

# CAYLEY GROUPS

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ABSTRACT. The classical Cayley map,  $X \mapsto (I_n - X)(I_n + X)^{-1}$ , is a birational isomorphism between the special orthogonal group  $\mathbf{SO}_n$  and its Lie algebra  $\mathfrak{so}_n$ , which is  $\mathbf{SO}_n$ -equivariant with respect to the conjugating and adjoint actions, respectively. We ask whether or not maps with these properties can be constructed for other algebraic groups. We show that the answer is usually “no”, with a few exceptions. In particular, we show that a Cayley map for the group  $\mathbf{SL}_n$  exists if and only if  $n \leq 3$ , answering an old question of LUNA.

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## 1. Introduction

The exponential map is a fundamental instrument of Lie theory that yields local linearization of various problems involving Lie groups and their actions; see [Bou<sub>1</sub>]. Let  $L$  be a real Lie group with Lie algebra  $\mathfrak{l}$ . As the differential at 0 of the exponential  $\exp : \mathfrak{l} \rightarrow L$  is bijective,  $\exp$  yields a diffeomorphism of an open neighborhood of 0 in  $\mathfrak{l}$  onto an open neighborhood  $U$  of the identity element  $e$  in  $L$ . The inverse diffeomorphism  $\lambda$  (logarithm) is equivariant with respect to the action of  $L$  on  $\mathfrak{l}$  via the adjoint representation  $\text{Ad}_L : L \rightarrow \text{Aut } \mathfrak{l}$  and on  $L$  by conjugation, i.e.,  $\lambda(gug^{-1}) = \text{Ad}_L g(\lambda(u))$  if  $g \in L$ ,  $u \in U$  and  $gug^{-1} \in U$ . This shows that the conjugating action of  $L$  on its underlying manifold is linearizable in a neighborhood of  $e$ .

In this paper we study what happens if  $L$  is replaced with a connected linear algebraic group  $G$  over an algebraically closed field  $k$ : what is a natural algebraic counterpart of  $\lambda$  for such  $G$  and for which  $G$  does it exist?

In what follows we assume that  $\text{char } k = 0$  (in fact in many places this assumption is either redundant or can be bypassed by modifying the relevant proof).

**1.1. The classical Cayley map.** Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . One way to look at the problem is to replace the Hausdorff topology in the Lie group setting by the étale topology, i.e., to define the algebraic counterpart of  $\lambda$  as a  $G$ -equivariant morphism  $G \rightarrow \mathfrak{g}$  étale at  $e$ . Then, at least for reductive groups, there is no existence problem: such morphisms always exist; see the Corollary to Lemma 10.3 below. Properties of some of them have been studied by KOSTANT and MICHOR in [KM]; see Example 10.4 below. Note also that a  $G$ -equivariant dominant morphism  $G \rightarrow \mathfrak{g}$  exists for every linear algebraic group  $G$ ; see Theorem 10.2 below.

In the present paper we look at the problem differently. Our point of view stems from a discovery made by CAYLEY in 1846, [Ca]; cf. [Weyl], [Pos]. It suggests that the most direct approach, i.e., replacing the Hausdorff topology by the Zariski topology, leads to something really interesting. Namely, let  $G$  be the special orthogonal group,

$$G = \mathbf{SO}_n := \{X \in \text{Mat}_{n \times n} \mid X^T X = I_n\},$$

where  $I_n$  is the identity  $n \times n$ -matrix. Then

$$\mathfrak{g} = \mathfrak{o}_n := \{Y \in \text{Mat}_{n \times n} \mid Y^T = -Y\},$$

and the adjoint representation  $\text{Ad}_G : G \rightarrow \text{Aut } \mathfrak{g}$  is given by

$$(1.1) \quad \text{Ad}_G g(Y) = gYg^{-1}, \quad g \in G, Y \in \mathfrak{g}.$$

CAYLEY discovered that there exists a birational isomorphism

$$(1.2) \quad \lambda: G \xrightarrow{\sim} \mathfrak{g}$$

equivariant with respect to the conjugating and adjoint actions of  $G$  on the underlying varieties of  $G$  and  $\mathfrak{g}$ , respectively, i.e., such that

$$(1.3) \quad \lambda(gXg^{-1}) = \text{Ad}_G g(\lambda(X))$$

if  $g$  and  $X \in G$  and both sides of (1.3) are defined. His proof is given by the explicit formula defining such  $\lambda$ :

$$(1.4) \quad \lambda: X \mapsto (I_n - X)(I_n + X)^{-1}$$

(one immediately deduces from (1.4) that  $Y \mapsto (I_n - Y)(I_n + Y)^{-1}$  is the inverse of  $\lambda$ , and from (1.1) that (1.3) holds).

**1.2. Basic definitions, main problem, and examples.** Inspired by this example, we introduce the following definition for an arbitrary connected linear algebraic group  $G$ .

**Definition 1.5.** A *Cayley map* for  $G$  is a birational isomorphism (1.2) satisfying (1.3). A group  $G$  is called a *Cayley group* if it admits a Cayley map. If  $G$  is defined over a subfield  $K$  of  $k$ , then a Cayley map defined over  $K$  is called a *Cayley  $K$ -map*. If  $G$  admits a Cayley  $K$ -map,  $G$  is called a *Cayley  $K$ -group*.

Our starting point was a question, posed in 1975 to the second-named author by LUNA, [Lun3]. Using Definition 1.5, it can be reformulated as follows:

**Question 1.6.** *For what  $n$  is the special linear group  $\mathbf{SL}_n$  a Cayley group?*

It is easy to show (see Example 1.16 below) that  $\mathbf{SL}_2$  is a Cayley group. POPOV in [Pop2] has proved that, contrary to what was expected, [Lun1, Remarque, p. 14],  $\mathbf{SL}_3$  is a Cayley group as well.

More generally, given Definition 1.5, it is natural to pose the following problem:

**Problem 1.7.** *Which connected linear algebraic groups are Cayley groups?*

Before stating our main results, we will discuss several examples. Set

$$\boldsymbol{\mu}_d := \{a \in \mathbf{G}_m \mid a^d = 1\}.$$

This is a cyclic subgroup of order  $d$  of the multiplicative group  $\mathbf{G}_m$ . Below we use the same notation  $\boldsymbol{\mu}_d$  for the central cyclic subgroup  $\{aI_n \mid a \in \boldsymbol{\mu}_d\}$  of  $\mathbf{GL}_n$ .

**Example 1.8.** If  $G_1, \dots, G_n$  are Cayley, then  $G := G_1 \times \dots \times G_n$  is Cayley (the converse is false; see Subsection 4.4). Indeed, if  $\mathfrak{g}_i$  is the Lie algebra of  $G_i$  and  $\lambda_i: G_i \xrightarrow{\sim} \mathfrak{g}_i$  a Cayley map, then  $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$  and  $\lambda_1 \times \dots \times \lambda_n: G \xrightarrow{\sim} \mathfrak{g}$  is a Cayley map.  $\square$

**Example 1.9.** Consider a finite-dimensional associative algebra  $A$  over  $k$  with identity element 1. Let  $\mathcal{L}_A$  be the Lie algebra whose underlying vector space is that of  $A$  and whose Lie bracket is given by

$$(1.10) \quad [X_1, X_2] := X_1X_2 - X_2X_1.$$

The group

$$G := A^*$$

of invertible elements of  $A$  is a connected linear algebraic group whose underlying variety is an open subset of that of  $A$ . This implies that  $\mathfrak{g}$  is naturally identified with  $\mathcal{L}_A$ , and

the adjoint action is given by formula (1.1). Hence the natural embedding  $\lambda : A^* \hookrightarrow \mathcal{L}_A$ ,  $X \mapsto X$ , is a Cayley map. Therefore  $G$  is a Cayley group.

Taking  $A = \text{Mat}_{n \times n}$ , we obtain that  $G := \mathbf{GL}_n$  is Cayley for every  $n \geq 1$ .  $\square$

**Example 1.11.** Maintain the notation of Example 1.9. For any element  $a \in A$ , denote by  $\text{tr } a$  the trace of the operator  $L_a$  of left multiplication of  $A$  by  $a$ . Since the algebra  $A$  is associative,  $a \mapsto L_a$  is a homomorphism of  $A$  to the algebra of linear operators on the underlying vector space of  $A$ . From this and (1.10), we deduce that  $k \cdot 1$  is an ideal of  $\mathcal{L}_A$ , the map

$$\tau : \mathcal{L}_A \rightarrow k \cdot 1, \quad a \mapsto \text{tr } a \cdot 1,$$

is a surjective homomorphism of Lie algebras, and

$$(1.12) \quad \mathcal{L}_A = \text{Ker } \tau \oplus k \cdot 1.$$

The subgroup  $k^* \cdot 1$  of  $A^*$  is normal; set

$$(1.13) \quad G := A^*/k^* \cdot 1.$$

As the Lie algebras of  $A^*$  and  $k^* \cdot 1$  are, respectively,  $\mathcal{L}_A$  and  $k \cdot 1$ , it follows from (1.12) that one can identify  $\mathfrak{g}$  with  $\text{Ker } \tau$ . Let  $A^* \rightarrow G$ ,  $a \mapsto [a]$ , be the natural projection. Then the formula

$$(1.14) \quad [a] \mapsto \frac{\text{tr } 1}{\text{tr } a} a - 1$$

defines a rational map  $\lambda : G \dashrightarrow \mathfrak{g} = \text{Ker } \tau$ . Since  $\text{tr } xax^{-1} = \text{tr } a$  for any  $a \in A$ ,  $x \in A^*$ , it follows from (1.14) that (1.3) holds. On the other hand, (1.14) clearly implies that

$$(1.15) \quad a \mapsto [a + 1]$$

is the inverse of  $\lambda$ . Thus  $G$  is a Cayley group.

If  $A$  is defined over a subfield  $K$  of  $k$ , then the group  $G$  and birational isomorphisms (1.14), (1.15) are defined over  $K$  as well. Hence  $G$  is a Cayley  $K$ -group.

For  $A = \text{Mat}_{n \times n}$  this shows that  $\mathbf{PGL}_n$  is a Cayley group for every  $n \geq 1$ . Note that in this case,  $\frac{\text{tr } 1}{\text{tr } a} = \frac{n}{\text{Tr } a}$ , where  $\text{Tr } a$  is the trace of matrix  $a$ . Let  $K$  be a subfield of  $k$ . Since every inner  $K$ -form  $G$  of  $\mathbf{PGL}_n$  is given by (1.13) for  $A = D \otimes_K k$ , where  $D$  is an  $n^2$ -dimensional central simple algebra over  $K$  and the  $K$ -structure of  $A$  is defined by  $D$ , cf. [Kn], all inner  $K$ -forms of  $\mathbf{PGL}_n$  are Cayley  $K$ -groups.

Setting  $A = \bigoplus_{i=1}^s \text{Mat}_{n_i \times n_i}$ , we conclude that  $\prod_{i=1}^s \mathbf{GL}_{n_i}/k^* I_{n_1+\dots+n_s}$  is a Cayley group. Here  $\prod_{i=1}^s \mathbf{GL}_{n_i}$  is block-diagonally embedded in  $\mathbf{GL}_{n_1+\dots+n_s}$ .  $\square$

**Example 1.16.** The following construction was noticed by WEIL in [Weil, p. 599]. Namely, maintain the notation of Example 1.9 (WEIL assumed that  $A$  is semisimple, but his construction, presented below, does not use this assumption). Let  $\iota$  be an involution (i.e., an involutory  $k$ -antiautomorphism) of the algebra  $A$ . Set

$$(1.17) \quad G := \{a \in A^* \mid a^\iota a = 1\}^\circ$$

(as usual,  $S^\circ$  denotes the identity component of an algebraic group  $S$ ). The Lie algebra of  $G$  is the subalgebra of odd elements of  $\mathcal{L}_A$  for  $\iota$ ,

$$\mathfrak{g} = \{a \in \mathcal{L}_A \mid a^\iota = -a\}.$$

The formula

$$(1.18) \quad a \mapsto (1 - a)(1 + a)^{-1}$$

defines an equivariant rational map  $\lambda: G \dashrightarrow \mathfrak{g}$ , and the formula

$$(1.19) \quad b \mapsto (1-b)(1+b)^{-1}$$

defines its inverse,  $\lambda^{-1}: \mathfrak{g} \dashrightarrow G$ . Thus  $\lambda$  is a Cayley map and  $G$  is a Cayley group.

If  $A$  and  $\iota$  are defined over a subfield  $K$  of  $k$ , then the group  $G$  and birational isomorphisms (1.18), (1.19) are defined over  $K$  as well. Hence  $G$  is a Cayley  $K$ -group.

For  $A = \text{Mat}_{n \times n}$  and the involution  $X \mapsto X^\top$ , this turns into the classical Cayley construction for  $G = \mathbf{SO}_n$ , proving that this group is Cayley for every  $n \geq 1$ . In particular, the following groups are Cayley:  $\mathbf{G}_m \simeq \mathbf{SO}_2$  (see Examples 1.9 and 1.20),  $\mathbf{PGL}_2 \simeq \mathbf{SL}_2/\mu_2 \simeq \mathbf{SO}_3$  (see Example 1.11),  $(\mathbf{SL}_2 \times \mathbf{SL}_2)/\mu_2 \simeq \mathbf{SO}_4$  (here  $\mathbf{SL}_2 \times \mathbf{SL}_2$  is block-diagonally embedded in  $\mathbf{SL}_4$ ),  $\mathbf{Sp}_4/\mu_2 \simeq \mathbf{SO}_5$ , and  $\mathbf{SL}_4/\mu_2 \simeq \mathbf{SO}_6$ .

For  $A = \text{Mat}_{2n \times 2n}$  and the involution  $X \mapsto J_{2n}^{-1} X^\top J_{2n}$ , where  $J_{2n} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ , we have

$$\begin{aligned} G &= \mathbf{Sp}_{2n} := \{X \in \text{Mat}_{2n \times 2n} \mid X^\top J_{2n} X = J_{2n}\}, \\ \mathfrak{g} &= \mathfrak{sp}_{2n} := \{Y \in \text{Mat}_{2n \times 2n} \mid Y^\top J_{2n} = -J_{2n} Y\}, \end{aligned}$$

so the construction shows that (1.4) is a Cayley map for  $\mathbf{Sp}_{2n}$ ; cf. [Pos, Examples 6, 7]. Thus  $\mathbf{Sp}_{2n}$  is Cayley for every  $n \geq 1$ . In particular,  $\mathbf{SL}_2 \simeq \mathbf{Spin}_3 \simeq \mathbf{Sp}_2$ ,  $\mathbf{Spin}_4 \simeq \mathbf{SL}_2 \times \mathbf{SL}_2$ , and  $\mathbf{Spin}_5 \simeq \mathbf{Sp}_4$  are Cayley. Below we shall prove that  $\mathbf{Spin}_n$  is not Cayley for  $n \geq 6$ .

Let  $K$  be a subfield of  $k$ . Since every  $K$ -form  $G$  of  $\mathbf{SO}_n$  or  $\mathbf{Sp}_{2n}$  is given by (1.17) for some algebra  $A$  and its involution  $\iota$ , both defined over  $K$ , see [Weil], [Kn], all  $K$ -forms of  $\mathbf{SO}_n$  and  $\mathbf{Sp}_{2n}$  are Cayley  $K$ -groups.  $\square$

**Example 1.20.** Every connected commutative linear algebraic group  $G$  is Cayley. In fact, in this case, condition (1.3) is vacuous, so the existence of (1.2) is equivalent to the property that the underlying variety of  $G$  is rational. CHEVALLEY in [Ch<sub>1</sub>] proved that over an algebraically closed field of characteristic zero this property holds for any connected linear algebraic group (not necessarily commutative). In particular, the algebraic torus  $\mathbf{G}_m^d$ , where

$$\mathbf{G}_m^d := \underbrace{\mathbf{G}_m \times \dots \times \mathbf{G}_m}_d \text{ if } d \geq 1, \quad \mathbf{G}_m^0 = e,$$

is a Cayley group for every  $d \geq 0$  (as  $\mathbf{G}_m = \mathbf{GL}_1$ , this also follows from Examples 1.8 and 1.9).

**Example 1.21.** Every unipotent linear algebraic group  $G$  is Cayley ( $G$  is automatically connected because  $\text{char } k = 0$ ). Indeed, we may assume without loss of generality that  $G \subset \mathbf{GL}_n$ , so that elements of  $G$  are unipotent  $n \times n$ -matrices, elements of  $\mathfrak{g}$  are nilpotent  $n \times n$ -matrices, and  $\text{Ad}_G$  is given by (1.1). So we have  $(I_n - X)^n = Y^n = 0$  for any  $X \in G$ ,  $Y \in \mathfrak{g}$ . Hence the exponential map is given by

$$\exp: \mathfrak{g} \longrightarrow G, \quad Y \mapsto \sum_{i=0}^{n-1} \frac{1}{i!} Y^i.$$

Therefore  $\exp$  is a  $G$ -equivariant morphism of algebraic varieties. Moreover, it is an isomorphism since the formula

$$(1.22) \quad \lambda := \ln: G \longrightarrow \mathfrak{g}, \quad X \mapsto -\sum_{i=1}^{n-1} \frac{1}{i} (I_n - X)^i,$$

defines its inverse.

More generally, by the Corollary of Proposition 4.2 below, every connected solvable linear algebraic group is Cayley.  $\square$

**1.3. Notational conventions.** In order to formulate our main results, we need some notation and definitions.

For any algebraic torus  $T$ , we denote by  $\widehat{T}$  its character group,

$$\widehat{T} := \mathrm{Hom}_{\mathrm{alg}}(T, \mathbf{G}_m),$$

written additively. It is a lattice (i.e., a free abelian group of finite rank).

Let  $T$  be a maximal torus of  $G$  and let

$$(1.23) \quad \begin{cases} N = N_{G,T} := \{g \in G \mid gTg^{-1} = T\}, \\ C = C_{G,T} := \{g \in G \mid gtg^{-1} = t \text{ for all } t \in T\}, \\ W = W_G = W_{G,T} := N/C \end{cases}$$

be, respectively, its normalizer, centralizer (which is the Cartan subgroup of  $G$ ), and the Weyl group. The group  $C$  is the identity component of  $N$ , and if  $G$  is reductive, then  $C = T$ ; see [Bor, 12.1, 13.17]. The finite group  $W$  naturally acts by automorphisms of  $\widehat{T}$ . Since all maximal tori in  $G$  are conjugate,  $W$  and the  $W$ -lattice  $\widehat{T}$  do not depend, up to isomorphism, on the choice of  $T$ .

**Definition 1.24.** The  $W$ -lattice  $\widehat{T}$  is called the *character lattice* of  $G$  and is denoted by  $\mathcal{X}_G$ .

**Remark 1.25.** The reader should be careful about this terminology: the elements of the character lattice of  $G$  are the characters of  $T$ , not of  $G$ .

**Definition 1.26.** A group  $G$  is called *stably Cayley* if  $G \times \mathbf{G}_m^d$  is Cayley for some  $d \geq 0$ . If  $G$  is defined over a subfield  $K$  of  $k$  and  $G \times \mathbf{G}_m^d$  is a Cayley  $K$ -group for some  $d \geq 0$ , then  $G$  is called a *stably Cayley  $K$ -group*.

It is easy to see that  $G$  is stably Cayley if and only if  $G \times A$  is Cayley for some connected abelian algebraic group  $A$ . In what follows we will sometimes use Definition 1.26 in this form.

**1.4. Main results.** Now we are ready to state our main results. In what follows we shall denote the generic torus of  $G$  by  $\mathbf{T}_G$ . (For the definition of  $\mathbf{T}_G$ , see [Vos], [CK] or Definition 3.7 in Subsection 3.2.)

**Theorem 1.27.** *Let  $G$  be a connected reductive algebraic group. Then the following implications hold:*

$$\mathcal{X}_G \text{ is sign-permutation} \xrightarrow{(a)} G \text{ is Cayley} \xrightarrow{(b)} \mathbf{T}_G \text{ is rational} \xrightarrow{(c)} \mathbf{T}_G \text{ is stably rational} \xleftrightarrow{(d)} \mathcal{X}_G \text{ is quasi-permutation} \xleftrightarrow{(e)} G \text{ is stably Cayley}.$$

*Moreover, the implications (a) and (b) cannot be reversed. In particular, not every stably Cayley group is Cayley.*

For the definitions of sign-permutation and quasi-permutation lattices, see Subsection 2.2. Note that it is a long-standing open question whether or not every stably rational torus is rational; see [Vos, p. 52]. In particular, we do not know whether or not implication (c) can be reversed. We also remark that (d) is well known; see, e.g., [Vos, Theorem 4.7.2].

A proof of Theorem 1.27 will be given in Subsection 3.3. In Section 4 we will partially reduce Problem 1.7 to the case where  $G$  is a simple group.

We will then use Theorem 1.27 to translate results about stable rationality of generic tori into statements about the existence (and, more often, the non-existence) of Cayley maps for various simple algebraic groups (i.e., groups having no proper connected normal

subgroups). In particular, LEMIRE and LORENZ in [LL] and CORTELLA and KUNYAVSKIĪ in [CK] have recently proved that the character lattice of  $\mathbf{SL}_n$  is quasi-permutation if and only if  $n \leq 3$ . (This result had been previously conjectured and proved for prime  $n$  by LE BRUYN in [LB<sub>1</sub>], [LB<sub>2</sub>].) Theorem 1.27 now tells us that  $\mathbf{SL}_n$  is not stably Cayley (and thus not Cayley) for any  $n \geq 4$ . On the other hand, Example 1.16 shows that  $\mathbf{SL}_2$  is Cayley, and POPOV in [Pop<sub>2</sub>] has proved that  $\mathbf{SL}_3$  is Cayley as well (an outline of the arguments from [Pop<sub>2</sub>] is reproduced in the Appendix; see also an explicit construction in Section 9). This settles Luna’s original Question 1.6 about  $\mathbf{SL}_n$ .

In a similar manner, we proceed to classify the connected simple groups  $G$  with quasi-permutation character lattices  $\mathcal{X}_G$ . For simply connected and adjoint groups this was done by CORTELLA and KUNYAVSKIĪ in [CK]. In Sections 6 and 8 we extend their results to all other connected simple groups. Combining this classification with Theorem 1.27, we obtain the following result.

**Theorem 1.28.** *Let  $G$  be a connected simple algebraic group. Then the following conditions are equivalent:*

- (a)  $G$  is stably Cayley;
- (b)  $G$  is one of the following groups:

$$(1.29) \quad \mathbf{SL}_n \text{ for } n \leq 3, \mathbf{SO}_n \text{ for } n \neq 2, 4, \mathbf{Sp}_{2n}, \mathbf{PGL}_n, \mathbf{G}_2.$$

**Remark 1.30.** The groups  $\mathbf{SO}_2$  and  $\mathbf{SO}_4$  are stably Cayley (and even Cayley; see Example 1.16) but they are excluded because they are not simple. Note also that, due to exceptional isomorphisms, some groups are listed twice in (1.29). (For example,  $\mathbf{Sp}_2 \simeq \mathbf{SL}_2$ .)

It is now natural to ask which of the stably Cayley simple groups listed in Theorem 1.28(b) are in fact Cayley. Here is the answer:

**Theorem 1.31.** *Let  $G$  be a connected simple algebraic group.*

- (a) *The following conditions are equivalent:*
  - (i)  $G$  is Cayley;
  - (ii)  $G$  is one of the following groups:

$$(1.32) \quad \mathbf{SL}_n \text{ for } n \leq 3, \mathbf{SO}_n \text{ for } n \neq 2, 4, \mathbf{Sp}_{2n}, \mathbf{PGL}_n.$$

- (b) *The group  $\mathbf{G}_2$  is not Cayley but the group  $\mathbf{G}_2 \times \mathbf{G}_m^2$  is Cayley.*

The first assertion of part (b) is based on the recent work of ISKOVSKIĪH [Isk<sub>4</sub>]. The groups  $\mathbf{SO}_n$ ,  $\mathbf{Sp}_{2n}$ , and  $\mathbf{PGL}_n$  were shown to be Cayley in Examples 1.16 and 1.11. The groups  $\mathbf{SL}_3$  and  $\mathbf{G}_2$  will be discussed in Section 9.

**Remark 1.33.** Question 1.6 was inspired by LUNA’s interest in the existence (for reductive  $G$ ) of “algebraic linearization” of the conjugating action in a Zariski neighborhood of the identity element  $e \in G$ , i.e., in the existence of  $G$ -isomorphic neighborhoods of  $e$  and  $0$  in  $G$  and  $\mathfrak{g}$ , respectively; cf. [Lun<sub>1</sub>]. In our terminology this is equivalent to the existence of a Cayley map (1.2) such that  $\lambda$  and  $\lambda^{-1}$  are defined at  $e$  and  $0$  respectively, and  $\lambda(e) = 0$ . Not all Cayley maps have this property. However, note that our proof of Theorem 1.31 (in combination with [Lun<sub>1</sub>, p. 13, Proposition]) shows that each of the simple groups listed in (1.32) admits a Cayley map with this property (and so does any direct product of these groups); see Examples 1.8, 1.9, 1.11, 1.16, 1.20, and 1.21, Subsections 9.1 and 9.2, and the Appendix.

Let  $K$  be a subfield of  $k$ . It follows from Theorems 1.28 and 1.31 and Examples 1.11 and 1.16 that classifying simple Cayley (respectively, stably Cayley)  $K$ -groups is reduced to classifying outer  $K$ -forms of  $\mathbf{PGL}_n$  for  $n \geq 3$  and  $K$ -forms of  $\mathbf{SL}_3$  (respectively, outer  $K$ -forms of  $\mathbf{PGL}_n$  for  $n \geq 3$  and  $K$ -forms of  $\mathbf{SL}_3$  and  $\mathbf{G}_2$ ) that are Cayley (respectively, stably Cayley)  $K$ -groups. Note that not all of these  $K$ -forms are Cayley (respectively, stably Cayley)  $K$ -groups. Indeed, Definitions 1.5 and 1.26 imply the following special property of Cayley (respectively, stably Cayley)  $K$ -groups: their underlying varieties are rational (respectively, stably rational) over  $K$ . For some of the specified  $K$ -forms this property does not hold:

**Example 1.34.** BERHUY, MONSURRÒ, and TIGNOL in [BMT] have shown that for every  $n \equiv 0 \pmod{4}$ , the group  $\mathbf{PGL}_n$  has a  $K$ -form  $G$  of outer type whose underlying variety is not stably rational over  $K$ . Hence  $G$  is not a stably Cayley  $K$ -group.  $\square$

**Remark 1.35.** The underlying varieties of all outer  $K$ -forms of  $\mathbf{PGL}_n$  with odd  $n$  are rational over  $K$ ; see [VK]. Note also that the underlying variety of any  $K$ -form of a linear algebraic group of rank at most 2 is rational over  $K$ ; e.g., see [Me, p. 189] and [Vos, 4.1, 4.9].

