{ed}(\text{Bun}_{\text{SL}_n}^{s,\xi})$ for $n = 1$. We may assume the result for all $r < n$. Let \mathcal{E} be an unstable bundle of rank n defined over a field extension L/\mathbb{k} . Let \mathcal{E}' be a maximal destabilizing subbundle of \mathcal{E} , so that we have an exact sequence

$$0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}/\mathcal{E}' \rightarrow 0$$

with $\mu(\mathcal{E}') > \mu(\mathcal{E})$. By the inductive hypothesis, both \mathcal{E}' and \mathcal{E}/\mathcal{E}' are defined over smaller fields. Taking the compositum of these two extensions we obtain an extension K with

$$\text{tr deg } K \leq h_g(\text{rank}(\mathcal{E}')) + h_g(\text{rank}(\mathcal{E}/\mathcal{E}')).$$

Set $W = \text{Ext}^1(\mathcal{E}/\mathcal{E}', \mathcal{E}')$. The bundle \mathcal{E} is defined over the function field K' of a subvariety of $\mathbb{P}(W)$. So

$$\begin{aligned} \text{tr deg } K' &\leq h_g(\text{rank}(\mathcal{E}')) + h_g(\text{rank}(\mathcal{E}/\mathcal{E}')) + \dim W - 1 \\ &\leq h_g(\text{rank}(\mathcal{E}')) + h_g(\text{rank}(\mathcal{E}/\mathcal{E}')) + \text{rank}(\mathcal{E}')(n - \text{rank}(\mathcal{E}'))g - 1. \end{aligned}$$

Hence it suffices to prove the following inequality.

If s, t are positive integers with $s + t = n$ then

$$h_g(s) + h_g(t) + stg - 1 \leq h_g(n).$$

We may assume that $s \geq t$. We use induction on t . For $t = 1$, the above inequality turns into

$$h_g(n-1) + (n-1)g \leq h_g(n).$$

This follows immediately from the recursive definition of h_g and the fact that $n^3 - n^2 \geq n^2$ for $n \geq 2$, as

$$\frac{n^2g}{4} \geq (n-1)g \quad \text{when } n \geq 2.$$

By induction, we may assume that

$$h_g(s) + h_g(t) + stg - 1 \leq h_g(n),$$

and we need to show that

$$h_g(s-1) + h_g(t+1) + (s-1)(t+1)g - 1 \leq h_g(n),$$

provided $s-1 \geq t+1$. We may as well prove the following inequality:

$$h_g(s-1) + h_g(t+1) + (s-1)(t+1)g - 1 \leq h_g(s) + h_g(t) + stg - 1.$$

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Rearranging things, we need to show that if $s \geq t + 2$ and $t \geq 2$ then

$$\begin{aligned} 0 &\leq h_g(s) - h_g(s-1) + h_g(t) - h_g(t+1) + (t-s)g + g \\ &= (s^3 - s^2) + \frac{s^2}{4}(g-1) + \frac{s}{2} + \frac{s^2 g^2}{4} \\ &\quad - ((t+1)^3 - (t+1)^2) - \frac{(t+1)^2}{4}(g-1) - \frac{t+1}{2} - \frac{(t+1)^2 g^2}{4} \\ &\quad + (t-s)g + g. \end{aligned}$$

The component functions in the above expression are all increasing, so pairing up like ones we deduce that it suffices to prove the following:

$$\frac{s^2 g^2}{4} - \frac{(t+1)^2 g^2}{4} + (t-s)g \geq 0.$$

We write $s = t + \delta$ with $\delta \geq 2$. Multiplying through by 4 the above becomes

$$\begin{aligned} (t+\delta)^2 g^2 - (t+1)^2 g^2 - 4\delta g &= (\delta^2 - 1)g^2 + 2g(tg(\delta-1) - 2\delta) \\ &\geq (\delta^2 - 1)g^2 + 4g(\delta-1) \geq 0. \end{aligned}$$

This is nonnegative as $t, \delta, g \geq 2$. \square

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