**1.5. Application to Cremona groups.** The Cremona group  $\mathrm{Cr}_d$ , i.e., the group of birational automorphisms of the affine space  $\mathbb{A}^d$ , is a classical object in algebraic geometry; see [Isk<sub>2</sub>] and the references therein. Classifying the subgroups of  $\mathrm{Cr}_d$  up to conjugacy is an important research direction originating in the works of BERTINI, ENRIQUES, FANO, and WIMAN. Most of the currently known results on Cremona groups relate to  $\mathrm{Cr}_2$  and  $\mathrm{Cr}_3$  (the case  $d = 1$  is trivial because  $\mathrm{Cr}_1 = \mathbf{PGL}_2$ ). For  $d \geq 4$  the groups  $\mathrm{Cr}_d$  are poorly understood, and any results that shed light on their structure are prized by the experts.

Our results provide some information about subgroups of  $\mathrm{Cr}_d$  by means of the following simple construction. Consider an action of an algebraic group  $G$  on a rational variety  $X$  of dimension  $d$ . Let  $G_0$  be the kernel of this action. Any birational isomorphism between  $X$  and  $\mathbb{A}^d$  gives rise to an embedding  $\iota_X: G/G_0 \hookrightarrow \mathrm{Cr}_d$ . A different birational isomorphism between  $X$  and  $\mathbb{A}^d$  gives rise to a conjugate embedding, so  $\iota_X$  is uniquely determined by  $X$  (as a  $G$ -variety) up to conjugacy in  $\mathrm{Cr}_d$ . If  $Y$  is another rational variety on which  $G$  acts, then the embeddings  $\iota_X$  and  $\iota_Y$  are conjugate if and only if  $X$  and  $Y$  are birationally isomorphic as  $G$ -varieties.

Now consider a special case of this construction, where  $G$  is a connected linear algebraic group,  $X$  is the underlying variety of  $G$  (with the conjugating  $G$ -action),  $Y = \mathfrak{g}$  (with the adjoint  $G$ -action), and the kernel  $G_0$  (for both actions) is the center of  $G$ ; see [Bor, 3.15]. Definition 1.5 can now be rephrased as follows: a connected algebraic group  $G$  is Cayley if and only if the embeddings  $\iota_G$  and  $\iota_{\mathfrak{g}}: G/G_0 = \mathrm{Ad}_G G \hookrightarrow \mathrm{Cr}_{\dim G}$  are conjugate in  $\mathrm{Cr}_{\dim G}$ . In this paper we show that many connected algebraic groups are not Cayley; each non-Cayley group  $G$  gives rise to a pair of non-conjugate embeddings of the form  $\iota_G, \iota_{\mathfrak{g}}: \mathrm{Ad}_G G \hookrightarrow \mathrm{Cr}_{\dim G}$ .

Definition 1.26 can be interpreted in a similar manner. For every  $d \geq 1$  consider the embedding  $\mathrm{Cr}_d \hookrightarrow \mathrm{Cr}_{d+1}$  given by writing  $\mathbb{A}^{d+1}$  as  $\mathbb{A}^d \times \mathbb{A}^1$  and sending an element  $g \in \mathrm{Cr}_d$  to  $g \times \mathrm{id}_{\mathbb{A}^1} \in \mathrm{Cr}_{d+1}$ . Denote the direct limit for the tower of groups  $\mathrm{Cr}_1 \hookrightarrow \mathrm{Cr}_2 \hookrightarrow \dots$  obtained in this way by  $\mathrm{Cr}_\infty$ . Suppose  $G$  is a group acting on rational varieties  $X$  and  $Y$  (possibly of different dimensions) with the same kernel  $G_0$ . Then it is easy to see that the embeddings  $\iota_X: G/G_0 \hookrightarrow \mathrm{Cr}_{\dim X}$  and  $\iota_Y: G/G_0 \hookrightarrow \mathrm{Cr}_{\dim Y}$  are conjugate in  $\mathrm{Cr}_\infty$  (or equivalently, in  $\mathrm{Cr}_m$  for some  $m \geq \max\{\dim X, \dim Y\}$ ) if and only if  $X$  and  $Y$  are stably isomorphic as  $G$ -varieties.

If  $V_1$  and  $V_2$  are vector spaces with faithful linear  $G$ -actions, then  $\iota_{V_1}$  and  $\iota_{V_2}$  are conjugate in  $\text{Cr}_\infty$  by the “no-name lemma”; cf. Subsection 2.4. We call an embedding  $G \hookrightarrow \text{Cr}_d$  *stably linearizable* if it is conjugate, in  $\text{Cr}_\infty$ , to  $\iota_V$  for some faithful linear  $G$ -action on a vector space  $V$ . Definition 1.26 and the “no-name lemma” now tell us that the following conditions are equivalent: (a)  $G$  is stably Cayley, (b) the embeddings  $\iota_G$  and  $\iota_{\mathfrak{g}}: \text{Ad}_G G \hookrightarrow \text{Cr}_{\dim G}$  are conjugate in  $\text{Cr}_\infty$ , and (c)  $\iota_G$  is stably linearizable. Once again, the results of this paper (and, in particular, Theorem 1.28) can be used to produce many examples of pairs of embeddings of the form  $\text{Ad}_G G \hookrightarrow \text{Cr}_{\dim G}$  that are not conjugate in  $\text{Cr}_\infty$ .

Now suppose that  $\Gamma$  is a finite group and  $L$  and  $M$  are faithful  $\Gamma$ -lattices; see Subsection 2.2. Then  $\Gamma$  acts on their dual tori, which we will denote by  $X$  and  $Y$ . It now follows from Lemma 2.5 that the embeddings  $\iota_X: \Gamma \hookrightarrow \text{Cr}_{\text{rank } L}$  and  $\iota_Y: \Gamma \hookrightarrow \text{Cr}_{\text{rank } M}$  are conjugate in  $\text{Cr}_\infty$  if and only if  $L$  and  $M$  are equivalent in the sense of Definition 2.2. Taking  $M$  to be a faithful permutation lattice, we conclude that the embedding  $\iota_X: \Gamma \hookrightarrow \text{Cr}_{\text{rank } X}$  is stably linearizable if and only if  $L$  is quasi-permutation (cf. Definition 2.4 and the Corollary to Lemma 2.5).

In the special case where  $L = \mathcal{X}_G$  is the character lattice of the algebraic group  $G$ ,  $\Gamma = W_G$  is the Weyl group, and  $X = T$  is a maximal torus with Lie algebra  $\mathfrak{t}$ , we see that the following conditions are equivalent: (a)  $G$  is stably Cayley, (b)  $\mathcal{X}_G$  is quasi-permutation, (c) the embeddings  $\iota_{\mathfrak{t}}$  and  $\iota_T: W \hookrightarrow \text{Cr}_{\dim T}$  are conjugate in  $\text{Cr}_\infty$ , and (d)  $\iota_T$  is stably linearizable. (Note that (a) and (b) are equivalent by Theorem 1.27, and (c) and (d) are equivalent because the  $W$ -action on  $\mathfrak{t}$  is linear.) Consequently, every reductive non-Cayley group  $G$  gives rise to a pair of embeddings  $\iota_T, \iota_{\mathfrak{t}}: W \hookrightarrow \text{Cr}_{\text{rank } G}$  which are not conjugate in  $\text{Cr}_\infty$ .

**Example 1.36.** Let  $G$  be a simple group of type  $A_{n-1}$  which is not stably Cayley, i.e.,  $G = \mathbf{SL}_n/\mu_d$ , where  $d|n$ ,  $d < n$ ,  $n \geq 4$ , and  $(n, d) \neq (4, 2)$ . Then the embeddings  $\iota_T$  and  $\iota_{\mathfrak{t}}: S_n \hookrightarrow \text{Cr}_{n-1}$  are not conjugate in  $\text{Cr}_\infty$ .

Assume further that  $n \neq 6$ . Then by Hölder’s theorem (see [Höl]),  $S_n$  has no outer automorphisms. Thus the images  $\iota_T(S_n)$  and  $\iota_{\mathfrak{t}}(S_n)$  are isomorphic finite subgroups of  $\text{Cr}_{n-1}$  which are not conjugate in  $\text{Cr}_\infty$ .  $\square$

## 2. Preliminaries

In this section we collect certain preliminary facts for subsequent use. Some of them are known and some are new. Throughout this section  $\Gamma$  will denote a group; starting from Subsection 2.2, it is assumed to be finite.

**2.1.  $\Gamma$ -fields and  $\Gamma$ -varieties.** In what follows we will use the following terminology. A  $\Gamma$ -field is a field  $K$  together with an action of  $\Gamma$  by automorphisms of  $K$ . Let  $K_1$  and  $K_2$  be  $\Gamma$ -fields containing a common  $\Gamma$ -subfield  $K_0$ . We say that  $K_1$  and  $K_2$  are *isomorphic as  $\Gamma$ -fields* (or  *$\Gamma$ -isomorphic*) *over  $K_0$*  if there is a  $\Gamma$ -equivariant field isomorphism  $K_1 \rightarrow K_2$  which is the identity on  $K_0$ . We say that  $K_1$  and  $K_2$  are *stably isomorphic as  $\Gamma$ -fields* (or *stably  $\Gamma$ -isomorphic*) *over  $K_0$*  if, for suitable  $n$  and  $m$ ,  $K_1(x_1, \dots, x_n)$  and  $K_2(y_1, \dots, y_m)$  are isomorphic as  $\Gamma$ -fields over  $K_0$ . Here,  $x_1, \dots, x_n$  and  $y_1, \dots, y_m$  are algebraically independent variables over  $K_1$  and  $K_2$ , respectively; these variables are assumed to be fixed by the  $\Gamma$ -action.

If  $\Gamma$  is an algebraic group, a  $\Gamma$ -variety is an algebraic variety  $X$  endowed with an algebraic (morphic) action of  $\Gamma$ . A  $\Gamma$ -equivariant morphism (respectively, rational map) of  $\Gamma$ -varieties is a  $\Gamma$ -morphism (respectively, *rational  $\Gamma$ -map*). If  $X_1$  and  $X_2$  are irreducible  $\Gamma$ -varieties,

then  $k(X_1)$  and  $k(X_2)$  are  $\Gamma$ -fields with respect to the natural actions of  $\Gamma$ . These fields are stably  $\Gamma$ -isomorphic over  $k$  if and only if there is a birational  $\Gamma$ -isomorphism  $X_1 \times \mathbb{A}^r \dashrightarrow X_2 \times \mathbb{A}^s$  for some  $r$  and  $s$ , where  $\Gamma$  acts on  $X_1 \times \mathbb{A}^r$  and  $X_2 \times \mathbb{A}^s$  via the first factors. In this case,  $X_1$  and  $X_2$  are called *stably birationally  $\Gamma$ -isomorphic*.

**2.2.  $\Gamma$ -lattices.** From now on we assume that  $\Gamma$  is a finite group.

A *lattice*  $L$  of rank  $r$  is a free abelian group of rank  $r$ . A  $\Gamma$ -*lattice* is a lattice equipped with an action of  $\Gamma$  by automorphisms. It is called *faithful* (respectively, *trivial*) if the homomorphism  $\Gamma \rightarrow \text{Aut}_{\mathbb{Z}} L$  defining the action is injective (respectively, trivial). If  $H$  is a subgroup of  $\Gamma$ , then  $L$  considered as an  $H$ -lattice is denoted by  $L|_H$ .

Given a group  $H$  and a ring  $R$ , we denote by  $R[H]$  the group ring of  $H$  over  $R$ . If  $K$  is a field and  $L$  is a  $\Gamma$ -lattice, we denote by  $K(L)$  the fraction field of  $K[L]$ ; both  $K[L]$  and  $K(L)$  inherit a  $\Gamma$ -action from  $L$ . We usually think of these objects multiplicatively, i.e., we consider the set of symbols  $\{x^a\}_{a \in L}$  as a basis of the  $K$ -vector space  $K[L]$ , and the multiplication being defined by  $x^a x^b = x^{a+b}$ . So  $\sigma \cdot x^a = x^{\sigma \cdot a}$  for any  $\sigma \in \Gamma$ . If  $a_1, \dots, a_r$  is a basis of  $L$  and  $x_i := x^{a_i}$ , then  $K[L] = K[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$  and  $K(L) = K(x_1, \dots, x_r)$ . Note that any group isomorphism  $L \rightarrow \widehat{\mathbf{G}_m^r}$  induces the  $K$ -isomorphisms of algebras  $K[L] \rightarrow K[\mathbf{G}_m^r]$  and fields  $K(L) \rightarrow K(\mathbf{G}_m^r)$ , and therefore it induces a  $K$ -defined algebraic action of  $\Gamma$  on the torus  $\mathbf{G}_m^r$  by its automorphisms. Any such action is obtained in this way.

An important example is  $L = \mathcal{X}_G$ , the character lattice of a connected algebraic group  $G$ , and  $\Gamma = W$ , the Weyl group of  $G$ . In this case,  $k(\mathcal{X}_G)$  is the field of rational functions on a maximal torus of  $G$ .

**Definition 2.1.** A  $\Gamma$ -lattice  $L$  is called *permutation* (respectively, *sign-permutation*) if it has a basis  $\varepsilon_1, \dots, \varepsilon_r$  such that the set  $\{\varepsilon_1, \dots, \varepsilon_r\}$  (respectively,  $\{\varepsilon_1, -\varepsilon_1, \dots, \varepsilon_r, -\varepsilon_r\}$ ) is  $\Gamma$ -stable.

If  $X$  is a finite set endowed with an action of  $\Gamma$ , we denote by  $\mathbb{Z}[X]$  the free abelian group generated by  $X$  and endowed with the natural action of  $\Gamma$ . Permutation lattices may be, alternatively, defined as those of the form  $\mathbb{Z}[X]$ . Since  $X$  is the union of  $\Gamma$ -orbits, any permutation lattice is isomorphic to some  $\bigoplus_{i=1}^s \mathbb{Z}[\Gamma/\Gamma_i]$ , where each  $\Gamma_i$  is a subgroup of  $\Gamma$ .

**Definition 2.2.** ([C-TS<sub>1</sub>]). Two  $\Gamma$ -lattices  $M$  and  $N$  are called *equivalent*, written  $M \sim N$ , if they become  $\Gamma$ -isomorphic after extending by permutation lattices, i.e., if there are exact sequences of  $\Gamma$ -lattices

$$(2.3) \quad 0 \longrightarrow M \longrightarrow E \longrightarrow P \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N \longrightarrow E \longrightarrow Q \longrightarrow 0$$

where  $P$  and  $Q$  are permutation lattices.

For a direct proof that this does indeed define an equivalence relation and for further background see [C-TS<sub>1</sub>, Lemma 8] or [Sw].

**Definition 2.4.** A  $\Gamma$ -lattice  $L$  is called *quasi-permutation* if  $L \sim 0$  under this equivalence relation, i.e.,  $L$  becomes permutation after extending by a permutation lattice. In other words,  $L$  is quasi-permutation if and only if there is an exact sequence of  $\Gamma$ -lattices

$$0 \longrightarrow L \longrightarrow P \longrightarrow Q \longrightarrow 0,$$

where  $P$  and  $Q$  are permutation lattices.

**Lemma 2.5.** *Let  $M$  and  $N$  be faithful  $\Gamma$ -lattices and let  $K$  be a field. Then the following properties are equivalent:*

- (i)  $K(M)$  and  $K(N)$  are stably isomorphic as  $\Gamma$ -fields over  $K$ ;
- (ii)  $M \sim N$ .

*Proof.* See [LL, Proposition 1.4]; this assertion is also implicit in [Sw], [C–TS<sub>1</sub>] and [Vos, 4.7].  $\square$

Lemma 2.5 and Definition 2.4 immediately imply the following.

**Corollary.** *Let  $L$  be a faithful  $\Gamma$ -lattice and let  $K$  be a field. Then the following properties are equivalent:*

- (i)  $K(L)$  is stably isomorphic to  $K(P)$  (as a  $\Gamma$ -field over  $K$ ) for some faithful permutation  $\Gamma$ -lattice  $P$ ;
- (ii)  $L$  is quasi-permutation.

**2.3. Stable equivalence and flasque resolutions.** In addition to the equivalence relation  $\sim$  on  $\Gamma$ -lattices, we will also consider a stronger equivalence relation  $\approx$  of stable equivalence. Two  $\Gamma$ -lattices  $L_1$  and  $L_2$  are called *stably equivalent* if  $L_1 \oplus P_1 \simeq L_2 \oplus P_2$  for suitable permutation  $\Gamma$ -lattices  $P_1$  and  $P_2$ .

A  $\Gamma$ -lattice  $L$  is called *flasque* if  $H^{-1}(S, L) = 0$  for all subgroups  $S$  of  $\Gamma$ . Every  $\Gamma$ -lattice  $L$  has a *flasque resolution*

$$(2.6) \quad 0 \longrightarrow L \longrightarrow P \longrightarrow Q \longrightarrow 0$$

with  $P$  a permutation  $\Gamma$ -lattice and  $Q$  a flasque  $\Gamma$ -lattice. Moreover,  $Q$  is determined by  $L$  up to stable equivalence: If  $0 \rightarrow L \rightarrow P' \rightarrow Q' \rightarrow 0$  is another flasque resolution of  $L$ , then  $Q \approx Q'$ . Following [C–TS<sub>1</sub>], we will denote the stable equivalence class of  $Q$  in the flasque resolution (2.6) by

$$\rho(L).$$

Note that by [C–TS<sub>1</sub>, Lemme 8], for  $\Gamma$ -lattices  $M, N$ ,

$$(2.7) \quad M \sim N \iff \rho(M) = \rho(N).$$

Dually, every  $\Gamma$ -lattice  $L$  has a *coflasque resolution*

$$(2.8) \quad 0 \longrightarrow R \longrightarrow P \longrightarrow L \longrightarrow 0$$

with  $P$  a permutation  $\Gamma$ -lattice and  $R$  a *coflasque*  $\Gamma$ -lattice; that is,  $H^1(S, R) = 0$  holds for all subgroups  $S$  of  $\Gamma$ . Similarly,  $R$  is determined by  $L$  up to stable equivalence. Note that the dual of a flasque resolution for  $L$  is a coflasque resolution for  $L^*$  since the finite abelian group  $H^1(S, L)$  is dual to  $H^{-1}(S, L^*)$ . For details, see [C–TS<sub>1</sub>, Lemme 5]. Note that since  $H^{\pm 1}$  is trivial for permutation modules,  $H^{\pm 1}(\Gamma, L)$  depends only on the stable equivalence class  $[L]$  of  $L$  and therefore is denoted by  $H^{\pm 1}(\Gamma, [L])$ .

Following COLLIOT–THÉLÈNE and SANSUC, [C–TS<sub>1</sub>, C–TS<sub>2</sub>], we define

$$\mathrm{III}^i(\Gamma, M) = \bigcap_{a \in \Gamma} \mathrm{Ker}(\mathrm{Res}_{\langle a \rangle}^{\Gamma}: H^i(\Gamma, M) \longrightarrow H^i(\langle a \rangle, M))$$

for any  $\mathbb{Z}[\Gamma]$ -module  $M$ . Of particular interest to us will be the case where  $M$  is a  $\Gamma$ -lattice  $L$  and  $i = 1$  or  $2$ .

The following lemma is extracted from [C–TS<sub>2</sub>, pp. 199–202]. For a proof, see also [LL, Lemma 4.2].

**Lemma 2.9.** (a) For any exact sequence of  $\mathbb{Z}[\Gamma]$ -modules

$$0 \longrightarrow M \longrightarrow P \longrightarrow N \longrightarrow 0$$

with  $P$  a permutation projective  $\Gamma$ -lattice,  $\mathbb{H}^2(\Gamma, M) \simeq \mathbb{H}^1(\Gamma, N)$ .

(b)  $\mathbb{H}^1(\Gamma, \rho(L)) \simeq \mathbb{H}^2(\Gamma, L)$  for any  $\Gamma$ -lattice  $L$ .

(c) If  $L$  is equivalent to a direct summand of a quasi-permutation  $\Gamma$ -lattice, then  $\mathbb{H}^2(S, L) = 0$  holds for all subgroups  $S$  of  $\Gamma$ .

In particular,  $\mathbb{H}^2(\Gamma, \cdot)$  is constant on  $\sim$ -classes.

The following technical proposition will help us show that certain  $\Gamma$ -lattices are equivalent.

**Proposition 2.10.** Let  $X$  and  $Y$  be  $\Gamma$ -lattices satisfying the exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0$$

where  $\Gamma$  acts trivially on  $\mathbb{Z}/d\mathbb{Z}$ .

(a) If  $(d, |\Gamma|) = 1$ , then  $X \oplus \mathbb{Z} \simeq Y \oplus \mathbb{Z}$  so that  $X \approx Y$  and  $X^* \approx Y^*$ .

(b) If the fixed point sequence

$$0 \longrightarrow X^S \longrightarrow Y^S \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0$$

is exact for all subgroups  $S$  of  $\Gamma$ , then  $X^* \sim Y^*$  as  $\Gamma$ -lattices.

*Proof.* (a) This follows directly from Roiter's form of Schanuel's Lemma [CR, 31.8] applied to the sequence of the proposition and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times d} \mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0.$$

(b) We claim that any coflasque resolution

$$0 \longrightarrow C_1 \longrightarrow P \longrightarrow X \longrightarrow 0$$

for  $X$  can be extended to a coflasque resolution

$$0 \longrightarrow C_2 \longrightarrow P \oplus Q \longrightarrow Y \longrightarrow 0$$

for  $Y$  so that the following diagram commutes and has exact rows and columns:

$$(2.11) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_1 & \longrightarrow & P & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_2 & \longrightarrow & P \oplus Q & \longrightarrow & Y \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U & \longrightarrow & Q & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Here  $C_1, C_2$  are  $\Gamma$ -coflasque and  $P, Q$  are  $\Gamma$ -permutation. Indeed, as is described in [C-TS<sub>1</sub>, Lemme 3], given a surjective homomorphism  $\pi_0$  from a permutation  $\Gamma$ -lattice  $P_0$  to a given  $\Gamma$ -lattice  $X$ , we form a coflasque resolution of  $X$  by defining a new permutation  $\Gamma$ -lattice  $P$  containing  $P_0$  as a  $\Gamma$ -sublattice and a new surjective homomorphism  $\pi : P \rightarrow X$  which extends  $\pi_0$  and such that  $\text{Ker } \pi$  is coflasque. Explicitly, take  $P = P_0 \oplus \bigoplus_S \mathbb{Z}[\Gamma/S] \otimes X^S$  where the sum is taken over all subgroups  $S$  of  $\Gamma$  for which  $\pi : P^S \rightarrow X^S$  is not a surjection and such that  $\Gamma$  acts on  $\mathbb{Z}[\Gamma/S] \otimes X^S$  via the first factor. Then we take  $\pi : P \rightarrow X$  to be

the unique  $\Gamma$ -map such that  $\pi|_{P_0} = \pi_0$  and such that for each  $S$ ,  $\pi(gS \otimes x) = x$  for  $g \in \Gamma$  and  $x \in X^S$ . Then  $\pi : P \rightarrow X$  is a surjective  $\Gamma$ -map which maps  $P^S$  surjectively onto  $X^S$  for all subgroups  $S$  of  $\Gamma$  so that  $H^1(S, \text{Ker } \pi) = 0$  as required. To obtain a compatible coflasque resolution for  $Y$ , extend the surjection from the permutation lattice  $P$  onto  $X$  to a surjection from the permutation lattice  $P \oplus Q_0$  onto  $Y$  and then adjust this surjection  $P \oplus Q_0 \rightarrow Y$  to one with a coflasque kernel  $P \oplus Q \rightarrow Y$  as above. Then the top two rows are exact and commutative. The bottom row is obtained via the Snake Lemma.

Let  $S$  be a subgroup of  $\Gamma$ . Taking  $S$ -fixed points in (2.11), we obtain

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & C_1^S & \longrightarrow & P^S & \longrightarrow & X^S \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & C_2^S & \rightarrow & P^S \oplus Q^S & \rightarrow & Y^S \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & U^S & \longrightarrow & Q^S & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Since  $C_1, C_2$  are coflasque, we find that the first two rows and columns are exact. By hypothesis, the third column is exact. Then a diagram chase shows that the bottom row is exact. But then this means that  $U$  is coflasque since

$$0 \rightarrow U^S \rightarrow Q^S \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow H^1(S, U) \rightarrow H^1(S, Q) = 0$$

is exact. Applying [LL, Lemma 1.1] to

$$0 \rightarrow U \rightarrow Q \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow 0,$$

we find that  $U$  is also quasi-permutation as it satisfies

$$0 \rightarrow U \rightarrow Q \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0.$$

So as  $U$  is coflasque, this sequence splits and  $U$  is in fact stably permutation with  $U \oplus \mathbb{Z} \simeq Q \oplus \mathbb{Z}$ . Combining this isomorphism with the sequence from the first column of the first commutative diagram gives us an exact sequence

$$0 \rightarrow C_1 \rightarrow C_2 \oplus \mathbb{Z} \rightarrow Q \oplus \mathbb{Z} \rightarrow 0.$$

Since  $C_1$  is coflasque and  $Q \oplus \mathbb{Z}$  is permutation, this new sequence splits so that  $C_1 \oplus Q \oplus \mathbb{Z} \simeq C_2 \oplus \mathbb{Z}$ . Since

$$0 \rightarrow X^* \rightarrow P \rightarrow C_1^* \rightarrow 0, \quad 0 \rightarrow Y^* \rightarrow P \oplus Q \rightarrow C_2^* \rightarrow 0$$

are flasque resolutions of  $X^*$  and  $Y^*$ , this implies  $\rho(X^*) = \rho(Y^*)$  (i.e., that the corresponding flasque lattices are stably equivalent). By [C-TS<sub>1</sub>, Lemme 8], we conclude that  $X^* \sim Y^*$ .  $\square$

**2.4. Speiser's Lemma.** Let  $\pi : Y \rightarrow X$  be an algebraic vector bundle. We call it an algebraic *vector  $\Gamma$ -bundle* if  $\Gamma$  acts on  $X$  and  $Y$ , the morphism  $\pi$  is  $\Gamma$ -equivariant, and  $g : \pi^{-1}(x) \rightarrow \pi^{-1}(g(x))$  is a linear map for every  $x \in X$  and  $g \in \Gamma$ .

The first of the following related rationality results is an immediate consequence of the classical Speiser's Lemma; the others follow from the first. In a broader context, when  $\Gamma$  is any algebraic group, results of this type appear in the literature under the names of "no-name method" ([Do]) and "no-name lemma" (see [C-T]).

**Lemma 2.12.** (a) *Suppose  $E$  is a  $\Gamma$ -field and  $K$  is a  $\Gamma$ -subfield of  $E$  such that  $\Gamma$  acts on  $K$  faithfully,  $E = K(x_1, \dots, x_m)$ , and  $Kx_1 + \dots + Kx_m$  is  $\Gamma$ -stable. Then  $E = K(t_1, \dots, t_m)$ , where  $t_1, \dots, t_m$  are  $\Gamma$ -invariant elements of  $Kx_1 + \dots + Kx_m$ .*

(b) *Let  $\pi: Y \rightarrow X$  be an algebraic vector  $\Gamma$ -bundle. Suppose that  $X$  is irreducible and the action of  $\Gamma$  on  $X$  is faithful. Then  $\pi$  is birationally  $\Gamma$ -trivial; i.e., there exists a birational  $\Gamma$ -isomorphism  $\varphi: Y \xrightarrow{\sim} X \times k^m$ , where  $\Gamma$  acts on  $X \times k^m$  via the first factor, such that the diagram*

$$\begin{array}{ccc} Y & \overset{\varphi}{\dashrightarrow} & X \times k^m \\ \pi \searrow & & \swarrow \pi_1 \\ & X & \end{array}$$

*is commutative ( $\pi_1$  denotes projection to the first factor).*

(c) *Let  $V_1$  and  $V_2$  be finite-dimensional vector spaces over  $k$  endowed with faithful linear actions of  $\Gamma$ . Then  $V_1$  and  $V_2$  are stably  $\Gamma$ -isomorphic.*

(d) *Suppose  $L$  is a field and*

$$0 \longrightarrow S \xrightarrow{\iota} N \xrightarrow{\tau} P \longrightarrow 0$$

*is an exact sequence of  $\Gamma$ -lattices, where  $S$  is faithful and  $P$  is permutation. Then the  $\Gamma$ -field  $L(N)$  is  $\Gamma$ -isomorphic over  $L$  to the  $\Gamma$ -field  $L(S)(t_1, \dots, t_r)$ , where the elements  $t_1, \dots, t_r$  are  $\Gamma$ -invariant and algebraically independent over  $L(S)$ .*

*Proof.* Part (a) follows from Speiser's Lemma, [Spe]; cf. [HK, Theorem 1] or [Sh, Appendix 3].

(b) Recall that, by definition, algebraic bundles are locally trivial in the étale topology, but algebraic vector bundles are automatically locally trivial in the Zariski topology; see [Se]. This implies that after replacing  $X$  by a  $\Gamma$ -stable dense open subset  $U$  and  $Y$  by  $\pi^{-1}(U)$ , we may assume that  $Y = X \times k^m$  (but we do not claim that  $\Gamma$  acts via the first factor!) and  $\pi$  is the projection to the first factor.

Using the projections  $Y \rightarrow X$  and  $Y \rightarrow k^m$ , we shall view  $k(X)$  and  $k(k^m)$  as subfields of  $k(Y)$ . Put  $E := k(Y)$ ,  $K := k(X)$  and let  $x_1, \dots, x_m$  be the standard coordinate functions on  $k^m$ . If  $g \in \Gamma$  and  $b \in X$ , then the definition of  $\Gamma$ -bundle implies that  $g(x_i)|_{\pi^{-1}(b)} \in kx_1|_{\pi^{-1}(b)} + \dots + kx_m|_{\pi^{-1}(b)}$ . In turn, this implies that the assumptions of (a) hold. Part (b) now follows from part (a).

(c) Applying part (b) to the projections  $V_1 \leftarrow V_1 \times V_2 \rightarrow V_2$ , we see that both  $V_1$  and  $V_2$  are stably  $\Gamma$ -isomorphic to  $V_1 \times V_2$ .

(d) Identify  $S$  with  $\iota(S)$ ; then  $K := L(S)$  is a  $\Gamma$ -subfield of  $E := L(N)$ . Put  $x_1 = 1 \in E$  and choose  $x_2, \dots, x_m \in N \subset E$  such that  $\tau(x_2), \dots, \tau(x_m)$  is a basis of  $P$  permuted by  $\Gamma$ . The elements  $x_2, \dots, x_m$  are algebraically independent over  $K$ . If  $g \in \Gamma$ , then for every  $i$  there is a  $j$  such that  $a_{ij} := g(x_i) - x_j \in \text{Ker } \tau = S \subset K$ ; so  $g(x_i) = a_{ij}x_1 + x_j$ . This shows that the assumptions of (a) hold. The claim (with  $r = m - 1$ ) now follows from part (a).  $\square$

**2.5. Homogeneous fiber spaces.** Let  $H$  be an algebraic group and let  $S$  be a closed subgroup of  $H$ . Consider an algebraic variety  $X$  endowed with an algebraic (morphic) action of  $S$  and the algebraic action of  $S$  on  $H \times X$  defined by

$$(2.13) \quad s(h, x) = (hs^{-1}, s(x)), \quad s \in S, (h, x) \in H \times X.$$

Assume that there exists a geometric quotient, [MFK], [PV, 4.2],

$$(2.14) \quad H \times X \longrightarrow (H \times X)/S.$$

This is always the case if every finite subset of  $X$  is contained in an affine open subset of  $X$  (note that this property holds if the variety  $X$  is quasi-projective) ([Se, 3.2]; cf. [PV, 4.8]). The variety  $(H \times X)/S$ , called a *homogeneous fiber space over  $H/S$  with fiber  $X$* , is denoted by  $H \times^S X$ . If  $H$  is connected and  $X$  is irreducible, then  $H \times^S X$  is irreducible. We denote by  $[h, x]$  the image of a point  $(h, x) \in H \times X$  under the morphism (2.14).

The group  $H$  acts on  $H \times X$  by left translations of the first factor. As this action commutes with the  $S$ -action (2.13), the universal property of geometric quotients implies that the corresponding  $H$ -action on  $H \times^S X$ ,

$$h'[h, x] = [h'h, x], \quad h', h \in H, \quad x \in X,$$

is algebraic. It also implies that since the composition of the projection  $H \times X \rightarrow H$  with the canonical morphism  $H \rightarrow H/S$  is constant on  $S$ -orbits of the action (2.13), this composition induces a morphism

$$(2.15) \quad \pi = \pi_{H,S,X} : H \times^S X \longrightarrow H/S, \quad [h, x] \mapsto hS.$$

This morphism is  $H$ -equivariant and its fiber over the point  $o \in H/S$  corresponding to  $S$  is  $S$ -stable and  $S$ -isomorphic to  $X$ ; in what follows we identify  $X$  with this fiber. Since  $H$  acts transitively on  $H/S$  and  $\pi$  is  $H$ -equivariant, the  $H$ -orbit of any point of  $H \times^S X$  intersects  $X$ . If  $Z$  is an open (respectively, closed)  $H$ -stable subset of  $X$ , and  $\iota : Z \hookrightarrow X$  is the identity embedding, then  $H \times^S Z \rightarrow H \times^S X$ ,  $[h, z] \mapsto [h, \iota(z)]$ , is the embedding of algebraic varieties whose image is an  $H$ -stable closed (respectively, open) subset of  $H \times^S X$ . Every  $H$ -stable closed (respectively, open) subset of  $H \times^S X$  is obtained in this way.

If the action of  $S$  on  $X$  is trivial, then  $H \times^S X = H/S \times X$  and  $\pi$  is the projection to the first factor.

The morphism  $\pi$  is a locally trivial fibration in the étale topology; i.e., each point of  $H/S$  has an open neighborhood  $U$  such that the pull back of  $\pi^{-1}(U) \xrightarrow{\pi} U$  over a suitable étale covering  $\tilde{U} \rightarrow U$  is isomorphic to the trivial fibration  $\tilde{U} \times X \rightarrow \tilde{U}$ ,  $(y, x) \mapsto x$ ; see [Se, §2], [PV, 4.8]. If  $X$  is a  $k$ -vector space and the action of  $S$  on  $X$  is linear, then (2.15) is an algebraic vector  $H$ -bundle, so  $\pi$  is locally trivial in the Zariski topology; i.e.,  $\pi^{-1}(U) \xrightarrow{\pi} U$  is isomorphic to  $U \times X \rightarrow U$ ,  $(u, x) \mapsto x$ , for a suitable  $U$ ; see [Se].

If  $\psi$  is a (not necessarily  $H$ -equivariant) morphism (respectively, rational map) of  $H \times^S X$  to  $H \times^S Y$  such that

$$(2.16) \quad \pi_{H,S,X} = \pi_{H,S,Y} \circ \psi,$$

then we say that  $\psi$  is a morphism (respectively, rational map) *over  $H/S$* .

**Lemma 2.17.** (a) *If  $\psi : H \times^S X \rightarrow H \times^S Y$  is an  $H$ -morphism over  $H/S$ , then  $\psi|_X$  is an  $S$ -morphism  $X \rightarrow Y$ . The map  $\psi \mapsto \psi|_X$  is a bijection between  $H$ -morphisms  $H \times^S X \rightarrow H \times^S Y$  over  $H/S$  and  $S$ -morphisms  $X \rightarrow Y$ . Moreover,  $\psi$  is dominant (respectively, an isomorphism) if and only if  $\psi|_X$  is dominant (respectively, an isomorphism).*

(b) *Let  $H$  be connected and let  $X$  and  $Y$  be irreducible. Then the statements in (a) hold with “morphism” and “isomorphism” replaced by, respectively, “rational map” and “birational isomorphism”.*

*Proof.* (a) Since  $X = \pi_{H,S,X}^{-1}(o)$ ,  $Y = \pi_{H,S,Y}^{-1}(o)$ , the first statement follows from (2.16). As every  $H$ -orbit in  $H \times^S X$  intersects  $X$  and  $\psi$  is  $H$ -equivariant,  $\psi$  is uniquely determined

by  $\psi|_X$ . If  $\varphi : X \rightarrow Y$  is an  $S$ -morphism, then  $H \times X \rightarrow H \times Y$ ,  $(h, x) \mapsto (h, \varphi(x))$ , is a morphism commuting with the actions of  $S$  (defined for  $H \times X$  by (2.13) and analogously for  $H \times Y$ ) and  $H$ . By the universal property of geometric quotients, the  $H$ -map  $\psi : H \times^S X \rightarrow H \times^S Y$ ,  $[h, x] \mapsto [h, \varphi(x)]$ , is a morphism over  $H/S$ . We have  $\psi|_X = \varphi$ . The same argument proves the last statement.

(b) Since  $\psi$  is  $H$ -equivariant, its indeterminacy locus is  $H$ -stable. As every  $H$ -orbit in  $H \times^S X$  intersects  $X$ , this locus cannot contain  $X$ . Consequently,  $\psi|_X : X \dashrightarrow H \times^S Y$  is a well-defined rational  $S$ -map. In view of (2.16), its image lies in  $Y$ . Now (b) follows from (a) because rational maps are the equivalence classes of morphisms of dense open subsets, and all  $H$ -stable open subsets in  $H \times^S X$  are of the form  $H \times^S Z$  where  $Z$  is an  $S$ -stable open subset of  $X$ .  $\square$

### 3. Cayley maps, generic tori, and lattices

**3.1. Restricting Cayley maps to Cartan subgroups.** Let  $G$  be a connected linear algebraic group and let  $T$  be its maximal torus. Consider the Cartan subgroup  $C$ , its normalizer  $N$ , and the Weyl group  $W$  defined by (1.23). Let  $\mathfrak{g}$ ,  $\mathfrak{t}$ , and  $\mathfrak{c}$  be the Lie algebras of  $G$ ,  $T$ , and  $C$ , respectively.

Since  $C$  is the identity component of  $N$  and the Cartan subgroups of  $G$  are all conjugate to each other, [Bor, 12.1], assigning to a point of  $G/N$  the identity component of its  $G$ -stabilizer (respectively, the Lie algebra of this  $G$ -stabilizer) yields a bijection between  $G/N$  and the set of all Cartan subgroups in  $G$  (respectively, all Cartan subalgebras in  $\mathfrak{g}$ ). So  $G/N$  can be considered as the *variety of all Cartan subgroups in  $G$*  (respectively, the *variety of all Cartan subalgebras in  $\mathfrak{g}$* ).

Moreover the Cartan subgroups in  $G$  (respectively, the Cartan subalgebras in  $\mathfrak{g}$ ) parametrized in this way by the points of  $G/N$  naturally “merge” to form a homogeneous fiber space over  $G/N$  with fiber  $C$  (respectively,  $\mathfrak{c}$ ). More precisely, consider the homogeneous fiber space  $G \times^N C$  over  $G/N$  defined by the conjugating action of  $N$  on  $C$  (respectively, the homogeneous fiber space  $G \times^N \mathfrak{c}$  over  $G/N$  defined by the adjoint action of  $N$  on  $\mathfrak{c}$ ). Then for any  $g \in G$ , the map  $\pi_{G,N,C}^{-1}(g(o)) \rightarrow gCg^{-1}$ ,  $[g, c] \mapsto gcg^{-1}$  (respectively, the map  $\pi_{G,N,\mathfrak{c}}^{-1}(g(o)) \rightarrow \text{Ad}_G g(\mathfrak{c})$ ,  $[g, x] \mapsto \text{Ad}_G g(x)$ ), is a well-defined isomorphism (we use the notation of Subsection 2.5 for  $H = G$ ,  $S = N$ ).

Consider the conjugating and adjoint actions, respectively, of  $G$  on  $G$  and  $\mathfrak{g}$ . Then the definition of homogeneous fiber space implies that

$$(3.1) \quad \gamma_C : G \times^N C \longrightarrow G, [g, c] \mapsto gcg^{-1}, \quad \gamma_{\mathfrak{c}} : G \times^N \mathfrak{c} \longrightarrow \mathfrak{g}, [g, x] \mapsto \text{Ad}_G g(x),$$

are well-defined  $G$ -equivariant maps, and the universal property of geometric factor implies that they are morphisms.

**Lemma 3.2.** (a) *The morphisms  $\gamma_C$  and  $\gamma_{\mathfrak{c}}$  in (3.1) are birational  $G$ -isomorphisms.*

(b) *Any rational  $G$ -maps  $G \times^N C \dashrightarrow G \times^N \mathfrak{c}$  and  $G \times^N \mathfrak{c} \dashrightarrow G \times^N C$  are rational maps over  $G/N$ .*

*Proof.* (a) Since the Cartan subgroups of  $G$  are all conjugate and every element of a dense open set  $U$  in  $G$  belongs to a unique Cartan subgroup, [Bor, §12], every fiber  $\gamma_C^{-1}(u)$ , where  $u \in U$ , is a single point. As  $\text{char } k = 0$ , this means that  $\gamma_C$  is a birational isomorphism. For  $\gamma_{\mathfrak{c}}$  the arguments are analogous because  $\mathfrak{c}$  is a Cartan subalgebra in  $\mathfrak{g}$ , Cartan subalgebras in  $\mathfrak{g}$  are all  $\text{Ad}_G G$ -conjugate and a general element of  $\mathfrak{g}$  is contained in a unique Cartan subalgebra, [Bou<sub>3</sub>, Ch. VII].

(b) Since a general element of  $T$  (respectively,  $\mathfrak{t}$ ) is regular,  $C$  (respectively,  $\mathfrak{c}$ ) is the unique Cartan subgroup (respectively, subalgebra) containing  $T$  (respectively,  $\mathfrak{t}$ ), [Bor, §13]; see [Bou<sub>3</sub>, Ch. VII]. This implies that  $C$  and  $\mathfrak{c}$  are the fixed point sets of the actions of  $T$  on  $G \times^N C$  and  $G \times^N \mathfrak{c}$ , respectively. Since the maps under consideration are  $G$ -equivariant, this immediately implies the claim.  $\square$

**Remark 3.3.** The group varieties of  $C$  and  $\mathfrak{c}$  are the “standard relative sections” of, respectively,  $G$  and  $\mathfrak{g}$  induced by the rational  $G$ -map  $\pi_{G,N,C} \circ \gamma_C^{-1} : G \dashrightarrow G/N$  and  $\pi_{G,N,\mathfrak{c}} \circ \gamma_{\mathfrak{c}}^{-1} : \mathfrak{g} \dashrightarrow G/N$ ; in particular, this yields the following isomorphisms of invariant fields:

$$(3.4) \quad k(G)^G \xrightarrow{\simeq} k(C)^N, f \mapsto f|_C, \quad k(\mathfrak{g})^G \xrightarrow{\simeq} k(\mathfrak{c})^N, f \mapsto f|_{\mathfrak{c}};$$

see [Pop<sub>3</sub>, Definition (1.7.6) and Theorem (1.7.5)].

**Lemma 3.5.** (a)  $G$  is Cayley if and only if  $C$  and  $\mathfrak{c}$  are birationally  $N$ -isomorphic.  
 (b)  $G$  is stably Cayley if and only if  $C$  and  $\mathfrak{c}$  are stably birationally  $N$ -isomorphic.

*Proof.* (a) By Lemma 2.17, the existence of a birational  $N$ -isomorphism  $\varphi : C \dashrightarrow \mathfrak{c}$  implies the existence of a birational  $G$ -isomorphism  $\psi : G \times^N C \dashrightarrow G \times^N \mathfrak{c}$ . Then Lemma 3.2 shows that  $\gamma_{\mathfrak{c}} \circ \psi \circ \gamma_C^{-1} : G \dashrightarrow \mathfrak{g}$  is a Cayley map.

Conversely, let  $\lambda : G \dashrightarrow \mathfrak{g}$  be a Cayley map. Then  $\psi := \gamma_{\mathfrak{c}}^{-1} \circ \lambda \circ \gamma_C : G \times^N C \dashrightarrow G \times^N \mathfrak{c}$  is a birational  $G$ -isomorphism. By Lemma 3.2,  $\psi$  is a rational map over  $G/N$ . Hence, by Lemma 2.17,  $\psi|_C : C \dashrightarrow \mathfrak{c}$  is a birational  $N$ -isomorphism.

(b) If  $C$  and  $\mathfrak{c}$  are stably birationally  $N$ -isomorphic, it follows from the rationality of the underlying variety of any linear algebraic torus that for some natural  $d$  there exists a birational  $N$ -isomorphism

$$(3.6) \quad C \times \mathbf{G}_m^d \dashrightarrow \mathfrak{c} \oplus k^d,$$

where  $k^d$  is the Lie algebra of  $\mathbf{G}_m^d$  and  $N$  acts on  $C \times \mathbf{G}_m^d$  and  $\mathfrak{c} \oplus k^d$  via  $C$  and  $\mathfrak{c}$ , respectively. Clearly  $C \times \mathbf{G}_m^d$  is the Cartan subgroup of  $G \times \mathbf{G}_m^d$  with normalizer  $N \times \mathbf{G}_m^d$  and Lie algebra  $\mathfrak{c} \oplus k^d$ , and the birational isomorphism (3.6) is  $N \times \mathbf{G}_m^d$ -equivariant. Now (a) implies that  $G \times \mathbf{G}_m^d$  is Cayley and hence  $G$  is stably Cayley.

Conversely, assume that  $G \times \mathbf{G}_m^d$  is Cayley for some  $d$ . Then the above arguments and (a) show that there exists a birational  $N$ -isomorphism (3.6). Since the group varieties of  $\mathbf{G}_m^d$  and  $k^d$  are rational, this means that  $C$  and  $\mathfrak{c}$  are stably birationally  $N$ -isomorphic.  $\square$

For reductive groups, Lemma 3.5 translates into the statement resulting also from [Lun<sub>1</sub>, p. 13, Proposition]:

**Corollary.** *Let  $G$  be a connected reductive linear algebraic group.*

- (a)  $G$  is Cayley if and only if  $T$  and  $\mathfrak{t}$  are birationally  $W$ -isomorphic.
- (b)  $G$  is stably Cayley if and only if  $T$  and  $\mathfrak{t}$  are stably birationally  $W$ -isomorphic.

*Proof.* Since  $G$  is reductive,  $C = T$  and  $\mathfrak{c} = \mathfrak{t}$ . As  $T$  is commutative, this implies that the actions of  $N$  on  $T$  and  $\mathfrak{t}$  descend to the actions of  $W$ . The claim now follows from Lemma 3.5.  $\square$

**3.2. Generic tori.** We now recall the definition of generic tori in a form suitable for our purposes; see [Vos, 4.1] or [CK, p. 772]. We maintain the notation of Subsections 2.5 and 3.1.

Assume that  $G$  is a connected reductive linear algebraic group; then  $C = T$  and  $\mathfrak{c} = \mathfrak{t}$ . According to the discussion in the previous subsection,  $G/N$  may be interpreted in two ways: first, as the *variety of all maximal tori in  $G$*  and, second, as the *variety of all maximal tori in  $\mathfrak{g}$* . The maximal torus in  $G$  (respectively, in  $\mathfrak{g}$ ) assigned to a point  $g(o) \in G/N$  is  $gTg^{-1}$  (respectively,  $\text{Ad}_G g(\mathfrak{t})$ ); it is naturally identified with the fiber over  $g(o)$  of the morphism  $\pi_{G,N,T} : G \times^N T \rightarrow G/N$  (respectively,  $\pi_{G,N,\mathfrak{t}} : G \times^N \mathfrak{t} \rightarrow G/N$ ).

**Definition 3.7.** The triples

$$\mathbf{T}_G := (G \times^N T, \pi_{G,N,T}, G/N) \quad \text{and} \quad \mathfrak{t}_{\mathfrak{g}} := (G \times^N \mathfrak{t}, \pi_{G,N,\mathfrak{t}}, G/N)$$

are called, respectively, the *generic torus of  $G$*  and the *generic torus of  $\mathfrak{g}$* .

We identify the field  $k(G/N)$  with its image in  $k(G \times^N T)$  under the embedding  $\pi_{G,N,T}^*$ .

**Definition 3.8.** The generic torus  $\mathbf{T}_G$  is called *rational* if  $k(G \times^N T)$  is a purely transcendental extension of  $k(G/N)$ . If  $\mathbf{T}_{G \times \mathbf{G}_m^d}$  is rational for some  $d$ , then  $\mathbf{T}_G$  is called *stably rational*.

Equivalently,  $\mathbf{T}_G$  is called rational if there exists a birational isomorphism

$$(3.9) \quad G \times^N T \xrightarrow{\sim} G/N \times \mathbb{A}^r$$

over  $G/N$  (then  $r = \dim T$ ). The arguments used in the proof of Lemma 3.5(b) show that stable rationality of  $\mathbf{T}_G$  is equivalent to the property that there exists a purely transcendental field extension  $E$  of  $k(G \times^N T)$  such that  $E$  is a purely transcendental extension of  $k(G/N)$ . There are groups  $G$  such that the generic torus  $\mathbf{T}_G$  is not stably rational (and hence not rational), [Vos], [CK].

Of course, for the generic torus  $\mathfrak{t}_{\mathfrak{g}}$  in  $\mathfrak{g}$ , one could also introduce the notions analogous to that in Definition 3.8. However in the Lie algebra context the rationality problem of generic tori is quite easy: since  $\pi_{G,N,\mathfrak{t}} : G \times^N \mathfrak{t} \rightarrow G/N$  is a vector bundle, it is locally trivial in the Zariski topology, and hence  $\mathfrak{t}_{\mathfrak{g}}$  is always rational; i.e., there exists a birational isomorphism

$$(3.10) \quad G \times^N \mathfrak{t} \xrightarrow{\sim} G/N \times \mathbb{A}^r$$

over  $G/N$ .

**3.3. Proof of Theorem 1.27.** *Implication (a):* By the Corollary of Lemma 3.5, it is enough to construct a  $W$ -equivariant birational isomorphism  $\varphi : T \xrightarrow{\sim} \mathfrak{t}$ .

Using the sign-permutation basis of  $\widehat{T}$ , we can  $W$ -equivariantly identify the maximal torus  $T$  with  $\mathbf{G}_m^r$ , where  $r$  is the rank of  $G$  and every  $w \in W$  acts on  $\mathbf{G}_m^r$  by

$$(3.11) \quad (t_1, \dots, t_r) \mapsto (t_{\sigma(1)}^{\varepsilon_1}, \dots, t_{\sigma(r)}^{\varepsilon_r}),$$

for some  $\sigma \in S_r$  and some  $\varepsilon_1, \dots, \varepsilon_r \in \{\pm 1\}$  (depending on  $w$ ). The Lie algebra  $\mathfrak{t}$  is the tangent space to  $\mathbf{G}_m^r$  at  $e = (1, \dots, 1)$ ; it follows from (3.11) that we can identify it with  $k^r$  where  $w$  acts by

$$(3.12) \quad (x_1, \dots, x_r) \mapsto (\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_r x_{\sigma(r)}).$$

From (3.11) and (3.12) we easily deduce that the formula

$$(t_1, \dots, t_r) \mapsto ((1 - t_1)(1 + t_1)^{-1}, \dots, (1 - t_r)(1 + t_r)^{-1})$$

defines a desired birational  $W$ -isomorphism  $\varphi: T \xrightarrow{\sim} \mathfrak{t}$ . This completes the proof of implication (a).

To see that implication (a) cannot be reversed, consider the group  $G := \mathbf{SL}_3$ . First note that this group is Cayley; see Proposition 9.1. On the other hand,  $W \simeq S_3$  and since the character lattice  $\mathcal{X}_G$  has rank 2, it cannot be sign-permutation. Indeed, if it were, then  $S_3$  would embed into  $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes S_2$ , which is impossible.

*Implication (b):* By the Corollary of Lemma 3.5, there is a birational  $N$ -isomorphism  $T \xrightarrow{\sim} \mathfrak{t}$ . By Lemma 2.17, this implies that there is a birational  $G$ -isomorphism  $G \times^N T \xrightarrow{\sim} G \times^N \mathfrak{t}$  over  $G/N$ . Its composition with the birational isomorphism (3.10) is a birational isomorphism (3.9) over  $G/N$ . Hence  $\mathbf{T}_G$  is rational.

To see that implication (b) cannot be reversed, consider the exceptional group  $\mathbf{G}_2$ . The generic torus of  $\mathbf{G}_2$  is rational; see [Vos, 4.9]. On the other hand,  $\mathbf{G}_2$  is not a Cayley group; see Proposition 9.10.

*Implication (c):* This is obvious from the definition.

*Equivalence (d):* This is well known; see, e.g., [Vos, Theorem 4.7.2].

*Equivalence (e):* Let  $V$  be any finite-dimensional faithful permutation  $W$ -module over  $k$  (for instance, the one determined by the regular representation of  $W$ ). Then clearly  $k(V) = k(P)$  for some permutation  $W$ -lattice  $P$ . Since the action of  $W$  on  $\mathfrak{t}$  is faithful, [Bor], we deduce from Lemma 2.12(c) that  $k(\mathfrak{t})$  and  $k(P)$  are stably  $W$ -isomorphic over  $k$ . Therefore, since  $k(T) = k(\widehat{T})$ , applying the Corollary of Lemma 3.5 implies that  $G$  is stably Cayley if and only if  $k(\widehat{T})$  and  $k(P)$  are stably  $W$ -isomorphic over  $k$ . On the other hand, the latter property holds if and only if the  $W$ -lattice  $\widehat{T}$  is quasi-permutation; see the Corollary of Lemma 2.5, whence the claim.  $\square$

**Example 3.13.** The character lattice  $\mathbb{Z}\mathbf{A}_{n-1}$  of  $\mathbf{PGL}_n$  is defined by the exact sequence

$$0 \longrightarrow \mathbb{Z}\mathbf{A}_{n-1} \longrightarrow \mathbb{Z}[\mathbf{S}_n/\mathbf{S}_{n-1}] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

where  $\epsilon$  is the augmentation map and the Weyl group  $W = S_n$  acts trivially on  $\mathbb{Z}$  and naturally on  $\mathbb{Z}[\mathbf{S}_n/\mathbf{S}_{n-1}]$ ; see Subsection 6.1. Thus  $\mathbb{Z}\mathbf{A}_{n-1}$  is quasi-permutation. By Theorem 1.27, we conclude that  $\mathbf{PGL}_n$  is stably Cayley. We know that in fact  $\mathbf{PGL}_n$  is even Cayley; see Example 1.11. Note though that  $\mathbb{Z}\mathbf{A}_{n-1}$  is not sign-permutation if  $n > 2$ . Indeed, from the sequence above, we can show that  $H^1(S_n, \mathbb{Z}\mathbf{A}_{n-1}) \simeq \mathbb{Z}/n\mathbb{Z}$ , whereas by [LL, Lemma 4.4], a sign-permutation  $\Gamma$ -lattice  $L$  would have  $H^1(\Gamma, L) \cong (\mathbb{Z}/2\mathbb{Z})^d$  for some  $d \geq 0$ .  $\square$

#### 4. Reduction theorems

The purpose of this section is to show that to a certain extent classifying arbitrary Cayley groups is reduced to classifying simple ones.

As before, let  $G$  be a connected linear algebraic group. Denote by  $R$  and  $R_u$ , respectively, the radical and the unipotent radical of  $G$ . Recall that a *Levi subgroup* of  $G$  is a connected subgroup  $L$ , necessarily reductive, such that  $G = L \ltimes R_u$ ; since  $\text{char } k = 0$ , Levi subgroups exist and are conjugate, [Bor, 11.22].

In this section we will address the following questions:

- (a) If a Levi subgroup of  $G$  is (stably) Cayley, is  $G$  (stably) Cayley?

- (b) Let  $G$  be reductive. If  $G/R$  is (stably) Cayley, is  $G$  (stably) Cayley?
- (c) Let  $G$  be reductive and let  $H_1, \dots, H_n$  be a complete list of its connected normal simple subgroups. What is the relation between the (stably) Cayley property of  $G$  and that of  $H_1, \dots, H_n$ ?

**4.1. Unipotent normal subgroups.** We will need a generalization of Example 1.21. Let  $U$  be a normal unipotent subgroup of  $G$ . Denote by  $\mathfrak{u}$  the Lie algebra of  $U$ . The group  $G$  acts on  $U$  by conjugation and on  $\mathfrak{u}$  by  $\text{Ad}_G|_{\mathfrak{u}}$ .

**Lemma 4.1.** *There exists a  $G$ -isomorphism of  $G$ -varieties  $U \rightarrow \mathfrak{u}$ .*

*Proof.* We may assume without loss of generality that  $G \subset \mathbf{GL}_n$ . Since  $\text{Ad}_G$  is given by (1.1), it follows from (1.22) that  $\ln: U \rightarrow \mathfrak{u}$  is a  $G$ -morphism. By Example 1.21, it is an isomorphism, whence the claim.  $\square$

#### 4.2. The Levi decomposition.

**Proposition 4.2.** *Let  $L$  be a Levi subgroup of  $G$ .*

- (a) *If  $L$  is Cayley, then so is  $G$ .*
- (b)  *$G$  is stably Cayley if and only if  $L$  is stably Cayley.*

*Proof.* Let  $T$  be a maximal torus of  $L$ . It is a maximal torus of  $G$  as well, [Bor, 11.20]. Using the notation of (1.23) and Subsection 3.1, we have  $C = T \times U$  where  $U$  is a unipotent group, [Bor, 12.1]. Let  $\mathfrak{u}$  be the Lie algebra of  $U$  and let  $d = \dim U$ . As  $T$  and  $U$  are, respectively, the semisimple and unipotent parts of the nilpotent group  $C$ , they are stable under the conjugating action of  $N$ , and  $C$ , as an  $N$ -variety, is the product of the  $N$ -varieties  $T$  and  $U$ . Consequently,  $\mathfrak{t}$  and  $\mathfrak{u}$  are stable under the adjoint action of  $N$ , and  $\mathfrak{c}$ , as an  $N$ -variety, is the product of  $N$ -varieties  $\mathfrak{t}$  and  $\mathfrak{u}$ . By Lemma 4.1, there exists an isomorphism of  $N$ -varieties

$$(4.3) \quad \tau: U \longrightarrow \mathfrak{u}.$$

(a) Assume that  $L$  is Cayley. Then by the Corollary of Lemma 3.5, there is a birational  $W_{L,T}$ -isomorphism  $\varphi: T \xrightarrow{\sim} \mathfrak{t}$ . Since the action of  $W_{L,T}$  on  $T$  (respectively,  $\mathfrak{t}$ ) is faithful,  $W_{L,T}$  can be considered as a transformation group of  $T$  (respectively,  $\mathfrak{t}$ ). By [Bor, 11.20], it coincides with the transformation group  $\{T \rightarrow T, t \mapsto ntn^{-1} \mid n \in N\}$  (respectively,  $\{\mathfrak{t} \rightarrow \mathfrak{t}, x \mapsto \text{Ad}_G n(x) \mid n \in N\}$ ). Therefore the map  $\varphi$  is  $N$ -equivariant. Hence

$$\varphi \times \tau: C = T \times U \dashrightarrow \mathfrak{t} \oplus \mathfrak{u} = \mathfrak{c}$$

is a birational  $N$ -isomorphism. Lemma 3.5 now implies that  $G$  is Cayley.

(b) Since  $L \times \mathbf{G}_m^d$  is the Levi subgroup of  $G \times \mathbf{G}_m^d$ , it follows from (a) that if  $L$  is stably Cayley, then  $G$  is stably Cayley.

To prove the converse, it suffices to show that if  $G$  is Cayley, then  $L$  is stably Cayley. In turn, Lemma 3.5 and its Corollary reduce this to proving that if there exists a birational  $N$ -isomorphism

$$\alpha: C = T \times U \xrightarrow{\sim} \mathfrak{t} \times \mathfrak{u} = \mathfrak{c},$$

then  $T$  and  $\mathfrak{t}$  are stably birationally  $W_{L,T}$ -isomorphic. We shall prove this last statement.

Since  $T$  is the identity component of  $N_{L,T} = N \cap L$  and  $T$  acts trivially on  $C$  and  $\mathfrak{c}$ , the actions of  $N_{L,T}$  on  $C$ ,  $\mathfrak{c}$ ,  $T$ ,  $\mathfrak{t}$ ,  $U$ , and  $\mathfrak{u}$  descend to actions of  $W_{L,T} = N_{L,T}/T$ . Moreover,  $C$  (respectively,  $\mathfrak{c}$ ), as a  $W_{L,T}$ -variety, is the product of  $W_{L,T}$ -varieties  $T$  and  $U$  (respectively,  $\mathfrak{t}$  and  $\mathfrak{u}$ ), and  $\alpha$  is a birational  $W_{L,T}$ -isomorphism.

Since  $W_{L,T}$  acts linearly on  $\mathfrak{u}$ , Lemma 2.12(b) implies that there are birational  $W_{L,T}$ -isomorphisms

$$\beta: T \times \mathbb{A}^d \xrightarrow{\sim} T \times \mathfrak{u} \quad \text{and} \quad \gamma: \mathfrak{t} \times \mathfrak{u} \xrightarrow{\sim} \mathfrak{t} \times \mathbb{A}^d,$$

where  $W_{L,T}$  acts on  $T \times \mathbb{A}^d$  and  $\mathfrak{t} \times \mathbb{A}^d$  via the first factors. Considering the composition of the following birational  $W_{L,T}$ -isomorphisms

$$T \times \mathbb{A}^d \xrightarrow{\beta} T \times \mathfrak{u} \xrightarrow{\text{id} \times \tau^{-1}} T \times U \xrightarrow{-\alpha} \mathfrak{t} \times \mathfrak{u} \xrightarrow{-\gamma} \mathfrak{t} \times \mathbb{A}^d,$$

we now see that  $T$  and  $\mathfrak{t}$  are indeed stably birationally  $W_{L,T}$ -isomorphic.  $\square$

**Remark 4.4.** The converse to Proposition 4.2(a) fails for  $G := \mathbf{G}_2 \times \mathbf{G}_a^2$ . Indeed, the first factor is the Levi subgroup of  $G$ . By Proposition 9.10, it is not Cayley. Consider the group  $H := \mathbf{G}_2 \times \mathbf{G}_m^2$ . Both  $G$  and  $H$  have the same Lie algebra  $\mathfrak{g}$ . By Proposition 9.11,  $H$  is Cayley; let  $\lambda: H \xrightarrow{\sim} \mathfrak{g}$  be a Cayley map. Fix a birational isomorphism of group varieties  $\delta: \mathbf{G}_a^2 \xrightarrow{\sim} \mathbf{G}_m^2$ . Since the second factors of  $G$  and  $H$  lie in the kernels of conjugating and adjoint actions,  $\lambda \circ (\text{id} \times \delta): G \xrightarrow{\sim} \mathfrak{g}$  is a Cayley map. Thus  $G$  is Cayley.

**Corollary.** *Every connected solvable linear algebraic group  $G$  is Cayley.*

*Proof.* A Levi subgroup  $L$  of  $G$  is a torus, [Bor, 10.6]. By Example 1.20,  $L$  is Cayley. Hence by Proposition 4.2(a),  $G$  is Cayley as well.  $\square$

### 4.3. From reductive to semisimple.

**Proposition 4.5.** *Let  $G$  be a connected reductive group and let  $Z$  be a connected closed central subgroup of  $G$ .*

- (a) *If  $G/Z$  is Cayley, then so is  $G$ .*
- (b)  *$G$  is stably Cayley if and only if  $G/Z$  is stably Cayley.*

*Proof.* Since  $G$  is reductive,  $R$  is a torus and the identity component of the center of  $G$ ; see [Bor, 11.21]. Thus  $Z$  is a subtorus of  $R$ . Let  $T$  be a maximal torus of  $G$ . We have  $R \subset T$  (see [Bor, 11.11]),  $T/Z$  is a maximal torus of  $G/Z$  and the natural epimorphism  $G \rightarrow G/Z$  identifies  $W$  with  $W_{G/Z, T/Z}$  (we use the notation of (1.23) and Subsection 3.1); see [Bor, 11.20]. Since  $Z$  is central, it is pointwise fixed with respect to the action of  $W$ . Thus we have the following exact sequence of  $W$ -homomorphisms of tori:

$$e \longrightarrow Z \longrightarrow T \longrightarrow T/Z \longrightarrow e$$

which in turn yields the exact sequence of  $W$ -lattices of character groups

$$0 \longrightarrow \widehat{T/Z} \longrightarrow \widehat{T} \longrightarrow \widehat{Z} \longrightarrow 0.$$

Note that  $W$  acts trivially on  $\widehat{Z}$ . In particular,  $\widehat{Z}$  is a permutation  $W$ -lattice, and the last exact sequence tells us that the character lattices  $\widehat{T}$  and  $\widehat{T/Z}$  are equivalent; see Definition 2.2. Thus if one of them is quasi-permutation, then so is the other. Part (b) now follows from Theorem 1.27.

Since the  $W$ -fields  $k(T)$  and  $k(T/Z)$  are  $W$ -isomorphic to  $k(\widehat{T})$  and  $k(\widehat{T/Z})$ , respectively, we deduce from Lemma 2.12(d) that  $T$  is birationally  $W$ -isomorphic to  $T/Z \times \mathbb{A}^m$ , where  $W$  acts on  $T/Z \times \mathbb{A}^m$  via the first factor and  $m = \dim Z$ .

On the other hand, let  $\mathfrak{f}$  and  $\mathfrak{z}$  be the Lie algebras of  $T/Z$  and  $Z$ , respectively. Then, since the Lie algebras  $\mathfrak{t}$  and  $\mathfrak{f} \oplus \mathfrak{z}$  are  $W$ -equivariantly isomorphic and  $W$  acts on  $\mathfrak{z}$  trivially,

we see that  $\mathfrak{t}$ , as a  $W$ -variety, is isomorphic to  $\mathfrak{f} \times \mathbb{A}^m$ , where  $W$  acts on  $\mathfrak{f} \times \mathbb{A}^m$  via the first factor.

Now to prove part (a), assume that  $G/Z$  is Cayley. Then by the Corollary of Lemma 3.5, there is a birational  $W$ -isomorphism  $\varphi : T/Z \xrightarrow{\sim} \mathfrak{f}$ . This gives a birational  $W$ -isomorphism  $T/Z \times \mathbb{A}^m \xrightarrow{\varphi \times \text{id}} \mathfrak{f} \times \mathbb{A}^m$ . Applying the Corollary of Lemma 3.5 once again, we conclude that  $G$  is Cayley. This completes the proof of part (a).  $\square$

Setting  $Z = R$ , we obtain

**Corollary.** *Let  $G$  be a connected reductive group and  $G_{\text{ss}} := G/R$ .*

- (a) *If  $G_{\text{ss}}$  is Cayley, then so is  $G$ .*
- (b)  *$G$  is stably Cayley if and only if  $G_{\text{ss}}$  is stably Cayley.*  $\square$

**Remark 4.6.** The converse to part (a) of the Corollary fails for  $G = \mathbf{G}_2 \times \mathbf{G}_m^2$ . Indeed,  $G$  is Cayley by Proposition 9.11 and  $G/R \simeq \mathbf{G}_2$  is not Cayley by Proposition 9.10.

**4.4. From semisimple to simple.** Let  $G_1, \dots, G_n$  be connected linear algebraic groups and let  $\mathfrak{g}_i$  be the Lie algebra of  $G_i$ . If each  $G_i$  is Cayley, then so is  $G_1 \times \dots \times G_n$ ; see Example 1.8. The converse fails for  $n = 2$ ,  $G_1 = \mathbf{G}_2$ ,  $G_2 = \mathbf{G}_m^2$ ; see Propositions 9.10 and 9.11.

**Lemma 4.7.**  *$G_1 \times \dots \times G_n$  is stably Cayley if and only if each  $G_i$  is stably Cayley.*

*Proof.* The “if” direction follows from Definition 1.26 and Example 1.8. To prove the converse, we use the fact that the underlying variety of each  $G_i$  is rational over  $k$ ; see [Ch<sub>1</sub>]. This implies that the underlying variety of  $G_1 \times \dots \times G_n$ , as a  $G_i$ -variety, is birationally isomorphic to  $G_i \times \mathbf{G}_m^{d_i}$  with the conjugating action via the first factor and  $d_i = \sum_{j \neq i} \dim G_j$ . The “only if” direction now follows from Definition 1.26 and the fact that the underlying variety of the Lie algebra of  $G_1 \times \dots \times G_n$ , as a  $G_i$ -variety, is isomorphic to  $\mathfrak{g}_i \oplus k^{d_i}$  with the adjoint action via the first summand.  $\square$

As usual, given subgroups  $X$  and  $Y$  of  $G$ , we denote by  $(X, Y)$  the subgroup generated by the commutators  $xyx^{-1}y^{-1}$  with  $x \in X$ ,  $y \in Y$ .

**Proposition 4.8.** *Assume  $G$  is a connected reductive group and let  $H_1, \dots, H_m$  be the connected closed normal subgroups of  $G$  such that*

- (i)  $(H_i, H_j) = e$  for all  $i \neq j$ ,
- (ii)  $G = H_1 \dots H_m$ .

*Let  $\tilde{H}_i$  be the subgroup of  $G$  generated by all  $H_j$ 's with  $j \neq i$ . If  $G$  is stably Cayley, then each  $G/\tilde{H}_i \simeq H_i/(H_i \cap \tilde{H}_i)$  is stably Cayley.*

*Proof.* Since  $H_1, \dots, H_m$  are connected, each  $\tilde{H}_i$  is connected; see [Bor, 2.2]. Since  $G$  is reductive, all  $H_i$  and  $\tilde{H}_i$  are reductive.

It follows from (i) and (ii) that

$$H_1 \times \dots \times H_m \rightarrow G, \quad (h_1, \dots, h_m) \mapsto h_1 \dots h_m,$$

is an epimorphism of algebraic groups. Let  $T_i$  be a maximal torus of  $H_i$ . Then  $T_1 \times \dots \times T_m$  is a maximal torus of  $H_1 \times \dots \times H_m$ . Therefore its image  $T := T_1 \dots T_m$  under the above epimorphism is a maximal torus of  $G$ ; see [Bor, 11.14]. The same argument shows that the group  $S_i$  of  $T$  generated by all  $T_j$ 's with  $j \neq i$  is a maximal torus of  $\tilde{H}_i$ .

It follows from (i) that  $S_i$  is pointwise fixed under the conjugating action of  $N_i := N_{H_i, T_i}$  on  $T$ . This action clearly descends to an action of  $W_i := W_{H_i, T_i} = N_i/T_i$ . Since  $H_i$  is connected reductive, any maximal torus of  $H_i$  coincides with its centralizer in  $H_i$ ; see [Bor, 13.17]. Consequently,  $T \cap H_i = T_i$  and  $W_i$ , considered as a transformation group of  $T$ , is the image of  $N_i$  under the natural projection  $N \rightarrow N/T = W$ . The natural epimorphism  $\pi_i : H_i \rightarrow H_i/(H_i \cap \tilde{H}_i)$  identifies  $W_i$  with  $W_{H_i/(H_i \cap \tilde{H}_i), \pi_i(T_i)}$ , so that the isomorphism  $T_i/(T_i \cap \tilde{H}_i) \rightarrow \pi_i(T_i)$  induced by  $\pi_i$  is  $W_i$ -equivariant; cf., e.g., [Bor, 11.20, 11.11].

The same argument applied to  $\tilde{H}_i$  and  $S_i$  instead of  $H_i$  and  $T_i$  shows that  $T \cap \tilde{H}_i = S_i$ ,

$$T_i \cap \tilde{H}_i = T_i \cap S_i,$$

and that a maximal torus of  $H_i/(H_i \cap \tilde{H}_i)$  is  $W_i$ -isomorphic to  $T_i/(T_i \cap S_i)$ . Now observe that  $T_i/(T_i \cap S_i)$  is  $W_i$ -isomorphic to  $T/S_i$  because  $T = T_i S_i$ . Therefore there is an exact sequence of  $W_i$ -homomorphisms of tori:

$$e \longrightarrow S_i \longrightarrow T \longrightarrow T_i/(T_i \cap S_i) \longrightarrow e.$$

Passing to the character groups, we deduce from it the following exact sequence of  $W_i$ -lattices:

$$0 \longrightarrow T_i/\widehat{(T_i \cap S_i)} \longrightarrow \hat{T} \longrightarrow \hat{S}_i \longrightarrow 0.$$

As the action of  $W_i$  on  $S_i$  is trivial,  $\hat{S}_i$  is a trivial and, in particular, a permutation  $W_i$ -lattice. Hence the above exact sequence shows that  $T_i/\widehat{(T_i \cap S_i)}$  and  $\hat{T}$  are equivalent  $W_i$ -lattices.

Assume now that  $G$  is stably Cayley. Then Theorem 1.27 implies that  $\hat{T}$  is quasi-permutation as a  $W$ -lattice, and hence as a  $W_i$ -lattice because  $W_i$  is a subgroup of  $W$ . Therefore the equivalent  $W_i$ -lattice  $T_i/\widehat{(S_i \cap T_i)}$  is quasi-permutation as well. Since the latter is the character lattice of  $H_i/(H_i \cap \tilde{H}_i)$ , Theorem 1.27 implies that  $H_i/(H_i \cap \tilde{H}_i)$  is stably Cayley.  $\square$

**Corollary.** *Let  $G$  be a connected semisimple group. Let  $H_1, \dots, H_m$  be the minimal elements among its connected closed normal subgroups. Define  $\tilde{H}_i$  as in Proposition 4.8. If  $G$  is stably Cayley, then each  $H_i/(H_i \cap \tilde{H}_i)$  is stably Cayley.*

*Proof.* By [Bor, 14.10], the assumptions of Proposition 4.8 hold.  $\square$

**Remark 4.9.** In Proposition 4.8, if  $G$  is stably Cayley,  $H_i$  is not necessarily stably Cayley. For example, take  $G = \mathbf{GL}_n$ ,  $m = 2$ ,  $H_1 = \mathbf{G}_m$  diagonally embedded in  $\mathbf{GL}_n$  and  $H_2 = \mathbf{SL}_n$ . Then  $G$  is Cayley by Example 1.9, and  $H_2$  is not stably Cayley for  $n > 3$  by Theorem 1.28.

## 5. Proof of Theorem 1.28: An overview

In this section we outline a strategy for proving Theorem 1.28; the technical parts of the proof will be carried out in Sections 6–8.

By Theorem 1.27, it will suffice to determine which connected simple groups have a stably rational generic torus (or, equivalently, a quasi-permutation character lattice). CORTELLA and KUNYAVSKIĬ in [CK, Theorem 0.1] have classified all simply connected and all adjoint connected simple groups that have a quasi-permutation character lattice. These are precisely  $\mathbf{SO}_{2n+1}$ ,  $\mathbf{Sp}_{2n}$ ,  $\mathbf{PGL}_n$ ,  $\mathbf{SL}_3$ , and  $\mathbf{G}_2$ . Therefore in order to complete the proof of Theorem 1.28, we need to determine which intermediate (i.e., neither simply connected nor adjoint) connected simple groups have a quasi-permutation character lattice.

Recall that intermediate connected simple groups exist only for types  $A_n$  and  $D_n$ . Connected simple groups of type  $A_{n-1}$  are precisely the groups  $\mathbf{SL}_n/\mu_d$ , where  $d$  is a divisor of  $n$ . Among them, intermediate groups are those with  $1 < d < n$ . In Section 7 we will prove the following.

**Proposition 5.1.** *Let  $d$  be a divisor of  $n$ , where  $1 < d < n$  and  $(n, d) \neq (4, 2)$ . Then the character lattice of the group  $\mathbf{SL}_n/\mu_d$  is not quasi-permutation.*

As we saw in Example 1.16, the group  $\mathbf{SL}_4/\mu_2$  is Cayley; in particular, by Theorem 1.27, its character lattice is quasi-permutation.

The intermediate connected simple groups of type  $D_n$  are  $\mathbf{SO}_{2n}$  for any  $n \geq 3$  and the half-spinor groups  $\mathbf{Spin}_{2n}^{1/2}$  for even  $n \geq 4$ . The latter are defined as follows. Consider the spinor group  $\mathbf{Spin}_{2n}$  for even  $n \geq 4$ . Its center is isomorphic to  $\mu_2 \times \mu_2$ , see [Ch<sub>2</sub>], [KMRT, §25], and consequently contains precisely three subgroups of order 2. One of them is the kernel of the vector representation, so the quotient of  $\mathbf{Spin}_{2n}$  modulo it is  $\mathbf{SO}_{2n}$ . Two others are the kernels of the half-spinor representations of  $\mathbf{Spin}_{2n}$ . They are mapped to each other by an outer automorphism of  $\mathbf{Spin}_{2n}$ , so the images of the half-spin representations are isomorphic to the same group that is  $\mathbf{Spin}_{2n}^{1/2}$ .

By Example 1.16, the groups  $\mathbf{SO}_{2n}$  are Cayley. If  $n = 4$ , the group of outer automorphisms of  $\mathbf{Spin}_{2n}$  is isomorphic to  $S_3$  (for  $n > 4$ , it is isomorphic to  $S_2$ ) and acts transitively on the set of all subgroups of order 2 of the center of  $\mathbf{Spin}_{2n}$ . Therefore  $\mathbf{Spin}_8^{1/2} \simeq \mathbf{SO}_8$ , whence it is Cayley. Thus we only need to consider the half-spin groups  $\mathbf{Spin}_{2n}^{1/2}$  for even  $n > 4$ . In Section 8 we will prove the following.

**Proposition 5.2.** *The character lattice of the half-spinor group  $\mathbf{Spin}_{2n}^{1/2}$  for even  $n > 4$  is not quasi-permutation.*

Thus in order to complete the proof of Theorem 1.28, we need to prove Propositions 5.1 and 5.2. This will be done in the next three sections.

## 6. The groups $\mathbf{SL}_n/\mu_d$ and their character lattices

**6.1. Lattices  $\mathcal{Q}_n(d)$ .** For any divisor  $d$  of  $n$ , the Weyl group  $W$  of the group  $G = \mathbf{SL}_n/\mu_d$  is isomorphic to the permutation group  $S_n$  of the set of integers  $\{1, \dots, n\}$ . The character lattice  $\mathcal{X}_G$  is described as follows.

Let  $\varepsilon_1, \dots, \varepsilon_n$  be the standard basis for the permutation  $S_n$ -lattice  $\mathbb{Z}[S_n/S_{n-1}]$  on which  $\sigma \in S_n$  acts via

$$(6.1) \quad \sigma(\varepsilon_i) = \varepsilon_{\sigma(i)} \quad \text{for all } i = 1, \dots, n.$$

We naturally embed  $\mathbb{Z}[S_n/S_{n-1}]$  into the  $\mathbb{Q}$ -vector space  $\mathbb{Z}[S_n/S_{n-1}] \otimes_{\mathbb{Z}} \mathbb{Q}$  endowed with the Euclidean structure such that  $\varepsilon_1, \dots, \varepsilon_n$  is the orthonormal basis and we naturally extend the action of  $S_n$  to this space.

The root system of type  $A_{n-1}$  is the subset

$$A_{n-1} := \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$$

of  $\mathbb{Z}[S_n/S_{n-1}] \otimes_{\mathbb{Z}} \mathbb{Q}$ . The Weyl group  $W(A_{n-1})$  of  $A_{n-1}$  is  $S_n$  acting by (6.1), and the standard base of  $A_{n-1}$  is  $\alpha_1, \dots, \alpha_{n-1}$ , where

$$(6.2) \quad \alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad i = 1, \dots, n-1;$$

see [Bou<sub>2</sub>]. The kernel of augmentation map

$$\mathbb{Z}[S_n/S_{n-1}] \xrightarrow{\epsilon} \mathbb{Z}, \quad \sum_{i=1}^n a_i \varepsilon_i \mapsto \sum_{i=1}^n a_i,$$

is the root  $S_n$ -lattice  $\mathbb{Z}\mathbf{A}_{n-1}$  of  $\mathbf{A}_{n-1}$ ,

$$(6.3) \quad \mathbb{Z}\mathbf{A}_{n-1} := \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_{n-1} = \left\{ \sum_{i=1}^n a_i \varepsilon_i \mid \sum_{i=1}^n a_i = 0 \right\}.$$

The character lattice of  $\mathbf{SL}_n/\boldsymbol{\mu}_d$  is isomorphic to the following  $S_n$ -lattice:

$$(6.4) \quad Q_n(d) := \mathbb{Z}\mathbf{A}_{n-1} + \mathbb{Z}d\varpi_1, \quad \text{where } \varpi_1 = \varepsilon_1 - \frac{1}{n} \sum_{i=1}^n \varepsilon_i.$$

The vector  $\varpi_1$  is the first fundamental dominant weight of the root system  $\mathbf{A}_{n-1}$  with respect to the base  $\alpha_1, \dots, \alpha_{n-1}$ .

Observe that the character lattice of  $\mathbf{SL}_n/\boldsymbol{\mu}_n = \mathbf{PGL}_n$  is the root  $S_n$ -lattice  $Q_n(n) = \mathbb{Z}\mathbf{A}_{n-1}$ , the character lattice of  $\mathbf{SL}_n/\boldsymbol{\mu}_1 = \mathbf{SL}_n$  is the weight  $S_n$ -lattice  $\Lambda_n$  of type  $\mathbf{A}_{n-1}$ , and that the following sequences of homomorphisms of  $S_n$ -lattices are exact:

$$(6.5) \quad 0 \longrightarrow \mathbb{Z}\mathbf{A}_{n-1} \longrightarrow Q_n(n/d) \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0,$$

$$(6.6) \quad 0 \longrightarrow Q_n(d) \longrightarrow \Lambda_n \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0.$$

Here  $\mathbb{Z}/d\mathbb{Z}$  denotes the cyclic group of order  $d$  with trivial  $S_d$ -action. Note that

$$(6.7) \quad Q_n(d)^* \simeq Q_n(n/d).$$

In this section we will prove a number of preliminary results about the lattices  $Q_n(d)$ . In the next section we will use these results to prove Proposition 5.1.

**6.2. Properties of  $Q_n(d)$ .** We begin by recalling a simple lemma which computes the cohomology  $H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1})$  for all subgroups  $\Gamma$  of  $S_n$ . The first part is extracted from [LL, Lemma 4.3].

**Lemma 6.8.** *For any subgroup  $\Gamma$  of  $S_n$ , we have*

$$H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1}) \simeq \mathbb{Z} / \sum_{\mathcal{O}} |\mathcal{O}| \mathbb{Z},$$

where  $\mathcal{O}$  runs over the orbits of  $\Gamma$  in  $\{1, \dots, n\}$ . More explicitly, the connecting homomorphism of the cohomology sequence induced by the augmentation sequence

$$(6.9) \quad 0 \longrightarrow \mathbb{Z}\mathbf{A}_{n-1} \longrightarrow \mathbb{Z}[S_n/S_{n-1}] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is given by

$$\mathbb{Z} = \mathbb{Z}[S_n/S_{n-1}] / \mathbb{Z}\mathbf{A}_{n-1} \xrightarrow{\partial} H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1}), \quad m\varepsilon_1 + \mathbb{Z}\mathbf{A}_{n-1} \mapsto [\sigma \mapsto m(\varepsilon_{\sigma(1)} - \varepsilon_1)],$$

where the image is the class of the given 1-cocycle from  $\Gamma$  to  $\mathbb{Z}\mathbf{A}_{n-1}$ .

*Proof.* From the cohomology sequence that is associated with (6.9), one obtains the exact sequence  $\mathbb{Z}[S_n/S_{n-1}]^\Gamma \xrightarrow{\epsilon} \mathbb{Z} \xrightarrow{\partial} H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1}) \rightarrow 0$  which implies the asserted description of  $H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1})$ . The calculation of the connecting homomorphism  $\partial$  follows directly from the identification of  $\mathbb{Z}$  with  $\mathbb{Z}[S_n/S_{n-1}] / \mathbb{Z}\mathbf{A}_{n-1}$  and an application of the Snake Lemma.  $\square$

**Lemma 6.10.** *For any subgroup  $\Gamma$  of  $S_n$ , the exact sequence (6.5) induces the following connecting homomorphism in cohomology:*

$$\mathbb{Z}/d\mathbb{Z} = Q_n(n/d) / \mathbb{Z}\mathbf{A}_{n-1} \xrightarrow{\partial} H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1}), \quad m + d\mathbb{Z} \mapsto \frac{mn}{d} + \sum_{\mathcal{O}} |\mathcal{O}| \mathbb{Z},$$

where the sum on the right runs over the orbits  $\mathcal{O}$  of  $\Gamma$  in  $\{1, \dots, n\}$ . In particular, if  $|H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1})|$  divides  $n/d$ , then  $\partial$  is the zero map.

*Proof.* Since  $Q_n(n/d)$  has  $\mathbb{Z}$ -basis  $\frac{n}{d}\varpi_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-2} - \varepsilon_{n-1}$  where  $\varpi_1$  is given by (6.4), we conclude that  $Q_n(n/d)/\mathbb{Z}\mathbf{A}_{n-1}$  is generated by  $\frac{n}{d}\varpi_1 + \mathbb{Z}\mathbf{A}_{n-1}$ . Using the Snake Lemma, one sees that the connecting homomorphism

$$\mathbb{Z}/d\mathbb{Z} = Q_n(n/d)/\mathbb{Z}\mathbf{A}_{n-1} \xrightarrow{\partial} H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1})$$

sends  $\frac{n}{d}\varpi_1 + \mathbb{Z}\mathbf{A}_{n-1}$  to the class of the 1-cocycle  $[\sigma \mapsto \frac{n}{d}(\varepsilon_{\sigma(1)} - \varepsilon_1)]$  in  $H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1})$ . An application of Lemma 6.8 and the identification  $\mathbb{Z}/d\mathbb{Z} = Q_n(n/d)/\mathbb{Z}\mathbf{A}_{n-1}$  completes the proof of the first statement. The second statement follows directly from the first.  $\square$

**Lemma 6.11.** *Let  $\Gamma$  be a subgroup of  $S_n$  which fixes at least one integer  $i \in \{1, \dots, n\}$ . Then  $H^1(\Gamma, Q_n(d)) = 0$ .*

*Proof.* Note that in this case,  $\{\varepsilon_t - \varepsilon_i \mid t \neq i\}$  is a permutation basis for  $\mathbb{Z}\mathbf{A}_{n-1}$  so that both  $\mathbb{Z}\mathbf{A}_{n-1}$  and  $\Lambda_n = (\mathbb{Z}\mathbf{A}_{n-1})^*$  are permutation  $\Gamma$ -lattices. This implies that  $H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1}) = 0 = H^1(\Gamma, \Lambda_n)$ . Observe that  $\nu_i = \varepsilon_i - \frac{1}{n} \sum_{t=1}^n \varepsilon_t \in \Lambda_n^\Gamma$  and that  $\nu_i + Q_n(d) = \varpi_1 + Q_n(d)$  since  $\nu_i - \varpi_1 = \varepsilon_i - \varepsilon_1 \in \mathbb{Z}\mathbf{A}_{n-1} \subseteq Q_n(d)$ . Then applying cohomology to the exact sequence (6.6), we obtain

$$\Lambda_n^\Gamma \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow H^1(\Gamma, Q_n(d)) \longrightarrow H^1(\Gamma, \Lambda_n) = 0.$$

Since  $\Lambda_n/Q_n(d) = \mathbb{Z}/d\mathbb{Z}$  is generated by  $\varpi_1 + Q_n(d)$ , the above argument shows that the map  $\Lambda_n^\Gamma \rightarrow \mathbb{Z}/d\mathbb{Z}$  is surjective so that  $H^1(\Gamma, Q_n(d)) = 0$ , as required.  $\square$

For a sequence of integers  $1 \leq i_1 < \dots < i_r \leq n$ , set

$$S_{\{i_1, \dots, i_r\}} := \{\sigma \in S_n \mid \sigma(j) = j \text{ for every } j \notin \{i_1, \dots, i_r\}\}.$$

This is a subgroup of  $S_n$ ; in particular,  $S_{\{1, \dots, n\}} = S_n$ . The map

$$\iota_{\{i_1, \dots, i_r\}} : S_r \longrightarrow S_{\{i_1, \dots, i_r\}}, \quad \iota_{\{i_1, \dots, i_r\}}(\sigma)(i_s) = i_{\sigma(s)} \quad \text{for all } \sigma \text{ and } s,$$

is an isomorphism. In what follows, the subgroup  $S_{\{1, \dots, m\}} \times S_{\{m+1, \dots, 2m\}}$  of  $S_{2m}$  is denoted simply by  $S_m \times S_m$ . For a sequence of integers

$$1 \leq i_1 < \dots < i_r < j_1 < \dots < j_r < \dots < l_1 < \dots < l_r \leq n,$$

the image of the embedding

$$S_r \longrightarrow S_n, \quad \sigma \mapsto \iota_{\{i_1, \dots, i_r\}}(\sigma) \iota_{\{j_1, \dots, j_r\}}(\sigma) \dots \iota_{\{l_1, \dots, l_r\}}(\sigma),$$

is called the *copy of  $S_r$  diagonally embedded in  $S_{\{i_1, \dots, i_r, j_1, \dots, j_r, \dots, l_1, \dots, l_r\}}$* .

**Lemma 6.12.** *Let  $n = td$ . Then the following properties hold:*

(a) *Let  $X_d$  be the copy of  $S_d$  diagonally embedded in  $S_n$ . Then*

$$\mathbb{Z}\mathbf{A}_{n-1}|_{X_d} \simeq \mathbb{Z}\mathbf{A}_{d-1} \oplus \mathbb{Z}[S_d/S_{d-1}]^{t-1}.$$

(b) *Let  $Y_d := S_{\{1, \dots, d\}} \times \tilde{X}_d$  where  $\tilde{X}_d$  is the copy of  $S_d$  diagonally embedded in  $S_{\{d+1, \dots, n\}}$ . Then*

$$\mathbb{Z}\mathbf{A}_{n-1}|_{Y_d} \simeq \mathbb{Z}\mathbf{A}_{2d-1}|_{S_d \times S_d} \oplus \mathbb{Z}[(S_d \times S_d)/(S_d \times S_{d-1})]^{t-2}.$$

*Proof.* For the first statement, note that

$$\mathcal{B}_1 = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{d-1} - \varepsilon_d\} \cup \{\varepsilon_i - \varepsilon_{d+i} \mid i = 1, \dots, (t-1)d\}$$

is a basis for  $\mathbb{Z}\mathbf{A}_{n-1}$ , since  $\mathcal{B}_0 = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n-1\}$  is a basis for  $\mathbb{Z}\mathbf{A}_{n-1}$ , and the equations

$$\varepsilon_i - \varepsilon_{d+i} = \sum_{t=i}^{d+i-1} \alpha_k$$

for  $i = 1, \dots, (t-1)d$  show that the change of coordinates matrix relating  $\mathcal{B}_1$  to  $\mathcal{B}_0$  is upper triangular with coefficients in  $\mathbb{Z}$  and diagonal entries 1. But then

$$\begin{aligned} \mathbb{Z}\mathbf{A}_{n-1}|_{X_d} &= \bigoplus_{i=1}^{d-1} \mathbb{Z}(\varepsilon_i - \varepsilon_{i+1}) \oplus \bigoplus_{r=1}^{t-1} \left( \bigoplus_{i=(r-1)d+1}^{rd} \mathbb{Z}(\varepsilon_i - \varepsilon_{d+i}) \right) \\ &\simeq \mathbb{Z}\mathbf{A}_{d-1} \oplus \mathbb{Z}[\mathbb{S}_d/\mathbb{S}_{d-1}]^{t-1}. \end{aligned}$$

For the second statement, similarly note that

$$\{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{2d-1} - \varepsilon_{2d}\} \cup \{\varepsilon_i - \varepsilon_{d+i} \mid i = d+1, \dots, (t-1)d\}$$

is a basis for  $\mathbb{Z}\mathbf{A}_{n-1}$  so that

$$\begin{aligned} \mathbb{Z}\mathbf{A}_{n-1}|_{Y_d} &= \bigoplus_{i=1}^{2d-1} \mathbb{Z}(\varepsilon_i - \varepsilon_{i+1}) \oplus \bigoplus_{r=2}^{t-1} \left( \bigoplus_{i=(r-1)d+1}^{rd} \mathbb{Z}(\varepsilon_i - \varepsilon_{d+i}) \right) \\ &\simeq \mathbb{Z}\mathbf{A}_{2d-1}|_{\mathbb{S}_d \times \mathbb{S}_d} \oplus \mathbb{Z}[(\mathbb{S}_d \times \mathbb{S}_d)/(\mathbb{S}_d \times \mathbb{S}_{d-1})]^{t-2}. \quad \square \end{aligned}$$

## 7. STABLY CAYLEY GROUPS OF TYPE $\mathbf{A}_n$

**7.1. Restricting  $Q_n(d)$  to some subgroups.** In this section we will prove Proposition 5.1. We will first show that  $Q_n(d)$  restricted to certain appropriate subgroups of  $\mathbb{S}_n$  is equivalent in each case to a smaller more manageable sublattice. We will then show that the smaller lattices are not quasi-permutation.

**Proposition 7.1.** *Suppose  $d|n$  and let  $p$  be a prime divisor of  $n/d$ . Let  $X_p$  be the copy of  $\mathbb{S}_p$  diagonally embedded in  $\mathbb{S}_n$ , and let  $Y_p = \mathbb{S}_{\{1, \dots, p\}} \times \tilde{X}_p$ , where  $\tilde{X}_p$  is the copy of  $\mathbb{S}_p$  diagonally embedded in  $\mathbb{S}_{\{p+1, \dots, n\}}$ . Then the following equivalencies hold:*

- (a)  $Q_n(d)|_{X_p} \sim \Lambda_p$ .
- (b)  $Q_n(d)|_{Y_p} \sim \Lambda_{2p}|_{\mathbb{S}_p \times \mathbb{S}_p}$ .

*Proof.* Recall that we have the exact sequence (6.5). The definition of  $p$  implies that  $n = lp$  for a positive integer  $l$ . By Lemma 6.12,

$$\begin{aligned} \mathbb{Z}\mathbf{A}_{n-1}|_{X_p} &\simeq \mathbb{Z}\mathbf{A}_{p-1} \oplus \mathbb{Z}[\mathbb{S}_p/\mathbb{S}_{p-1}]^{l-1}, \\ \mathbb{Z}\mathbf{A}_{n-1}|_{Y_p} &\simeq \mathbb{Z}\mathbf{A}_{2p-1}|_{\mathbb{S}_p \times \mathbb{S}_p} \oplus \mathbb{Z}[(\mathbb{S}_p \times \mathbb{S}_p)/(\mathbb{S}_p \times \mathbb{S}_{p-1})]^{l-2}. \end{aligned}$$

Using this and Lemma 6.8, we see that  $H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1}) = H^1(\Gamma, \mathbb{Z}\mathbf{A}_{p-1}) = 0$  or  $\mathbb{Z}/p\mathbb{Z}$  for all subgroups  $\Gamma$  of  $X_p$  and that  $H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1}) = H^1(\Gamma, \mathbb{Z}\mathbf{A}_{2p-1}) = 0$  or  $\mathbb{Z}/p\mathbb{Z}$  for all subgroups  $\Gamma$  of  $Y_p$ . Then Lemma 6.10 and the fact that  $p$  divides  $n/d$  show that the connecting homomorphism  $\mathbb{Z}/d\mathbb{Z} \rightarrow H^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1})$  is zero for all subgroups  $\Gamma$  of  $X_p$  or of  $Y_p$ . But then the sequence (6.5) restricted to  $X_p$  or  $Y_p$  satisfies the conditions of Proposition 2.10(b). This means that

$$\begin{aligned} Q_n(d)|_{X_p} &= Q_n(n/d)^*|_{X_p} \sim (\mathbb{Z}\mathbf{A}_{n-1})^*|_{X_p} \sim (\mathbb{Z}\mathbf{A}_{p-1})^* = \Lambda_p, \\ Q_n(d)|_{Y_p} &= Q_n(n/d)^*|_{Y_p} \sim (\mathbb{Z}\mathbf{A}_{n-1})^*|_{Y_p} \sim (\mathbb{Z}\mathbf{A}_{2p-1})^*|_{\mathbb{S}_p \times \mathbb{S}_p} = \Lambda_{2p}|_{\mathbb{S}_p \times \mathbb{S}_p}. \quad \square \end{aligned}$$

**7.2. Lattices  $\Lambda_p$  and  $\Lambda_{2p}$ .** The following lemma is essentially a rephrasing of a result proved by BESSENRODT and LE BRUYN in [BLB]:

**Lemma 7.2.** *Let  $p > 3$  be prime. Then  $\Lambda_p$  is not a quasi-permutation  $S_p$ -lattice.*

*Proof.* Tensoring the augmentation sequence for  $\mathbb{Z}[S_n/S_{n-1}]$  with  $\mathbb{Z}\mathbf{A}_{n-1}$ , we obtain the exact sequence

$$(7.3) \quad 0 \longrightarrow (\mathbb{Z}\mathbf{A}_{n-1})^{\otimes 2} \longrightarrow \mathbb{Z}\mathbf{A}_{n-1} \otimes \mathbb{Z}[S_n/S_{n-1}] \xrightarrow{\tau} \mathbb{Z}\mathbf{A}_{n-1} \longrightarrow 0.$$

We have

$$\mathbb{Z}\mathbf{A}_{n-1} \otimes \mathbb{Z}[S_n/S_{n-1}] \simeq \mathbb{Z}[S_n/S_{n-2}].$$

One can show that  $\{(\varepsilon_i - \varepsilon_j) \otimes \varepsilon_j \mid i \neq j\}$  is the set of elements of a permutation basis for  $\mathbb{Z}\mathbf{A}_{n-1} \otimes \mathbb{Z}[S_n/S_{n-1}]$ . The map  $\tau$  then sends  $(\varepsilon_i - \varepsilon_j) \otimes \varepsilon_j$  to  $\varepsilon_i - \varepsilon_j$ .

For  $p$  prime, BESSENRODT and LE BRUYN in [BLB] show that

$$0 \longrightarrow (\mathbb{Z}\mathbf{A}_{p-1})^{\otimes 2} \longrightarrow \mathbb{Z}[S_p/S_{p-2}] \longrightarrow \mathbb{Z}\mathbf{A}_{p-1} \longrightarrow 0$$

is a coflasque resolution of  $\mathbb{Z}\mathbf{A}_{p-1}$  as an  $S_p$ -lattice. They also show that  $(\mathbb{Z}\mathbf{A}_{p-1})^{\otimes 2}$  is permutation projective as an  $S_p$ -lattice but is only  $S_p$ -stably permutation if  $p = 2, 3$ . By duality, the stable equivalence class of  $((\mathbb{Z}\mathbf{A}_{p-1})^{\otimes 2})^*$  is  $\rho(\Lambda_p)$ ; see Subsection 2.3). The statements above then imply that  $\Lambda_p$  is not a quasi-permutation  $S_p$ -lattice for any  $p > 3$ .  $\square$

**Proposition 7.4.** *Let  $p$  be a prime and let*

$$\Gamma := \langle (1, \dots, p), (p+1, \dots, 2p) \rangle \leq S_p \times S_p \leq S_{2p}.$$

*Then the following hold.*

- (a)  $\text{III}^2(\Gamma, \Lambda_{2p}) = 0$ . *In particular, a lattice in the stable equivalence class  $\rho(\Lambda_{2p})$  is coflasque as an  $\Gamma$ -lattice.*
- (b) *If  $p$  is odd,  $\Lambda_{2p}$  is not quasi-permutation as an  $\Gamma$ -lattice and hence is not quasi-permutation as an  $S_p \times S_p$ -lattice.*

*Proof.* (a) The second statement follows from the first. Note that any proper subgroup of  $\Gamma$  is cyclic, so that by the claim  $\text{III}^2(S, \Lambda_{2p}) = 0$  for all subgroups  $S$  of  $\Gamma$ . Then if

$$0 \longrightarrow \Lambda_{2p} \longrightarrow Q \longrightarrow M \longrightarrow 0$$

is a flasque resolution of  $\Lambda_{2p}$  considered as an  $S$ -lattice, then  $H^1(S, M) = \text{III}^2(S, \Lambda_{2p}) = 0$  by Lemma 2.9.

To prove the first statement, we need to first compute  $H^1(\Gamma, \Lambda_{2p})$  and  $H^2(\Gamma, \Lambda_{2p})$ .

We have  $H^1(\Gamma, \Lambda_{2p}) = H^{-1}(\Gamma, \mathbb{Z}\mathbf{A}_{2p-1})$  by duality. Then

$$H^{-1}(\Gamma, \mathbb{Z}\mathbf{A}_{2p-1}) = \text{Ker}_{\mathbb{Z}\mathbf{A}_{2p-1}}(N_\Gamma)/I_\Gamma\mathbb{Z}\mathbf{A}_{2p-1},$$

where  $N_\Gamma$  is the endomorphism  $l \mapsto \sum_{a \in \Gamma} al$ , and  $I_\Gamma$  is the augmentation ideal of  $\mathbb{Z}[\Gamma]$  ([Br]). We need to compute  $N_\Gamma$  on a basis for  $\mathbb{Z}\mathbf{A}_{2p-1}$ : we have  $N_\Gamma(\varepsilon_i - \varepsilon_{i+1}) = 0$  for  $i = 1, \dots, p-1, p+1, \dots, 2p-1$ , and  $N_\Gamma(\varepsilon_p - \varepsilon_{p+1}) = p(\varepsilon_1 + \dots + \varepsilon_p - \varepsilon_{p+1} - \dots - \varepsilon_{2p})$ . Then

$$\text{Ker } N_\Gamma = \text{Span}\{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{p-1} - \varepsilon_p, \varepsilon_{p+1} - \varepsilon_{p+2}, \dots, \varepsilon_{2p-1} - \varepsilon_{2p}\}.$$

But  $I_\Gamma\mathbb{Z}\mathbf{A}_{2p-1} = \text{Ker } N_\Gamma$  as  $((1, \dots, p) - \text{id})(\varepsilon_{p+1} - \varepsilon_i) = \varepsilon_i - \varepsilon_{i+1}$ ,  $i = 1, \dots, p-1$ ,  $((p+1, \dots, 2p) - \text{id})(\varepsilon_1 - \varepsilon_i) = \varepsilon_i - \varepsilon_{i+1}$ ,  $i = p+1, \dots, 2p-1$ . This shows that  $H^1(\Gamma, \Lambda_{2p}) = H^{-1}(\Gamma, \mathbb{Z}\mathbf{A}_{2p-1}) = 0$ .

To determine  $H^2(\Gamma, \Lambda_{2p})$ , we use the restriction of the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[S_{2p}/S_{2p-1}] \longrightarrow \Lambda_{2p} \longrightarrow 0$$

to  $\Gamma$ . Let

$$(7.5) \quad C_1 = \langle (1, \dots, p) \rangle, \quad C_2 = \langle (p+1, \dots, 2p) \rangle \quad \text{and} \quad P_1 = \mathbb{Z}[\Gamma/C_2], \quad P_2 = \mathbb{Z}[\Gamma/C_1].$$

Then we have the following exact sequence of  $\Gamma$ -lattices:

$$0 \longrightarrow \mathbb{Z} \longrightarrow P_1 \oplus P_2 \longrightarrow \Lambda_{2p} \longrightarrow 0.$$

Taking cohomology of this sequence, we get

$$\begin{aligned} 0 = H^1(\Gamma, \Lambda_{2p}) &\longrightarrow H^2(\Gamma, \mathbb{Z}) \longrightarrow H^2(\Gamma, P_1) \oplus H^2(\Gamma, P_2) \\ &\longrightarrow H^2(\Gamma, \Lambda_{2p}) \longrightarrow H^3(\Gamma, \mathbb{Z}) \longrightarrow H^3(\Gamma, P_1 \oplus P_2). \end{aligned}$$

But by Shapiro's Lemma, we have  $H^2(\Gamma, P_i) = H^2(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$  and  $H^3(\Gamma, P_i) = H^3(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) = 0$  for  $i = 1, 2$ . Also, by the Künneth formula, [Weib, p. 166],

$$\begin{aligned} H^n(\Gamma, \mathbb{Z}) &= \bigoplus_{i+j=n} H^i(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \otimes H^j(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}) \\ &\quad \oplus \bigoplus_{i+j=n+1} \text{Tor}_{\mathbb{Z}}^1(H^i(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}), H^j(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})), \end{aligned}$$

so that, in particular,  $H^3(\Gamma, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$  and  $H^2(\Gamma, \mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^2$ . This all yields an exact sequence

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow H^2(\Gamma, \Lambda_{2p}) \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0,$$

and so  $H^2(\Gamma, \Lambda_{2p}) = \mathbb{Z}/p\mathbb{Z}$ .

To show that  $\text{III}^2(\Gamma, \Lambda_{2p}) = 0$ , it would suffice to find a cyclic subgroup  $C$  of  $\Gamma$  for which  $\text{Res}_C^\Gamma : H^2(\Gamma, \Lambda_{2p}) \rightarrow H^2(C, \Lambda_{2p})$  is injective.

Take  $C = C_1$ . Since  $H^1(\Gamma, \Lambda_{2p}) = 0$ , we have that the sequence

$$0 \longrightarrow H^2(\Gamma/C, \Lambda_{2p}^C) \xrightarrow{\text{Inf}} H^2(\Gamma, \Lambda_{2p}) \xrightarrow{\text{Res}} H^2(C, \Lambda_{2p})$$

is exact. So it suffices to show that  $H^2(\Gamma/C, \Lambda_{2p}^C) = 0$ .

The fundamental dominant weights for  $\Lambda_{2p}$  are

$$\varpi_t = \sum_{i=1}^t \varepsilon_i - \frac{t}{2p} \sum_{i=1}^{2p} \varepsilon_i, \quad t = 1, \dots, 2p-1.$$

Let  $\nu_i = \varepsilon_i - \frac{1}{2p} \sum_{i=1}^{2p} \varepsilon_i$ ,  $i = 1, \dots, 2p$ . Note that

$$\nu_1 = \varpi_1, \quad \nu_t = \varpi_t - \varpi_{t-1}, \quad t = 2, \dots, 2p-1, \quad \nu_{2p} = -\varpi_{2p-1}.$$

This shows that  $\nu_1, \dots, \nu_p, \varpi_{p+1}, \dots, \varpi_{2p-1}$  is another basis for  $\Lambda_{2p}$  and that

$$\Lambda_{2p}|_C = \bigoplus_{i=1}^p \mathbb{Z}\nu_i \oplus \bigoplus_{i=p+1}^{2p-1} \mathbb{Z}\varpi_i \simeq \mathbb{Z}[C] \oplus \mathbb{Z}^{p-1}.$$

This shows that

$$\Lambda_{2p}^C = \mathbb{Z}(\sum_{i=1}^p \nu_i) \oplus \bigoplus_{i=p+1}^{2p-1} \mathbb{Z}\varpi_i = \bigoplus_{i=p}^{2p-1} \mathbb{Z}\varpi_i = \bigoplus_{i=p+1}^{2p} \mathbb{Z}\nu_i.$$

But  $\Gamma/C$  permutes  $\nu_{p+1}, \dots, \nu_{2p}$  cyclically so that  $\Lambda_{2p}^C \simeq \mathbb{Z}[\Gamma/C]$ . This implies that  $H^2(\Gamma/C, \Lambda_{2p}^C) = 0$  as required.

(b) To prove that  $\Lambda_{2p}$  is not  $\Gamma$ -quasi-permutation, we will construct a coflasque  $\Gamma$ -resolution of  $\mathbb{Z}\Lambda_{2p-1}$ . By duality, this will give us a flasque resolution of  $\Lambda_{2p}$ . We will then show that the lattice in the stable equivalence class  $\rho(\Lambda_{2p})$  is not permutation projective as a  $\Gamma$ -lattice.

As  $\alpha_1, \dots, \alpha_{p-1}$  and  $\alpha_{p+1}, \dots, \alpha_{2p-1}$  are the standard bases of the root subsystems of type  $A_{p-1}$ , we denote the  $\Gamma$ -sublattice of  $\mathbb{Z}A_{2p-1}$  generated by them simply by  $\mathbb{Z}A_{p-1} \oplus \mathbb{Z}A_{p-1}$ . Let  $\iota$  be its natural embedding into  $\mathbb{Z}A_{2p-1}$ . It is easily seen that  $\alpha_p + \mathbb{Z}A_{p-1} \oplus \mathbb{Z}A_{p-1}$  is  $\Gamma$ -stable. This implies that there is an exact sequence of  $\Gamma$ -lattices

$$0 \longrightarrow \mathbb{Z}A_{p-1} \oplus \mathbb{Z}A_{p-1} \xrightarrow{\iota} \mathbb{Z}A_{2p-1} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

A coflasque resolution of the  $\Gamma$ -lattice  $\mathbb{Z}A_{p-1} \oplus \mathbb{Z}A_{p-1}$  is given by

$$0 \longrightarrow \mathbb{Z}^2 \longrightarrow P_1 \oplus P_2 \longrightarrow \mathbb{Z}A_{p-1} \oplus \mathbb{Z}A_{p-1} \longrightarrow 0$$

where  $P_1$  and  $P_2$  are defined by (7.5) and the generator of the  $\Gamma$ -lattice  $P_1$  (respectively,  $P_2$ ) is sent to  $\alpha_1$  (respectively,  $\alpha_{p+1}$ ).

We now extend  $\iota$  to a coflasque resolution of the  $\Gamma$ -lattice  $\mathbb{Z}A_{2p-1}$ . Let

$$P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}A_{2p-1}$$

be a map of  $\Gamma$ -lattices where  $\pi_{P_1 \oplus P_2} = \iota$ ,  $\pi$  sends  $1 \in \mathbb{Z}[\Gamma]$  to  $\alpha_p$ , and  $\pi$  sends the  $1 \in \mathbb{Z}$  to  $\sum_{i=1}^p \varepsilon_i - \sum_{i=p+1}^{2p} \varepsilon_i = 2\varpi_p$ . It is easily verified that  $\pi$  is surjective (in fact  $\pi|_{\mathbb{Z}[\Gamma]}$  is surjective).

Let  $L = \text{Ker } \pi$ . To check that  $L$  is coflasque and hence that

$$0 \longrightarrow L \longrightarrow P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}A_{2p-1} \longrightarrow 0$$

is a coflasque resolution of  $\mathbb{Z}A_{2p-1}$ , we need only verify that for  $R := P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z}$ , we have  $\pi(R^K) = (\mathbb{Z}A_{2p-1})^K$  for all subgroups  $K$  of  $\Gamma$ .

For  $K = \Gamma$  or a cyclic subgroup generated by a disjoint product of two  $p$ -cycles,  $(\mathbb{Z}A_{2p-1})^K = \mathbb{Z}2\varpi_p$  so that  $\pi(\mathbb{Z}^K) = \pi(\mathbb{Z}) = (\mathbb{Z}A_{2p-1})^K$  and so  $\pi(R^K) = (\mathbb{Z}A_{2p-1})^K$ .

The only other subgroups are  $C_1$  and  $C_2$ . As the arguments are similar, we just consider  $C_1$ : the lattice  $(\mathbb{Z}A_{2p-1})^{C_1}$  has basis  $2\varpi_p, \alpha_{p+1}, \dots, \alpha_{2p-1}$ , and we have  $\pi(\mathbb{Z}) = \mathbb{Z}2\varpi_2$  and  $\pi(P_2^{C_1}) = \pi(P_2) = \bigoplus_{i=p+1}^{2p-1} \mathbb{Z}\alpha_i$ . This shows that

$$0 \longrightarrow L \longrightarrow P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}A_{2p-1} \longrightarrow 0$$

is a coflasque resolution. Dualizing, we obtain a flasque resolution for  $\Lambda_{2p}$ :

$$0 \longrightarrow \Lambda_{2p} \longrightarrow P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} \longrightarrow L^* \longrightarrow 0.$$

We have  $H^1(\Gamma, L^*) = \text{III}^2(\Gamma, \Lambda_{2p}) = 0$ . This shows that  $L$  is flasque and coflasque as a  $\Gamma$ -lattice.

We have the following commutative diagram with exact rows and columns:

$$(7.6) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & P_1 \oplus P_2 & \xrightarrow{\iota} & \mathbb{Z}A_{p-1} \oplus \mathbb{Z}A_{p-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L & \longrightarrow & P_1 \oplus P_2 \oplus \mathbb{Z}[\Gamma] \oplus \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}A_{2p-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U(p) & \longrightarrow & \mathbb{Z}[\Gamma] \oplus \mathbb{Z} & \xrightarrow{\theta} & \mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where  $U(p)$  is the kernel of the induced map  $\theta$ . Now  $2\varpi_p = \sum_{i=1}^{p-1} i(\alpha_i + \alpha_{2p-i}) + p\alpha_p$ . So  $\theta$  sends  $1 \in \mathbb{Z}[\Gamma]$  to  $\overline{\alpha_p}$  and sends  $1 \in \mathbb{Z}$  to  $p\overline{\alpha_p}$ . This shows that

$$\{(h-1, 0) \mid h \in \Gamma\} \cup \{(-p, 1)\}$$

is a  $\mathbb{Z}$ -basis for  $U(p)$ . Note that  $U(p)$  also satisfies

$$0 \longrightarrow U(p) \longrightarrow \mathbb{Z}[\Gamma] \longrightarrow \mathbb{Z}/p\mathbb{Z},$$

so that  $\mathbb{Q}U(p) \simeq \mathbb{Q}[\Gamma]$ .

From the above diagram, we then see that  $\mathbb{Q}L \simeq \mathbb{Q}[\Gamma] \oplus \mathbb{Q}^2$ . By [CW, Lemmas 2 and 3], to determine whether or not  $L$  is permutation projective is equivalent to checking whether  $\mathbb{F}_p L$  is a permutation module for  $\mathbb{F}_p[\Gamma]$ .

Tensoring the diagram (7.6) with  $\mathbb{F}_p$  leaves it exact so we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{F}_p^2 & \longrightarrow & \mathbb{F}_p P_1 \oplus \mathbb{F}_p P_2 & \xrightarrow{\text{id} \otimes \iota} & \mathbb{F}_p \mathbf{A}_{p-1} \oplus \mathbb{F}_p \mathbf{A}_{p-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{F}_p L & \longrightarrow & \mathbb{F}_p P_1 \oplus \mathbb{F}_p P_2 \oplus \mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p & \xrightarrow{\text{id} \otimes \pi} & \mathbb{F}_p \mathbf{A}_{2p-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{F}_p U(p) & \longrightarrow & \mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p & \xrightarrow{\text{id} \otimes \theta} & \mathbb{F}_p \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Suppose that  $\mathbb{F}_p L$  is permutation. Then since  $L$  is coflasque, the sequence

$$0 \longrightarrow L^\Gamma \xrightarrow{p} L^\Gamma \longrightarrow (L/pL)^\Gamma \longrightarrow 0$$

is exact so that  $(\mathbb{F}_p L)^\Gamma = L^\Gamma/pL^\Gamma$ . Since  $\mathbb{Q}[L] \simeq \mathbb{Q}[\Gamma] \oplus \mathbb{Q}^2$ ,  $\text{rank } L^\Gamma = 3$ . But then  $\dim_{\mathbb{F}_p} (\mathbb{F}_p L)^\Gamma = 3$ . This means that  $\mathbb{F}_p L$  must then have three transitive components. Since  $\text{rank } L = p^2 + 2$  and  $p > 2$ , this means that  $\mathbb{F}_p L = \mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p^2$ .

Looking at the  $\mathbb{Z}$ -basis for  $U(p)$  given above, it is clear that  $\mathbb{F}_p U(p) \simeq \mathbb{F}_p \oplus \mathbb{F}_p I_\Gamma$  where  $\mathbb{F}_p I_\Gamma$  is the augmentation ideal of  $\mathbb{F}_p[\Gamma]$ . Then the left column of the last commutative diagram implies that we have a surjective map  $\mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p^2 \rightarrow \mathbb{F}_p \oplus \mathbb{F}_p I_\Gamma$ . Since  $(\mathbb{F}_p I_\Gamma)^\Gamma = 0$ , this would imply that we have a surjective map  $\mathbb{F}_p[\Gamma] \rightarrow \mathbb{F}_p I_\Gamma$  or equivalently that  $\mathbb{F}_p I_\Gamma$  is a cyclic  $\mathbb{F}_p[\Gamma]$ -module. But since  $\Gamma$  is a finite  $p$ -group,  $\mathbb{F}_p[\Gamma]$  is a local ring with unique maximal ideal  $\mathbb{F}_p I_\Gamma$  by [Car, Corollary 1.4]. Then Nakayama's Lemma implies that  $\mathbb{F}_p I_\Gamma$  is a cyclic  $\mathbb{F}_p[\Gamma]$ -module if and only if  $\mathbb{F}_p I_\Gamma / (\mathbb{F}_p I_\Gamma)^2$  is generated by one element over  $\mathbb{F}_p$ . Since  $\Gamma \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ , we may use the Künneth formula to show that

$$\mathbb{F}_p I_\Gamma / \mathbb{F}_p I_\Gamma^2 = H_1(\Gamma, \mathbb{F}_p) \cong H_1(\mathbb{Z}/p\mathbb{Z}, \mathbb{F}_p)^2 \cong \mathbb{F}_p^2.$$

Alternatively, for any  $p$ -group  $H$ , we may show that

$$\mathbb{F}_p I_H / \mathbb{F}_p I_H^2 \longrightarrow H/H^p[H, H], \quad \overline{h-1} \mapsto \overline{h}$$

is a group isomorphism so that, in our case, we again have

$$\mathbb{F}_p I_\Gamma / \mathbb{F}_p I_\Gamma^2 \cong \mathbb{F}_p^2.$$

Then the above discussion shows that  $\mathbb{F}_p I_\Gamma$  is not a cyclic  $\mathbb{F}_p[\Gamma]$  module so that there is no surjective map from  $\mathbb{F}_p[\Gamma]$  to  $\mathbb{F}_p I_\Gamma$ . This implies that  $\mathbb{F}_p L$  is not permutation and hence

$L$  is not permutation projective as a  $\mathbb{Z}[\Gamma]$ -module. This implies in turn that  $\Lambda_{2p}$  is not quasi-permutation as a  $\Gamma$ -lattice or as an  $S_p \times S_p$ -lattice.  $\square$

**Remark 7.7.** Note that this argument fails for  $p = 2$ . Indeed, we showed that  $\text{rank } L = p^2 + 2$  and if  $\mathbb{F}_p L$  were permutation, it would have three transitive components. For  $p > 2$ , we used these facts to conclude that  $\mathbb{F}_p L = \mathbb{F}_p[\Gamma] \oplus \mathbb{F}_p^2$ . For  $p = 2$ , this is not so; here  $\mathbb{F}_2 L$  may have three permutation components, each of rank 2. Indeed, if  $\Gamma = \langle g, h \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , then one can define a surjective  $\mathbb{F}_2[\Gamma]$ -homomorphism

$$\mathbb{F}_2[\Gamma/\langle g \rangle] \oplus \mathbb{F}_2[\Gamma/\langle h \rangle] \oplus \mathbb{F}_2[\Gamma/\langle gh \rangle] \rightarrow \mathbb{F}_2 I_\Gamma \oplus \mathbb{F}_2$$

by sending the generator of the first component to  $(1 + g, 0)$ , the generator of the second component to  $(1 + h, 0)$ , and that of the third component to  $(0, 1)$ .

In fact, by Proposition 7.1, we see that  $Q_4(2)|_\Gamma \sim \Lambda_4|_\Gamma$ . Since  $Q_4(2)$  is the character lattice of the Cayley group  $\mathbf{SL}_4/\mu_2 \simeq \mathbf{SO}_6$ , by Theorem 1.27 it must be quasi-permutation as an  $S_4$ -lattice and hence as a  $\Gamma$ -lattice. Alternatively, one could show directly that  $Q_4(2)$  is a sign-permutation  $S_4$ -lattice and hence it is quasi-permutation.

**7.3. Completion of the proof of Proposition 5.1.** It now suffices to prove the following proposition to complete the proof of Proposition 5.1:

**Proposition 7.8.** *Suppose  $n/d$  is divisible by a prime  $p$ .*

- (a) *If  $p > 2$ , then the  $S_n$ -lattice  $Q_n(d)$  is not quasi-permutation.*
- (b) *If  $n > p^2$ , then the  $S_n$ -lattice  $Q_n(d)$  is not quasi-permutation.*

Indeed, by part (a), the  $S_n$ -lattice  $Q_n(d)$  is not quasi-permutation if the prime factorization of  $n/d$  includes a prime larger than 2. On the other hand, if  $n/d = 2^k$ , then, by part (b), the  $S_n$ -lattice  $Q_n(d)$  is not quasi-permutation, for any  $(n, d) \neq (4, 2)$ , and Proposition 5.1 follows.

*Proof.* (a) Proposition 7.1 shows that  $Q_n(d)|_{Y_p}$  is equivalent to  $\Lambda_{2p}|_{S_p \times S_p}$  which is not quasi-permutation by Proposition 7.4. Thus  $Q_n(d)$  is not quasi-permutation as a  $Y_p$ -lattice and hence as an  $S_n$ -lattice as well.

(b) We have  $n = tp$  with  $t > p$ . Following the proof of Proposition 4.1(i) in [LL], we define a subgroup  $\Gamma \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$  of  $S_n$  as follows. Arrange the numbers from 1 to  $n$  into a rectangular table with  $p$  columns and  $t$  rows, so that the first row is  $1, \dots, p$ , the second row is  $p + 1, \dots, 2p$ , etc. Let  $\sigma_i$  be the  $p$ -cycle that cyclically permutes the  $i$ th row and leaves elements of all other rows fixed. Note that  $\sigma_1, \dots, \sigma_t$  are commuting  $p$ -cycles; explicitly

$$\sigma_i = ((i-1)p + 1, (i-1)p + 2, \dots, ip).$$

We now set  $\Gamma := \langle \alpha, \beta \rangle$ , where

$$\alpha := \prod_{i=1}^{t-1} \sigma_i \quad \text{and} \quad \beta := \prod_{i=1}^{p-1} \sigma_i^{-i} \cdot \prod_{i=p+1}^t \sigma_i.$$

The subgroup  $\Gamma$  has orbits  $\mathcal{O}_i = \{(i-1)p + 1, (i-1)p + 2, \dots, ip\}$ ,  $i = 1, \dots, t$ , all of length  $p$  and every cyclic subgroup  $C$  of  $\Gamma$  has fixed points. This means that by Lemma 6.8

$$H^1(\Gamma, \mathbb{Z}A_{n-1}) \simeq \mathbb{Z}/p\mathbb{Z} \quad \text{but} \quad H^1(C, \mathbb{Z}A_{n-1}) = 0.$$

Also by Lemma 6.11, we find that

$$H^1(C, Q_n(n/d)) = 0.$$

Then Lemma 6.10 and the fact that  $p$  divides  $n/d$  show that  $\mathbb{Z}/d\mathbb{Z} \xrightarrow{\partial} \mathrm{H}^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1})$  is the zero map. The following commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}/d\mathbb{Z} & \xrightarrow{0} & \mathrm{H}^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1}) & \longrightarrow & \mathrm{H}^1(\Gamma, Q_n(n/d)) \\ \downarrow \text{Res} & & \downarrow \text{Res} & & \downarrow \text{Res} \\ \prod_{a \in \Gamma} \mathbb{Z}/d\mathbb{Z} & \xrightarrow{0} & \prod_{a \in \Gamma} \mathrm{H}^1(\langle a \rangle, \mathbb{Z}\mathbf{A}_{n-1}) = 0 & \longrightarrow & \prod_{a \in \Gamma} \mathrm{H}^1(\langle a \rangle, Q_n(n/d)) = 0 \end{array}$$

shows that

$$\mathbb{Z}/p\mathbb{Z} \simeq \mathrm{III}^1(\Gamma, \mathbb{Z}\mathbf{A}_{n-1}) \leq \mathrm{III}^1(\Gamma, Q_n(n/d)).$$

Now if  $M$  were a flasque lattice with  $\rho(Q_n(d)) =$  the stable equivalence class of  $M$ , then  $M^*$  is a coflasque lattice satisfying

$$0 \longrightarrow M^* \longrightarrow P \longrightarrow Q_n(n/d) \longrightarrow 0,$$

so that by Lemma 2.9(a),  $\mathrm{III}^2(\Gamma, M^*) \simeq \mathrm{III}^1(\Gamma, Q_n(n/d)) \neq 0$ . Lemma 2.9(c) now shows that  $M^*$  cannot be a direct summand of a quasi-permutation lattice and hence is not stably permutation. This implies that  $M$  cannot be stably permutation and so  $Q_n(d)$  cannot be quasi-permutation.  $\square$

## 8. STABLY CAYLEY GROUPS OF TYPE $\mathbf{D}_n$

**8.1. Root system of type  $\mathbf{D}_n$ .** Let  $\varepsilon_1, \dots, \varepsilon_n$  be the same as in Subsection 6.1. The root system of type  $\mathbf{D}_n$  is the set

$$\mathbf{D}_n = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}.$$

It has a base  $\alpha_1, \dots, \alpha_n$ , where  $\alpha_1, \dots, \alpha_{n-1}$  are given by (6.2) and  $\alpha_n = \varepsilon_{n-1} + \varepsilon_n$ . The fundamental dominant weights of  $\mathbf{D}_n$  with respect to this base are  $\varpi_i = \varepsilon_1 + \dots + \varepsilon_i$  for  $i = 1, \dots, n-2$ ,

$$\varpi_{n-1} = \frac{1}{2} \sum_{i=1}^{n-1} \varepsilon_i - \frac{1}{2} \varepsilon_n \quad \text{and} \quad \varpi_n = \frac{1}{2} \sum_{i=1}^{n-1} \varepsilon_i + \frac{1}{2} \varepsilon_n.$$

The Weyl group  $W(\mathbf{D}_n)$  of  $\mathbf{D}_n$  is  $(\mathbb{Z}/2\mathbb{Z})^{n-1} \times S_n$ , where  $(\mathbb{Z}/2\mathbb{Z})^{n-1}$  consists of all even numbers of sign changes on  $\{\varepsilon_1, \dots, \varepsilon_n\}$  and  $S_n$  acts via (6.1). The root and weight  $W(\mathbf{D}_n)$ -lattices of  $\mathbf{D}_n$  are, respectively,  $\mathbb{Z}\mathbf{D}_n$  and  $\Lambda(\mathbf{D}_n) := \mathbb{Z}\varpi_1 \oplus \dots \oplus \mathbb{Z}\varpi_n$ .

**8.2. Lattices  $\mathbf{Y}_{2m}$  and  $\mathbf{Z}_{2m}$ .** As we explained in Section 5, we are interested in the case where  $n$  is even,  $n = 2m$ ,  $m > 2$ . There are precisely the following three lattices between  $\Lambda(\mathbf{D}_{2m})$  and  $\mathbb{Z}\mathbf{D}_{2m}$ :

$$(8.1) \quad X_{2m} := \mathbb{Z}\mathbf{D}_{2m} + \mathbb{Z}\varpi_1, \quad Y_{2m} := \mathbb{Z}\mathbf{D}_{2m} + \mathbb{Z}\varpi_{2m-1}, \quad \text{and} \quad Z_{2m} := \mathbb{Z}\mathbf{D}_{2m} + \mathbb{Z}\varpi_{2m}.$$

The character lattice of  $\mathbf{Spin}_{4m}^{1/2}$  (see Section 5) is isomorphic to either of the lattices  $Y_{2m}$  and  $Z_{2m}$  while  $X_{2m}$  is isomorphic to the character lattice of  $\mathbf{SO}_{4m}$ . Note that  $\varepsilon_1, \dots, \varepsilon_n$  is the sign-permutation basis for  $X_{2m}$ ; this is consistent with the fact that  $\mathbf{SO}_{4m}$  is Cayley; see Theorem 1.27(a). Also note that

$$\left\{ \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4), \frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4), \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \right\}$$

is the sign-permutation basis for  $Y_4$ , and

$$\left\{ \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4), \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4), \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4) \right\}$$

is that for  $Z_4$ ; this is consistent with the fact that  $\mathbf{Spin}_8^{1/2}$  is Cayley (see Section 5).

Our goal is to prove Proposition 5.2. In view of the aforesaid, this is equivalent to proving the following.

**Proposition 8.2.** *The  $W(D_{2m})$ -lattices  $Y_{2m}$  and  $Z_{2m}$  are not quasi-permutation for any  $m > 2$ .*

*Proof.* For the subgroup  $S_{2m}$  of  $W(D_{2m})$  acting by (6.1), we consider the  $S_{2m}$ -lattices  $Y_{2m}|_{S_{2m}}$  and  $Z_{2m}|_{S_{2m}}$  and compare them to the  $S_{2m}$ -lattice  $Q_{2m}(m)$  defined by (6.4) and (6.3),

$$(8.3) \quad Q_{2m}(m) = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_{2m-1} + \mathbb{Z}\beta, \quad \text{where } \beta := m\varepsilon_1 - \frac{1}{2} \sum_{i=1}^{2m} \varepsilon_i,$$

that is isomorphic to the character lattice of  $\mathbf{SL}_{2m}/\mu_m$ ; see Subsection 6.1.

First we observe that

$$\alpha_1, \dots, \alpha_{2m-2}, \gamma, \varepsilon_{2m-2} + \varepsilon_{2m-1}, \quad \text{where } \gamma := \frac{1}{2} \sum_{i=1}^m \varepsilon_i - \frac{1}{2} \sum_{i=m+1}^{2m} \varepsilon_i,$$

is a basis for  $Y_{2m}$  if  $m$  is odd and for  $Z_{2m}$  if  $m$  is even. Since  $\alpha_1, \dots, \alpha_{2m-2}, \varepsilon_{2m-2} + \varepsilon_{2m-1}$  is a basis for  $\mathbb{Z}D_{2m-1}$ , (8.1) implies that proving this claim is equivalent to proving the equality

$$(8.4) \quad \mathbb{Z}D_{2m-1} + \mathbb{Z}\gamma = \begin{cases} \mathbb{Z}D_{2m} + \mathbb{Z}\varpi_{2m-1} & \text{if } m \text{ is odd,} \\ \mathbb{Z}D_{2m} + \mathbb{Z}\varpi_{2m} & \text{if } m \text{ is even.} \end{cases}$$

Note that

$$\begin{aligned} \varpi_{2m-1} - \gamma &= \sum_{i=m+1}^{2m-1} \varepsilon_i \in \mathbb{Z}D_{2m-1} & \text{if } m \text{ is odd,} \\ \varpi_{2m} + \gamma &= \sum_{i=1}^m \varepsilon_i \in \mathbb{Z}D_{2m-1} & \text{if } m \text{ is even.} \end{aligned}$$

Therefore proving (8.4) is equivalent to proving the inclusion

$$\mathbb{Z}D_{2m} \subseteq \mathbb{Z}D_{2m-1} + \mathbb{Z}\gamma,$$

which in turn is equivalent to proving the inclusions

$$\varepsilon_{2m-1} \pm \varepsilon_{2m} \in \mathbb{Z}D_{2m-1} + \mathbb{Z}\gamma.$$

Finally, the last inclusions indeed hold as we have

$$\begin{aligned} 2\gamma + (\varepsilon_{2m-1} + \varepsilon_{2m}) &= \sum_{i=1}^m \varepsilon_i - \sum_{i=m+1}^{2m-2} \varepsilon_i \in \mathbb{Z}D_{2m-1}, \\ 2\gamma - (\varepsilon_{2m-1} - \varepsilon_{2m}) &= \sum_{i=1}^{m-1} (\varepsilon_i - \varepsilon_{m+i}) + (\varepsilon_m - \varepsilon_{2m-1}) \in \mathbb{Z}D_{2m-1}. \end{aligned}$$

Thus the claim is proved.

Furthermore, the easily checked equalities

$$\begin{aligned} \beta &= \gamma + \sum_{i=1}^{m-1} (m-i)\alpha_i, \\ \alpha_{2m-1} &= 2\gamma - \sum_{i=1}^m i\alpha_i - \sum_{i=1}^{m-2} (m-i)\alpha_{m+i}, \end{aligned}$$

and (8.3) imply that  $\alpha_1, \dots, \alpha_{2m-2}, \gamma$  is a  $\mathbb{Z}$ -basis for  $Q_{2m}(m)$ .

We thus obtain the following exact sequences of  $S_{2m}$ -lattices:

$$0 \longrightarrow Q_{2m}(m) \longrightarrow Y_{2m}|_{S_{2m}} \longrightarrow \mathbb{Z} \longrightarrow 0$$

if  $m$  is odd and

$$0 \longrightarrow Q_{2m}(m) \longrightarrow Z_{2m}|_{S_{2m}} \longrightarrow \mathbb{Z} \longrightarrow 0$$

if  $m$  is even. Here the  $S_{2m}$ -lattice  $\mathbb{Z}$  is generated by  $\varepsilon_{2m-2} + \varepsilon_{2m-1}$  modulo  $Q_{2m}(m)$ . We claim that the  $S_{2m}$ -action on this lattice is trivial. Indeed, on the one hand, the alternating subgroup of  $S_{2m}$  has to act on this lattice trivially because it has no non-trivial

one-dimensional representations. On the other hand, as  $m > 2$ , the transposition  $(1, 2)$  acts trivially on  $\varepsilon_{2m-2} + \varepsilon_{2m-1}$ . Since the alternating subgroup and the transposition  $(1, 2)$  generate  $S_{2m}$ , this proves the claim.

The above exact sequences thus tell us that  $Y_{2m}|_{S_{2m}} \sim Q_{2m}(m)$  if  $m$  is odd and  $Z_{2m}|_{S_{2m}} \sim Q_{2m}(m)$  if  $m$  is even. By Proposition 5.1, the  $S_{2m}$ -lattice  $Q_{2m}(m)$  is not quasi-permutation for any  $m > 2$ . Thus for  $m > 2$ , the  $W(D_{2m})$ -lattice  $Y_{2m}$  is not quasi-permutation if  $m$  is odd, and the  $W(D_{2m})$ -lattice  $Z_{2m}$  is not quasi-permutation if  $m$  is even, as their restrictions to  $S_{2m}$  are not quasi-permutation. Since  $Y_{2m} \simeq Z_{2m}$  as  $W(D_{2m})$ -lattices, this completes the proof.  $\square$

### 9. Which stably Cayley groups are Cayley?

In this section we will prove Theorem 1.31. The groups  $G = \mathbf{SO}_n$ ,  $\mathbf{Sp}_{2n}$ , and  $\mathbf{PGL}_n$  are shown to be Cayley in Examples 1.16 and 1.11. It thus remains to consider  $\mathbf{SL}_3$  and  $\mathbf{G}_2$ .

#### 9.1. The group $\mathbf{SL}_3$ .

**Proposition 9.1.** *The group  $\mathbf{SL}_3$  is Cayley.*

The proof below is based on analysis of the explicit formulas in [Vos, 4.9] and the geometric ideas of the proof of Proposition 9.1 given in [Pop<sub>2</sub>]. We present it in a form that will also help us prove that  $\mathbf{G}_2 \times \mathbf{G}_m^2$  is Cayley; see Proposition 9.11 below. On the other hand, the spirit of the arguments in [Pop<sub>2</sub>] is close to that in [Isk<sub>4</sub>]. Since [Isk<sub>4</sub>] is the main ingredient we will use in showing that  $\mathbf{G}_2$  is not Cayley, see Lemma 9.9 and Proposition 9.10 below, we will give an outline of the proof of Proposition 9.1 from [Pop<sub>2</sub>] in the Appendix.

*Proof.* The Weyl group  $W$  of  $\mathbf{SL}_3$  is  $S_3$ . Consider the following subalgebra  $D$  of  $\text{Mat}_{3 \times 3}$ :

$$(9.2) \quad D := \{\text{diag}(a_1, a_2, a_3) \in \text{Mat}_{3 \times 3} \mid a_i \in k\}$$

and the action of  $S_3$  on  $D$  given by

$$(9.3) \quad \sigma(\text{diag}(a_1, a_2, a_3)) := \text{diag}(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) \quad \text{where } \sigma \in S_3.$$

The  $S_3$ -stable subvarieties

$$(9.4) \quad T = \{X \in D \mid \det X = 1\} \quad \text{and} \quad \mathfrak{t} = \{Y \in D \mid \text{tr } Y = 0\}$$

are, respectively, the maximal torus of  $\mathbf{SL}_3$  and its Lie algebra, considered as  $W$ -varieties. By the Corollary of Lemma 3.5, it suffices to show that  $T$  and  $\mathfrak{t}$  are birationally  $S_3$ -isomorphic.

Let  $D \setminus \{0\} \rightarrow \mathbb{P}(D)$ ,  $X \mapsto [X]$ , be the natural projection. Denote by  $\mathbb{P}_{S_3\text{-natural}}^2$  and  $\mathbb{P}_{S_3\text{-twisted}}^2$  the projective plane  $\mathbb{P}(D)$  endowed, respectively, with the natural and “twisted” rational actions of  $S_3$  given by

$$\sigma([X]) := [\sigma(X)] \quad \text{and} \quad \sigma([X]) := [\sigma(X)^{\text{sign } \sigma}], \quad \text{where } \sigma \in S_3, X \in D.$$

Let  $\pi : \mathbf{SL}_3 \rightarrow \mathbf{PGL}_3$  be the natural projection. Since  $d_e \pi$  is an isomorphism between the Lie algebras of  $\mathbf{SL}_3$  and  $\mathbf{PGL}_3$  and since  $\mathbf{PGL}_3$  is a Cayley group, see Example 1.11, the Corollary of Lemma 3.5 tells us that  $\mathfrak{t}$  is birationally  $S_3$ -isomorphic to the maximal torus  $\pi(T)$  of  $\mathbf{PGL}_3$ . In turn, we have the following birational  $S_3$ -isomorphisms of  $S_3$ -varieties:

$$\begin{aligned} \pi(T) &\xrightarrow{\simeq} \mathbb{P}_{S_3\text{-natural}}^2, & \pi(X) &\mapsto [X], \\ \mathbb{P}_{S_3\text{-twisted}}^2 &\xrightarrow{\simeq} T, & [\text{diag}(a_1, a_2, a_3)] &\mapsto \text{diag}(a_2/a_3, a_3/a_1, a_1/a_2). \end{aligned}$$

Thus we only need to show that  $\mathbb{P}_{\mathbb{S}_3\text{-natural}}^2$  and  $\mathbb{P}_{\mathbb{S}_3\text{-twisted}}^2$  are birationally  $\mathbb{S}_3$ -isomorphic. We shall establish this in three steps.

*Step 1.* Consider the action of  $\mathbb{S}_3$  on  $\mathfrak{t} \times \mathfrak{t}$  given by

$$(9.5) \quad \sigma(Y, Z) := \begin{cases} (\sigma(Y), \sigma(Z)) & \text{if } \sigma \text{ is even,} \\ (\sigma(Z), \sigma(Y)) & \text{if } \sigma \text{ is odd,} \end{cases} \quad \text{where } \sigma \in \mathbb{S}_3, Y, Z \in \mathfrak{t}.$$

It determines the action of  $\mathbb{S}_3$  on the surface  $\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t})$ . Denote resulting  $\mathbb{S}_3$ -surface by  $(\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{\mathbb{S}_3\text{-twisted}}$ .

We claim that the  $\mathbb{S}_3$ -varieties  $\mathbb{P}_{\mathbb{S}_3\text{-twisted}}^2$  and  $(\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{\mathbb{S}_3\text{-twisted}}$  are birationally  $\mathbb{S}_3$ -isomorphic. Indeed, it is immediately seen that the rational map

$$\varphi: \mathbb{P}_{\mathbb{S}_3\text{-twisted}}^2 \dashrightarrow (\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{\mathbb{S}_3\text{-twisted}}, \quad [X] \mapsto \left( \left[ X - \frac{\text{tr}(X)}{3} I_3 \right], \left[ X^{-1} - \frac{\text{tr}(X^{-1})}{3} I_3 \right] \right),$$

is  $\mathbb{S}_3$ -equivariant and we shall now construct a rational map inverse to  $\varphi$ . Note that for  $Y, Z \in \mathfrak{t}$  in general position,  $Y, Z, I_3$  form a basis of the vector space  $D$ . Thus there are unique  $\alpha, \beta, \gamma \in k$  such that

$$\alpha Z + \beta Y + \gamma I = -YZ.$$

Note that  $\alpha, \beta$ , and  $\gamma$  are, in fact, bihomogeneous rational functions of  $Y$  and  $Z$  of bidegree  $(1, 0)$ ,  $(0, 1)$ , and  $(1, 1)$ , respectively. We now consider the map

$$\psi: (\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{\mathbb{S}_3\text{-twisted}} \dashrightarrow \mathbb{P}_{\mathbb{S}_3\text{-twisted}}^2, \quad ([Y], [Z]) \mapsto [Y + \alpha I_3].$$

To compute  $\psi \circ \varphi$ , note that if  $Y = X - \frac{\text{tr}(X)}{3} I_3$  and  $Z = X^{-1} - \frac{\text{tr}(X^{-1})}{3} I_3$ , then expanding

$$I_3 = \left( Y + \frac{\text{tr}(X)}{3} I_3 \right) \left( Z + \frac{\text{tr}(X^{-1})}{3} I_3 \right),$$

we see that  $\alpha = \frac{\text{tr}(X)}{3}$  and thus  $\psi([Y], [Z]) = [X]$ . Thus  $\psi \circ \varphi = \text{id}$ , and hence  $\varphi$  is a birational  $\mathbb{S}_3$ -isomorphism.

*Step 2.* We now consider the linear action of  $\mathbb{S}_3$  on  $\mathfrak{t} \otimes \mathfrak{t}$  determined by the action (9.5) and the corresponding action of  $\mathbb{S}_3$  on  $\mathbb{P}(\mathfrak{t} \otimes \mathfrak{t})$ . Then the Segre embedding

$$(\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{\mathbb{S}_3\text{-twisted}} \hookrightarrow \mathbb{P}(\mathfrak{t} \otimes \mathfrak{t})$$

is  $\mathbb{S}_3$ -equivariant. Its image is a quadric  $Q$  in  $\mathbb{P}(\mathfrak{t} \otimes \mathfrak{t})$  described as follows. Choose a basis  $D_1 := \text{diag}(1, \zeta, \zeta^2)$ ,  $D_2 := \text{diag}(1, \zeta^2, \zeta)$  of  $\mathfrak{t}$ , where  $\zeta$  is a primitive cube root of unity. Set  $D_{ij} = D_i \otimes D_j$ . Then

$$(9.6) \quad Q = \{(\alpha_{11} : \alpha_{12} : \alpha_{21} : \alpha_{22}) \mid \alpha_{11}\alpha_{22} = \alpha_{12}\alpha_{21}\},$$

where  $(\alpha_{11} : \alpha_{12} : \alpha_{21} : \alpha_{22})$  is the point of  $\mathbb{P}(\mathfrak{t} \otimes \mathfrak{t})$  corresponding to  $\alpha_{11}D_{11} + \alpha_{12}D_{12} + \alpha_{21}D_{21} + \alpha_{22}D_{22} \in \mathfrak{t} \otimes \mathfrak{t}$ .

*Step 3.* Decomposing  $\mathfrak{t} \otimes \mathfrak{t}$  as a sum of  $\mathbb{S}_3$ -submodules, we obtain

$$(9.7) \quad \mathfrak{t} \otimes \mathfrak{t} = V_1 \oplus V_2 \oplus V_3,$$

where  $V_1 = kD_{11} + kD_{22}$  is a simple 2-dimensional submodule and  $V_2 = kD_{12}$ ,  $V_3 = kD_{21}$  are trivial 1-dimensional submodules. Since the  $\mathbb{S}_3$ -fixed point  $(0 : 0 : 1 : 0) \in \mathbb{P}(\mathfrak{t} \otimes \mathfrak{t})$  corresponding to  $V_3$  lies on  $Q$ , the stereographic projection  $Q \dashrightarrow \mathbb{P}(V_1 \oplus V_2)$  from this point is a birational  $\mathbb{S}_3$ -isomorphism.

Finally, the  $\mathbb{S}_3$ -module  $D$  is isomorphic to  $V_1 \oplus V_2$ . Hence  $\mathbb{P}(V_1 \oplus V_2)$  and  $\mathbb{P}_{\mathbb{S}_3\text{-natural}}^2$  are  $\mathbb{S}_3$ -isomorphic.

To sum up, we have established the existence of the following birational  $S_3$ -isomorphisms:

$$\mathbb{P}_{S_3\text{-twisted}}^2 \xrightarrow{\sim \text{Step 1}} (\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{S_3\text{-twisted}} \xrightarrow{\sim \text{Step 2}} Q \xrightarrow{\sim \text{Step 3}} \mathbb{P}_{S_3\text{-natural}}^2.$$

This completes the proof of Proposition 9.1.  $\square$

**9.2. The group  $G_2$ .** The Weyl group of  $G_2$  is the dihedral group  $S_3 \times S_2$  of order 12. The maximal torus of  $G_2$  and its Lie algebra are  $S_3 \times S_2$ -isomorphic, respectively, to  $T$  and  $\mathfrak{t}$  given by (9.4), where the action of the first factor of  $S_3 \times S_2$  is defined, as in the case of  $SL_3$ , by (9.3), and that of the non-trivial element  $\theta$  of the second factor by

$$(9.8) \quad \theta(X) := X^{-1} \text{ for } X \in T \quad \text{and} \quad \theta(Y) := -Y \text{ for } Y \in \mathfrak{t}.$$

We begin with the following surprising recent result due to ISKOVSKIKH, [Isk<sub>4</sub>].

**Lemma 9.9.** *The  $S_3 \times S_2$ -varieties  $T$  and  $\mathfrak{t}$  are not birationally  $S_3 \times S_2$ -isomorphic.*

*Proof outline.* Since  $T$  and  $\mathfrak{t}$  are rational surfaces, the theory of rational  $G$ -surfaces, due to MANIN [Ma<sub>1</sub>] and ISKOVSKIKH [Isk<sub>1</sub>], [Isk<sub>3</sub>], can be applied; this is precisely what is done in [Isk<sub>4</sub>] (see also [Isk<sub>5</sub>]). Minimal rational  $S_3 \times S_2$ -surfaces are known, and any equivariant birational isomorphism between two such surfaces can be written as a composition of so-called “elementary links”, which are completely enumerated in [Isk<sub>3</sub>]. The argument in [Isk<sub>4</sub>] and [Isk<sub>5</sub>] amounts to constructing suitable minimal models for  $T$  and  $\mathfrak{t}$  and explicitly checking that it is impossible to get from one to the other by a sequence of elementary links.  $\square$

**Proposition 9.10.**  *$G_2$  is not a Cayley group.*

*Proof.* By the Corollary of Lemma 3.5, this follows from Lemma 9.9.  $\square$

The following result illustrates how delicate the matter is.

**Proposition 9.11.**  *$G_2 \times G_m^2$  is a Cayley group.*

*Proof.* By the Corollary of Lemma 3.5, it suffices to show that  $T \times \mathbb{A}^2$  and  $\mathfrak{t} \times \mathbb{A}^2$  are birationally  $S_3 \times S_2$ -isomorphic, where in both cases  $S_3 \times S_2$  acts via the first factor. We shall define a birational  $S_3 \times S_2$ -isomorphism between them in three steps.

*Step 1.* Let  $(\mathfrak{t} \times \mathfrak{t})_{S_3 \times S_2\text{-twisted}}$  be the variety  $\mathfrak{t} \times \mathfrak{t}$  endowed with the following  $S_3 \times S_2$ -action:

$$(9.12) \quad (\sigma, \varepsilon)(Y, Z) := \begin{cases} \text{sign}(\sigma)(\sigma(Y), \sigma(Z)) & \text{if } \text{sign}(\sigma) = \text{sign}(\varepsilon), \\ \text{sign}(\sigma)(\sigma(Z), \sigma(Y)) & \text{otherwise,} \end{cases}$$

for any  $(\sigma, \varepsilon) \in S_3 \times S_2$  and  $Y, Z \in \mathfrak{t}$ . This action descends to  $\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t})$ ; denote the resulting  $S_3 \times S_2$ -variety by  $(\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{S_3 \times S_2\text{-twisted}}$ . We claim that  $(\mathfrak{t} \times \mathfrak{t})_{S_3 \times S_2\text{-twisted}}$  is birationally isomorphic to  $(\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{S_3 \times S_2\text{-twisted}} \times \mathbb{A}^2$  as an  $S_3 \times S_2$ -variety. Here  $S_3 \times S_2$  acts trivially on  $\mathbb{A}^2$ .

To prove the claim, let  $\mathfrak{t}'$  be  $\mathfrak{t}$  blown up at the origin. The  $S_3 \times S_2$ -action (9.12) on  $\mathfrak{t} \times \mathfrak{t}$  lifts to  $\mathfrak{t}' \times \mathfrak{t}'$ ; we shall denote the resulting  $S_3 \times S_2$ -variety by  $(\mathfrak{t}' \times \mathfrak{t}')_{S_3 \times S_2\text{-twisted}}$ . The natural projection  $\mathfrak{t} \dashrightarrow \mathbb{P}(\mathfrak{t})$  (which is only a rational map, not defined at the origin) lifts to a regular map  $\mathfrak{t}' \rightarrow \mathbb{P}(\mathfrak{t})$ . Moreover, the natural projection

$$(\mathfrak{t}' \times \mathfrak{t}')_{S_3 \times S_2\text{-twisted}} \longrightarrow (\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{S_3 \times S_2\text{-twisted}}$$

is an algebraic vector  $S_3 \times S_2$ -bundle of rank 2. Since  $S_3 \times S_2$  acts on  $(\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{S_3 \times S_2\text{-twisted}}$  faithfully, Lemma 2.12(b) shows that  $(\mathfrak{t}' \times \mathfrak{t}')_{S_3 \times S_2\text{-twisted}}$  is birationally isomorphic, as an

$S_3 \times S_2$ -variety, to  $(\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{S_3 \times S_2\text{-twisted}} \times \mathbb{A}^2$  (where  $S_3 \times S_2$  acts via the first factor, as above). Since  $(\mathfrak{t} \times \mathfrak{t})_{S_3 \times S_2\text{-twisted}}$  and  $(\mathfrak{t}' \times \mathfrak{t}')_{S_3 \times S_2\text{-twisted}}$  are birationally  $S_3 \times S_2$ -isomorphic, this proves the claim.

*Step 2.* Let  $\mathbb{P}_{S_3 \times S_2\text{-twisted}}^2$  be the projective plane  $\mathbb{P}(D)$  endowed with the action of  $S_3 \times S_2$  given by

$$(\sigma, \varepsilon)([X]) := [\sigma(X)^{\text{sign } \sigma \text{ sign } \varepsilon}], \quad \text{where } (\sigma, \varepsilon) \in S_3 \times S_2, X \in D.$$

Then the rational maps

$$\mathbb{P}_{S_3 \times S_2\text{-twisted}}^2 \dashrightarrow T, \quad [\text{diag}(a_1, a_2, a_3)] \mapsto \text{diag}(a_2/a_3, a_3/a_1, a_1/a_2),$$

and

$$\begin{aligned} \mathbb{P}_{S_3 \times S_2\text{-twisted}}^2 &\dashrightarrow (\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{S_3 \times S_2\text{-twisted}}, \\ [X] &\mapsto \left( \left[ X - \frac{\text{tr}(X)}{3} I_3 \right], \left[ X^{-1} - \frac{\text{tr}(X^{-1})}{3} I_3 \right] \right), \end{aligned}$$

are birational  $S_3 \times S_2$ -isomorphisms—the arguments are similar to those in the proof of Proposition 9.1.

*Step 3.* The definition of  $(\mathfrak{t} \times \mathfrak{t})_{S_3 \times S_2\text{-twisted}}$  in Step 1 shows that the map

$$(\mathfrak{t} \times \mathfrak{t})_{S_3 \times S_2\text{-twisted}} \longrightarrow \mathfrak{t}, \quad (t_1, t_2) \mapsto t_1 - t_2,$$

is  $S_3 \times S_2$ -equivariant. Hence, this map may be viewed as an algebraic  $S_3 \times S_2$ -vector bundle of rank 2. Since  $S_3 \times S_2$  acts on  $\mathfrak{t}$  faithfully, applying Lemma 2.12(b) once again, we conclude that  $(\mathfrak{t} \times \mathfrak{t})_{S_3 \times S_2\text{-twisted}}$  is birationally  $S_3 \times S_2$ -isomorphic to  $\mathfrak{t} \times \mathbb{A}^2$ , where  $S_3 \times S_2$  acts via the first factor.

To sum up, we have established the existence of the following birational  $S_3 \times S_2$ -isomorphisms:

$$T \times \mathbb{A}^2 \xrightarrow{\text{Step 2}} (\mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}))_{S_3 \times S_2\text{-twisted}} \times \mathbb{A}^2 \xrightarrow{\text{Step 1}} (\mathfrak{t} \times \mathfrak{t})_{S_3 \times S_2\text{-twisted}} \xrightarrow{\text{Step 3}} \mathfrak{t} \times \mathbb{A}^2.$$

This completes the proof of Proposition 9.11.  $\square$

**Remark 9.13.** We do not know whether or not  $\mathbf{G}_2 \times \mathbf{G}_m$  is a Cayley group.

## 10. Generalization

The notions of Cayley map and Cayley group naturally lead to generalizations which will be considered in this section.

**10.1. Generalized Cayley maps.** Let  $G$  be a connected linear algebraic group and let  $\mathfrak{g}$  be its Lie algebra. We consider  $G$  and  $\mathfrak{g}$  as  $G$ -varieties with respect to the conjugating and adjoint actions, respectively, and denote by  $\text{Rat}_G(G, \mathfrak{g})$  the set of all rational  $G$ -maps  $G \dashrightarrow \mathfrak{g}$  endowed with the natural structure of a vector space over  $k(G)^G$ . Set  $\text{Mor}_G(G, \mathfrak{g}) := \{\varphi \in \text{Rat}_G(G, \mathfrak{g}) \mid \varphi \text{ is a morphism}\}$ .

**Definition 10.1.** An element  $\varphi \in \text{Rat}_G(G, \mathfrak{g})$  (respectively,  $\varphi \in \text{Mor}_G(G, \mathfrak{g})$ ) is called a *generalized Cayley map* (respectively, *generalized Cayley morphism*) of  $G$  if  $\varphi$  is a dominant map.

We are now ready to state the main result of this subsection.

**Theorem 10.2.** *Every connected linear algebraic group admits a generalized Cayley morphism.*

Our proof of Theorem 10.2 will proceed in three steps. First we will construct a generalized Cayley morphism for every reductive group (Corollary to Lemma 10.3), and then a generalized Cayley map for an arbitrary linear algebraic group (Proposition 10.5), then a generalized Cayley morphism for an arbitrary linear algebraic group.

Our construction in the case of reductive groups relies on the following known fact; see [Lun<sub>2</sub>, Lemme III.1] and cf. [PV, 6.3].

**Lemma 10.3.** *Assume that the group  $G$  is reductive. Let  $X$  be an affine algebraic variety endowed with an algebraic action of  $G$  and let  $x \in X$  be a non-singular fixed point of  $G$ . Let  $T_x$  be the tangent space of  $X$  at  $x$  endowed with the natural action of  $G$ . Then there is a  $G$ -morphism  $\varepsilon : X \rightarrow T_x$  étale at  $x$  (hence dominant) and such that  $\varepsilon(x) = 0$ .*

*Proof.* We may assume without loss of generality that  $X$  is a  $G$ -stable subvariety of a finite-dimensional algebraic  $G$ -module  $V$ ; see [PV, Theorem 1.5]. Since  $x$  is a fixed point of  $G$ , we can replace  $X$  by its image under the parallel translation  $v \mapsto v - x$  and assume that  $x = 0$ . The tangent space  $T_x$  is identified with a submodule of  $V$ . Since  $G$  is reductive, the  $G$ -module  $V$  is semisimple. Hence  $V = T_x \oplus M$  for some submodule  $M$ . Now we can take  $\varepsilon = \pi|_X$ , where  $\pi : V \rightarrow T_x$  is the projection parallel to  $M$ .  $\square$

Taking  $X = G$  with the conjugating action and  $x = e$ , we obtain the following.

**Corollary.** *Assume that  $G$  is reductive. Then there is a generalized Cayley morphism  $\varphi$  of  $G$  étale at  $e$  and such that  $\varphi(e) = 0$ .*

The following special case of this construction was considered by KOSTANT and MICHOR, [KM].

**Example 10.4.** Assume that  $G$  is reductive. Consider an algebraic homomorphism  $\nu : G \rightarrow \mathbf{GL}(S)$ , where  $S$  is a finite-dimensional vector space over  $k$ . Then the  $k$ -vector space  $V := \text{End}(S)$  has a natural  $G$ -module structure defined by  $g(h) := \nu(g)h\nu(g)^{-1}$  for every  $g \in G$  and  $h \in V$ . If  $\nu$  is injective, identify  $G$  with the image of  $\iota \circ \nu$ , where  $\iota : \mathbf{GL}(S) \hookrightarrow V$  is the natural embedding. Then  $G$  is a  $G$ -stable subvariety of  $V$  and the restriction to  $\mathfrak{g} = T_e$  of the  $G$ -invariant inner product  $(x, y) \mapsto \text{tr } xy$  on  $V$  is non-degenerate. This yields the  $G$ -module decomposition  $V = \mathfrak{g} \oplus \mathfrak{g}^\perp$ , where  $\mathfrak{g}^\perp$  is the orthogonal complement to  $\mathfrak{g}$  with respect to  $(, )$ . The restriction to  $G$  of the projection  $V \rightarrow \mathfrak{g}$  parallel to  $\mathfrak{g}^\perp$  is a generalized Cayley morphism  $\varphi : G \rightarrow \mathfrak{g}$  étale at  $e$  such that  $\varphi(e) = 0$ .  $\square$

**Proposition 10.5.** *Every connected linear algebraic group  $G$  admits a generalized Cayley map.*

*Proof.* We use the notation of Proposition 4.2 and its proof. The group  $W_{L,T}$  is finite, hence reductive, and  $e \in T$  is its fixed point. Therefore Lemma 10.3 implies that there is a dominant  $W_{L,T}$ -morphism  $\varepsilon : T \rightarrow \mathfrak{t}$ . The arguments in the proof of part (a) of Proposition 4.2 show that  $\varepsilon$  is  $N$ -equivariant. Consider an  $N$ -isomorphism (4.3). Then

$$\varepsilon \times \tau : C = T \times U \longrightarrow \mathfrak{t} \oplus \mathfrak{u} = \mathfrak{c}$$

is a dominant  $N$ -morphism. Hence by Lemma 2.17, there is a dominant  $G$ -morphism

$$\theta : G \times^N C \longrightarrow G \times^N \mathfrak{c}$$

such that  $\theta|_C = \varepsilon \times \tau$ . Now, since, by Lemma 3.2, the  $G$ -morphisms  $\gamma_C$  and  $\gamma_{\mathfrak{c}}$  given by (3.1) are birational  $G$ -isomorphisms,  $\gamma_{\mathfrak{c}} \circ \theta \circ \gamma_C^{-1} \in \text{Rat}_G(G, \mathfrak{g})$  is a generalized Cayley map.  $\square$

Our next task is to deduce Theorem 10.2 from Proposition 10.5. Our argument will rely on the following simple lemma.

**Lemma 10.6.** *Every semi-invariant for the conjugating action of  $G$  on itself is, in fact, an invariant.*

*Proof.* Suppose  $t \in k[G]$  is a semi-invariant. That is, there exists an algebraic character  $\chi: G \rightarrow \mathbf{G}_m$  such that  $t(ghg^{-1}) = \chi(g)t(h)$  for every  $g, h \in G$ . We may assume  $t$  is not identically zero. Setting  $g = h$  in the above formula, we obtain

$$t(g) = \chi(g)t(g) \text{ for every } g \in G.$$

Since  $G$  is connected and  $t$  is not identically zero, this implies that  $\chi(g) = 1$  for every  $g \in G$ , i.e.,  $t \in k[G]^G$ .  $\square$

Theorem 10.2 is now an immediate consequence of Proposition 10.5 and Proposition 10.7 below.

**Proposition 10.7.** *Let  $\varphi \in \text{Rat}_G(G, \mathfrak{g})$ . Then there is  $f \in k[G]^G$  such that*

- (i)  $\{g \in G \mid f(g) = 0\}$  is the indeterminacy locus of  $\varphi$ ,
- (ii)  $f\varphi \in \text{Mor}_G(G, \mathfrak{g})$ .

Moreover, if  $\varphi$  is a generalized Cayley map of  $G$ , then (ii) may be replaced by

- (ii)'  $f\varphi$  is a generalized Cayley morphism  $G \rightarrow \mathfrak{g}$ .

*Proof.* We may assume that  $\varphi$  is not a morphism. Then the indeterminacy locus of  $\varphi$  is an unmixed closed subset  $X$  of  $G$  of codimension 1. Since, by [Pop<sub>1</sub>, Theorem 6], the Picard group of the underlying variety of  $G$  is finite, this implies that there is  $t \in k[G]$  such that  $\{g \in G \mid t(g) = 0\} = X$ . As  $\varphi$  is  $G$ -equivariant,  $X$  is  $G$ -stable. Hence, by [PV, Theorem 3.1],  $t$  is a semi-invariant of  $G$  and therefore  $t \in k[G]^G$  by Lemma 10.6. Consequently the function  $f = t^m$  satisfies (i) and (ii) for a sufficiently large positive integer  $m$ . The second assertion of the proposition follows from Lemma 10.8 below.  $\square$

**Lemma 10.8.** *Let  $\psi: X \dashrightarrow V$  be a dominant rational map, where  $X$  is an irreducible algebraic variety,  $V$  a vector space over  $k$ , and  $\dim X = \dim V$ . Then for every non-zero function  $t \in k(X)$ , at least one of the maps  $\alpha := t\psi$  and  $\beta := t^2\psi$  is dominant.*

*Proof.* Put  $h_i := \psi^*(x_i) \in k(X)$ , where  $x_1, \dots, x_n$  are the coordinate functions on  $V$  with respect to some basis. Then  $K := \psi^*(k(V)) = k(h_1, \dots, h_n)$ ,  $K_1 := \alpha^*(k(\overline{\alpha(X)})) = k(th_1, \dots, th_n)$  and  $K_2 := \beta^*(k(\overline{\beta(X)})) = k(t^2h_1, \dots, t^2h_n)$ , where the bar denotes the closure in  $V$ . All three fields contain the subfield  $K_0 := k(\dots, h_i/h_j, \dots)$ . We have  $\text{trdeg}_k K = n$ . Therefore  $\text{trdeg}_k K_0 = n - 1$ .

Assume the contrary: neither  $t\psi$  nor  $t^2\psi$  is dominant. Then  $\text{trdeg}_k K_1 = \text{trdeg}_k K_2 = n - 1$ . Since  $K_1 = K_0(th_i)$  and  $K_2 = K_0(t^2h_i)$  for any  $i$ , this implies that both  $th_i$  and  $t^2h_i$  are algebraic over  $K_0$ . Hence  $h_i = (th_i)^2/t^2h_i$  is algebraic over  $K_0$ . Thus  $K$  is algebraic over  $K_0$ . Hence  $\text{trdeg}_k K = \text{trdeg}_k K_0 = n - 1$ , a contradiction.  $\square$

**10.2. The Cayley degree.** Note that every generalized Cayley map  $\varphi: G \dashrightarrow \mathfrak{g}$  has finite degree, i.e.,  $\deg \varphi := [k(G) : \varphi^*(k(\mathfrak{g}))] < \infty$ . By Definition 1.5, Cayley maps are exactly generalized Cayley maps of degree 1. This naturally leads to the following definition of a “measure of non-Cayleyness” of  $G$ .

**Definition 10.9.** The *Cayley degree* of  $G$  is the number  $\text{Cay}(G) := \min_{\varphi} \deg \varphi$ , where  $\varphi$  runs through all generalized Cayley maps of  $G$ .

Clearly  $G$  is a Cayley group if and only if  $\text{Cay}(G) = 1$ . Theorem 1.31 may thus be interpreted as a classification of connected simple algebraic groups of Cayley degree 1 and, consequently, as a first step towards the solution of the following general problem:

**Problem 10.10.** *Find the Cayley degrees of connected simple algebraic groups.*

For example, composing the natural projection  $\mathbf{Spin}_n \rightarrow \mathbf{SO}_n$  with the classical Cayley map  $\mathbf{SO}_n \xrightarrow{\cong} \mathfrak{so}_n$  yields a generalized Cayley map  $\mathbf{Spin}_n \rightarrow \mathbf{SO}_n \dashrightarrow \mathfrak{so}_n = \mathfrak{spin}_n$  of degree 2. Combining this with Theorem 1.28, we conclude that

$$\text{Cay}(\mathbf{Spin}_n) = \begin{cases} 2 & \text{for } n \geq 6, \\ 1 & \text{for } n \leq 5. \end{cases}$$

Other examples can be found in [LPR, Section 10]. Note that Definition 10.9 and Problem 10.10 have natural analogues in the case where  $G$  is defined over a subfield  $K$  of  $k$  (here we consider only generalized Cayley maps  $\varphi$  defined over  $K$ ).

### Appendix. Alternative Proof of Proposition 9.1: An outline

*Step 1.* Consider  $D$ , see (9.2), as an open subset of  $\mathbb{P}^3$  given by  $x_0 \neq 0$ , and extend the  $S_3$ -action (9.3) up to  $\mathbb{P}^3$  by

$$\sigma(a_0 : a_1 : a_2 : a_3) = (a_0 : a_{\sigma(1)} : a_{\sigma(2)} : a_{\sigma(3)}), \quad \text{where } \sigma \in S_3.$$

The closure  $X$  of  $T$  in  $\mathbb{P}^3$ , see (9.4), is the rational cubic surface given by  $x_1x_2x_3 - x_0^3 = 0$ . It has exactly three fixed points

$$a_i := (1 : \varepsilon^i : \varepsilon^i : \varepsilon^i), \quad i = 1, 2, 3, \quad \varepsilon^3 = 1, \quad \varepsilon \neq 1,$$

and three singular (double) points

$$s_1 = (0 : 1 : 0 : 0), \quad s_2 = (0 : 0 : 1 : 0), \quad s_3 = (0 : 0 : 0 : 1).$$

The hyperplane section of  $X$  given by  $x_0 = 0$  is  $H := l_1 + l_2 + l_3$ , where  $l_i$  is the line given by  $x_0 = x_i = 0$ .

Since  $H$  is  $S_3$ -invariant, the  $S_3$ -action on  $X$  lifts to the surface  $\tilde{X}$  obtained from  $X$  by the simultaneous blowing up  $\mu : \tilde{X} \rightarrow X$  of  $s_1, s_2, s_3$ . The surface  $\tilde{X}$  is smooth and  $T$  is its open  $S_3$ -stable subset.

*Step 2.* We have  $\mu^*(H) = \sum_i t_i + \sum_{ij} m_{ij}$  where  $t_i$  is the proper inverse image of  $l_i$  and  $\mu^{-1}(s_i) = m_{ij} \cup m_{ir}$ ,  $\{i, j, r\} = \{1, 2, 3\}$ . The curves  $t_i, m_{ij}$  are isomorphic to  $\mathbb{P}^1$  and form a 9-gon as shown on the figure below. Their intersections are transversal and the self-intersection indices are  $(t_i, t_i) = -1$ ,  $(m_{ij}, m_{ij}) = -2$ .

Computing the canonical classes gives  $K_X = -H$  and  $K_{\tilde{X}} = \mu^*(-H)$ . Hence

$$(A1) \quad (K_{\tilde{X}}, K_{\tilde{X}}) = (-H, -H) = \deg X = 3.$$

*Step 3.* By the Castelnuovo criterion, the curves  $t_i$  are exceptional, so they can be simultaneously blown down:  $\nu: \tilde{X} \rightarrow Y$ . The surface  $Y$  is smooth, and the  $S_3$ -invariance of  $t_1 + t_2 + t_3$  implies that the action of  $S_3$  on  $\tilde{X}$  descends to  $Y$ . We can consider  $T$  as an open  $S_3$ -stable subset of  $Y$ .

It follows from (A1) that

$$(A2) \quad (K_Y, K_Y) = 6,$$

and  $\text{Pic } T = 0$  implies that  $(\text{Pic } Y)^{S_3}$  is generated by

$$P := \nu_*(\sum_{ij} m_{ij}).$$

Hence  $K_Y = nP$  for some non-zero integer  $n$ . Rationality of  $Y$  implies  $n < 0$ . If  $C$  is a positive divisor on  $Y$ , then  $\sum_{\sigma \in S_3} \sigma(C) = cP$  for some positive integer  $c$ . Using (A2), we then obtain

$$\begin{aligned} (-K_Y, C) &= (-K_Y, \sum_{\sigma \in S_3} \sigma(C))/6 \\ &= -cn(P, P)/6 = -c(K_Y, K_Y)/6n = -c/n > 0. \end{aligned}$$

By the Nakai–Moishezon criterion, this implies that  $-K_Y$  is ample, i.e.,  $Y$  is a Del Pezzo surface. From (A2) it then follows (see, e.g., [Ma2, § 24]) that  $|-K_Y|$  defines an embedding of  $Y$  into  $\mathbb{P}^6$  equivariant with respect to a certain action of  $S_3$  on  $\mathbb{P}^6$ . We keep the notation  $Y$  for its image.

*Step 4.* Consider on  $Y$  the linear system  $|R|$  of all hyperplane sections in  $\mathbb{P}^6$  containing the fixed point  $a_1 \in T \subseteq Y$  and which is singular at  $a_1$ . These are precisely sections by hyperplanes tangent to  $Y$  at  $a_1$ , so

$$(A3) \quad \dim |R| = 4.$$

The system  $|R|$  is an  $S_3$ -stable subsystem of  $|-K_Y|$ . By Bertini's theorem, its general element  $R$  is an irreducible curve. We have

$$(A4) \quad p_a(R) = 1 + (R, (R + K_Y))/2 = 1 + (R, (R - R))/2 = 1.$$

On the other hand,  $p_a(R) = g + \sum_x \delta_x$ , where  $g$  is the geometric genus of the normalization of  $R$ , the sum is taken over all singular points  $x$  of  $R$ , and  $\delta_x > 0$ . This and (A4) imply that  $R$  is a rational curve whose singular locus is the double point  $a_1$ .

The system  $|R|$  has no fixed components. Indeed, if  $F$  were such a component, then  $\dim H^0(Y, \mathcal{O}(F)) = 1$  and, by the Riemann–Roch theorem,

$$(A5) \quad \dim H^0(Y, \mathcal{O}(K_Y - F)) \geq ((F, F) - (F, K_Y))/2.$$

Let  $-K_Y = F + E$ . Since  $F > 0$  and  $E > 0$ , the left-hand side of (A5) is zero, whence  $0 \geq (F, F) + (F, E)/2$ . Since  $(F, E) \geq 0$ , this yields  $0 \geq (F, F)$ . But  $F = mP$  for some non-zero integer  $m$ . Therefore

$$0 \geq (F, F) = m^2(P, P) = 6m^2/n^2 > 0,$$

a contradiction.

From (A3) we deduce that  $a_1$  is a unique base point of  $|R|$ .

*Step 5.* Let  $\gamma: \tilde{Y} \rightarrow Y$  be the blowing up of  $a_1$ . The action of  $S_3$  lifts to  $\tilde{Y}$ . The proper inverse image  $|\tilde{R}|$  of  $|R|$  is a 4-dimensional  $S_3$ -stable linear system on  $\tilde{Y}$ . It has no base points and separates points of an open subset of  $\tilde{Y}$ . Hence  $|\tilde{R}|$  defines an  $S_3$ -equivariant morphism  $\tilde{Y} \rightarrow \mathbb{P}^3$  with respect to a certain  $S_3$ -action on  $\mathbb{P}^3$ . Let  $Z$  be its image. This morphism then yields an  $S_3$ -equivariant birational isomorphism  $\psi: \tilde{Y} \rightarrow Z$ .

Let  $l = \gamma^{-1}(a_1)$  and let  $\tilde{R}$  be the proper inverse image of  $R$ . Then  $(l, l) = -1$  and, since  $a_1$  is a double point of  $R$ , we have  $\gamma^*(R) = \tilde{R} + 2l$  and  $(l, \tilde{R}) = 2$ . This yields

$$6 = (R, R) = (\tilde{R}, \tilde{R}) + 4(l, \tilde{R}) + 4(l, l) = (\tilde{R}, \tilde{R}) + 4,$$

so  $(\tilde{R}, \tilde{R}) = 2$ . Since  $\deg Z = (\tilde{R}, \tilde{R})$ , this means that  $Z$  is an  $S_3$ -stable quadric in  $\mathbb{P}^3$ .

*Step 6.* Since the point  $a'_2 := \psi \circ \gamma^{-1}(a_2) \in Z$  is fixed by  $S_3$ , it follows from the complete reducibility of representations of reductive groups that there is an  $S_3$ -stable plane  $L \simeq \mathbb{P}^2$  in  $\mathbb{P}^3$  not passing through  $a'_2$ . Consider the stereographic projection  $\pi: Z \dashrightarrow L$  from  $a'_2$ ; it is birational and  $S_3$ -equivariant. The map  $\pi$  is defined at  $\psi \circ \gamma^{-1}(a_3)$  and  $a'_3 := \pi \circ \psi \circ \gamma^{-1}(a_3) \in L$  is a fixed point of  $S_3$ . Using the complete reducibility argument again, we conclude that there is an  $S_3$ -stable line  $l \subset L$  such that  $a'_3 \in L \setminus l$ . Thus we obtain a faithful linear action of  $S_3$  on  $\mathbb{A}^2 \simeq L \setminus l$ . But there is a unique 2-dimensional faithful linear representation of  $S_3$ , namely that on  $\mathfrak{t}$  given by (9.3), (9.4).

In summary, we have constructed the following chain of birational equivariant maps of  $S_3$ -varieties:

$$\mathfrak{t} \hookrightarrow L \xleftarrow{\pi} Z \xleftarrow{\psi} \tilde{Y} \xrightarrow{\gamma} Y \xleftarrow{\nu} \tilde{X} \xrightarrow{\mu} X \hookrightarrow T.$$

This shows that  $T$  and  $\mathfrak{t}$  are birationally isomorphic as  $S_3$ -varieties, thus completing the proof of Proposition 9.1.  $\square$

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## REFERENCES

- [BLB] C. BESSENRODT, L. LE BRUYN, *Stable rationality of certain  $PGL_n$ -quotients*, Invent. Math. **104** (1991), 179–199.
- [BMT] G. BERHUY, M. MONSURRÒ, J.–P. TIGNOL, *Cohomological invariants and  $R$ -triviality of adjoint classical groups*, Math. Z. **248** (2004), 313–323.
- [Bor] A. BOREL, *Linear Algebraic Groups: Second Enlarged Edition*, Graduate Texts in Mathematics, Vol. 126, Springer-Verlag, 1991.
- [Bou<sub>1</sub>] N. BOURBAKI, *Groupes et algèbres de Lie*, Chap. II, III, Hermann, Paris, 1972.
- [Bou<sub>2</sub>] N. BOURBAKI, *Groupes et algèbres de Lie*, Chap. IV, V, VI, Hermann, Paris, 1968.
- [Bou<sub>3</sub>] N. BOURBAKI, *Groupes et algèbres de Lie*, Chap. VII, VIII, Hermann, Paris, 1975.
- [Br] K. S. BROWN, *Cohomology of Groups*, Graduate Texts in Mathematics, Vol. 87, Springer-Verlag, New York, 1982.
- [Car] J. F. CARLSON, *Modules and Group Algebras*, Notes by R. SUTER, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1996.

- [Ca] A. CAYLEY, *Sur Quelques propriétés des déterminants gauches*, J. Rein. Angew. Math. (Crelle) **32** (1846), 119–123. Reprinted in: *The Coll. Math. Papers of Arthur Cayley*, Vol. I, No. 52, Cambridge University Press, 1889, 332–336.
- [Ch<sub>1</sub>] C. CHEVALLEY, *On algebraic group varieties*, J. Math. Soc. Japan **6** (1954), 303–324.
- [Ch<sub>2</sub>] C. CHEVALLEY, *The Algebraic Theory of Spinors and Clifford Algebras*, Springer, 1991.
- [CW] G. CLIFF, A. WEISS, *Summands of permutation lattices for finite groups*, Proc. of the AMS **110** (1999), no. 1, 17–20.
- [C–TS<sub>1</sub>] J.–L. COLLIOT–THÉLÈNE, J.–J. SANSUC, *La R-équivalence sur les tores*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 175–229.
- [C–TS<sub>2</sub>] J.–L. COLLIOT–THÉLÈNE, J.–J. SANSUC, *Principal homogeneous spaces under flasque tori: Applications*, J. Algebra **106** (1987), 148–205.
- [C–T] J.–L. COLLIOT–THÉLÈNE (with the collaboration of J.–J. Sansuc) *The rationality problem for fields under linear algebraic groups (with special regards to the Brauer group)*, IX Escuela Latinoamericana de Matemáticas, Santiago de Chile, July 1988.
- [CK] A. CORTELLA, B. KUNYAVSKIĬ, *Rationality problem for generic tori in simple groups*, J. Algebra **225** (2000), no. 2, 771–793.
- [CM] V. CHERNOUSOV, A. MERKURJEV, *R-equivalence and special unitary groups*, J. Algebra **209** (1998), 175–198.
- [CR] C. CURTIS, I. REINER, *Methods of Representation Theory with Applications to Finite Groups and Orders*, Vol. I, Reprint of the 1981 original, Wiley Classics Library, A Wiley Interscience Publication, John Wiley and Sons, Inc, New York, 1990.
- [Di] J. DIEUDONNÉ, *La géométrie des groupes classiques*, Erg. Math. Grenz., Bd. 5, Springer-Verlag, 1971.
- [Do] I. V. DOLGACHEV, *Rationality of fields of invariants*, in: *Algebraic Geometry, Bowdoin*, 1985 (Brunswick, Maine, 1985), Amer. Math. Soc., Providence, RI, 1987, pp. 3–16.
- [HK] M. HAJJA, M. C. KANG, *Some actions of symmetric groups*, J. Algebra **177** (1995), no. 2, 511–535.
- [Höl] O. HÖLDER, *Die Gruppen der Ordnungen  $p^3$ ,  $pq^2$ ,  $pqr$ ,  $p^4$* , Math. Ann. **43** (1893), 301–412.
- [Isk<sub>1</sub>] V. A. ISKOVSIKH, *Minimal models of rational surfaces over arbitrary fields*, Math. USSR-Izv. **14** (1980), no. 1, 17–39.
- [Isk<sub>2</sub>] V. A. ISKOVSIKH, *Cremona group*, in: Math. Encyclopaedia, Vol. 3, Sov. Encycl., Moscow, 1982, (in Russian), p. 95.
- [Isk<sub>3</sub>] V. A. ISKOVSIKH, *Factorization of birational mappings of rational surfaces from the point of view of Mori theory*, Russian Math. Surveys **51** (1996), no. 4, 585–652.
- [Isk<sub>4</sub>] V. A. ISKOVSIKH, *Two non-conjugate embeddings of  $S_3 \times \mathbb{Z}_2$  into the Cremona group*, Proc. Steklov Inst. of Math. **241** (2003), 93–97.
- [Isk<sub>5</sub>] V. A. ISKOVSIKH, *Two non-conjugate embeddings of  $S_3 \times \mathbb{Z}_2$  into the Cremona group*, II, [arXiv:math.AG/0508484](https://arxiv.org/abs/math/0508484).
- [KM] B. KOSTANT, P. W. MICHOR, *The generalized Cayley map for an algebraic group to its Lie algebra*, in: *The Orbit Method in Geometry and Physics (Marseille, 2000)*, Progr. Math., Vol. 213, Birkhäuser Boston, MA, 2003, pp. 259–296.
- [KMRT] M.–A. KNUS, A. MERKURJEV, M. ROST, J.–P. TIGNOL, *The Book of Involutions*, Colloquium Publications, Vol. 44, AMS, 1998.
- [Kn] M. KNESER, *Lectures on Galois Cohomology of Classical Groups*, Tata Inst. of Fund. Research Lectures on Math., Vol. 47, Bombay, 1969.
- [LB<sub>1</sub>] L. LE BRUYN, *Centers of generic division algebras, the rationality problem 1965–1990*, Israel J. Math. **76** (1991), no. 1–2, 97–111.
- [LB<sub>2</sub>] L. LE BRUYN, *Generic norm one tori*, Nieuw Arch. Wisk. (4) **13** (1995), no. 3, 401–407.
- [LL] N. LEMIRE, M. LORENZ, *On certain lattices associated with generic division algebras*, J. Group Theory **3** (2000), no. 4, 385–405.
- [LPR] N. LEMIRE, V. L. POPOV, Z. REICHSTEIN, *Cayley groups*, Forschungsinstitut für Mathematik ETH Zürich preprint, August 2004.
- [Lun<sub>1</sub>] D. LUNA, *Sur la linéarité au voisinage d'un point fixe*, Unpublished manuscript, May 1972.
- [Lun<sub>2</sub>] D. LUNA, *Slices étales*, Bull. Soc. Math. France, Mémoire **33** (1973), 81–105.
- [Lun<sub>3</sub>] D. LUNA, *Letter to V. L. Popov*, January 18, 1975.
- [Ma<sub>1</sub>] YU. I. MANIN, *Rational surfaces over perfect fields II*, Math. USSR, Sb. **1** (1968), 141–168.

- [Ma<sub>2</sub>] YU. I. MANIN, *Cubic Forms: Algebra, Geometry, Arithmetic*, North-Holland Mathematical Library, Vol. 4. North-Holland Publ. Co., Amsterdam, London; Amer. Elsevier Publ. Co., New York, 1974.
- [Me] A. MERKURJEV, *R-equivalence and rationality problem for semisimple adjoint classical algebraic groups*, Publ. Math. IHES **46** (1996), 189–213.
- [MFK] D. MUMFORD, J. FOGARTY, F. KIRWAN, *Geometric Invariant Theory: Third Enlarged Edition*, Erg. Math. Grenzg., 3 Folge, 34, Springer-Verlag, 1994.
- [Pop<sub>1</sub>] V. L. POPOV, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector bundles*, Math. USSR-Izv. **8** (1974), 301–327.
- [Pop<sub>2</sub>] V. L. POPOV, *Letter to D. Luna*, March 12, 1975.
- [Pop<sub>3</sub>] V. L. POPOV, *Sections in invariant theory*, in: *The Sophus Lie Memorial Conference (Oslo, 1992)*, Scand. Univ. Press, Oslo, 1994, pp. 315–361.
- [PV] V. L. POPOV, E. B. VINBERG, *Invariant Theory*, in: *Algebraic Geometry IV*, Encycl. Math. Sci., Vol. 55, Springer-Verlag, 1994, pp. 123–284.
- [Pos] M. POSTNIKOV, *Lie Groups and Lie Algebras. Lectures in Geometry. Semester V*, Mir, Moscow, 1986.
- [Sa] D. J. SALTMAN, *Invariant fields of linear groups and division algebras*, in: *Perspectives in Ring Theory, Antwerp, 1987*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 233, Kluwer Acad. Publ., Dordrecht, 1988, pp. 279–297.
- [Se] J.-P. SERRE, *Espaces fibrés algébriques*, in: Séminaire C. Chevalley E.N.S., 1958, *Anneaux de Chow et applications*, Exposé no. 1, Secr. math. 11 rue Pierre Curie, Paris 5e, 1958, 1-01–1-37. Reprinted in: *Exposés de Séminaires 1950–1999*, Doc. Math. **1**, Soc. Math. France, 2001, 107–139.
- [Sh] I. R. SHAFAREVICH, *Basic Algebraic Geometry*, Vol. 1, Second Ed., Springer-Verlag, Berlin, 1994.
- [Spe] A. SPEISER, *Zahlentheoretische Sätze aus der Gruppentheorie*, Math. Zeitschrift **5** (1919), 1–6.
- [Spr] T. A. SPRINGER, *Linear Algebraic Groups: Second Ed.*, Progress in Math., Vol. 9, Birkhäuser, 1998.
- [Sw] R. G. SWAN, *Invariant rational functions and a problem of Steenrod*, Invent. Math. **7** (1969), 148–158.
- [Vos] V. E. VOSKRESENSKIĬ, *Algebraic Groups and Their Birational Invariants*, AMS, Providence, RI, 1998.
- [VK] V. E. VOSKRESENSKIĬ, A. A. KLYACHKO, *Toroidal Fano varieties and root systems*, Math. USSR-Izv. **24** (1985), 221–244.
- [Weib] C. A. WEIBEL, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, Cambridge, 1994.
- [Weil] A. WEIL, *Algebras with involution and the classical groups*, J. Indian Math. Soc., (N.S.) **24** (1961), 589–623.
- [Weyl] H. WEYL, *The Classical Groups, Their Invariants and Representations*, Princeton, N.J., Princeton Univ. Press, 1935.

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