ON NORI'S OBSTRUCTION TO UNIVERSAL BUNDLES.

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Abstract. Let G be $\mathrm{SL}_n, \mathrm{Sp}_{2n}$ or SO_{2n}. We consider the moduli space M of semistable principal G-bundles over a curve X. Our main result is that if U is a Zariski open subset of M then there is no universal bundle on $U \times X$.

1. Introduction

We work over an algebraically closed field k of characteristic zero. Let X be a smooth projective curve over k. If G is a semisimple algebraic group, a moduli space of semistable G-bundles over X is constructed in various works, for example [Ram96a], [Ram96b] and [Sch02]. We outline the construction below in section 2. The construction of the moduli space depends upon fixing a faithful representation $\rho : G \to \mathrm{SL}_n$. We will denote the obtained moduli space by $\mathrm{M}(\rho, G)$.

The main result of this paper is that a universal bundle cannot exist over the generic point of $\mathrm{M}(\rho, G)^0$ when G is a classical group, not of adjoint type, with its standard representation. (Here $\mathrm{M}(\rho, G)^0$ denotes the connected component containing the trivial bundle. It is an irreducible algebraic variety.) When G is of adjoint type it is known that a universal bundle does indeed exist. Our result is closely related to the work of Balaji, Biswas, Nagaraj and Newstead in [BBNN06]. They show that for a connected semisimple algebraic group, a universal bundle exists over M' if and only if H is of adjoint type. Here M' is the open subset parameterising bundles with automorphism group the centre of H. Using [Mil80, IV Corollary 2.6] one sees that our results are a proper subset of those in [BBNN06]. The real interest in the results of this paper comes from the methods. The proof of the theorem in [BBNN06] requires a detour in the world of topology while our proof is entirely in the realm of algebra. The reason we are stuck with characteristic zero arises from the fact that we are applying Luna's slice theorem.

The method of proof that we use originates from some ideas of M. Nori. The kernel of our argument can be found in [Ses82, Part 6]. After completing our work we learned that N. Hoffman had announced different proofs valid for all reductive groups with non-trivial centres. This work is in the process of being written.

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2. Construction of $\mathrm{M}(\rho, G)$

We will follow the construction in [Sch02], however we do not need all the subtleties of the construction that come from wanting to compactify a moduli space. For the convenience of the reader, we outline the construction of an open subset of the moduli space in [Sch02] that we will denote by $\mathrm{M}(\rho, G)$.

2.1. **Reductions of structure group.** Recall that we have fixed a faithful representation $\rho : G \to \mathrm{SL}_n$. If $\mathcal{E}$ is a rank n vector bundle then the associated GL_n-principal bundle is the scheme of isomorphisms

$$\underline{\mathrm{Isom}}(\mathcal{O}_X^n, \mathcal{E}).$$

We are interested in reductions of structure group of this principal bundle to G. This is nothing but a section of the fibration

$$\underline{\mathrm{Isom}}(\mathcal{O}_X^n, \mathcal{E})/G \to X.$$

There is an inclusion

$$\underline{\mathrm{Isom}}(\mathcal{O}_X^n, \mathcal{E})/G \subseteq \underline{\mathrm{Hom}}(\mathcal{O}_X^n, \mathcal{E})/\!\!/G = \underline{\mathrm{Spec}}(\mathrm{Sym}^*(\mathcal{O}_X^n \otimes \mathcal{E}^\vee)^G).$$

Hence a reduction of structure group is just an algebra homomorphism

$$\tau : \mathrm{Sym}^*(\mathcal{O}_X^n \otimes \mathcal{E}^\vee)^G \to \mathcal{O}_X$$

such that the induced section of

$$\underline{\mathrm{Hom}}(\mathcal{O}_X^n, \mathcal{E})/\!\!/G \to X$$

has image inside of the isomorphism locus.

2.2. **Semistability.** We recall the notions of stability and semistability using [GLSS08] as our reference. Let $\gamma : \mathbb{G}_m \to G$ be a one parameter subgroup of G. Together with the representation $\rho : G \to \mathrm{SL}(W)$ it defines a weight space decomposition

$$W = \oplus_{n\in\mathbb{Z}} W_n.$$

We may order the weights $n_1 < n_2 < \cdots < n_k$. There is an induced flag

$$F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k = W$$

where

$$F_i = \oplus_{j<i} W_{n_j}.$$

The parabolic subgroup of G preserving this flag will be denoted by $Q(\gamma)$. We also set $\alpha_i = \frac{n_{i+1}-n_i}{\dim W}$.

Let $P \to X$ be a principal G-bundle with associated vector bundle $\rho_*(P)$. A reduction of structure group τ of P to $Q(\gamma)$ yields a filtration

$$0 \subsetneq \mathcal{A}_1 \subset \mathcal{A}_2 \cdots \subset \mathcal{A}_k = \rho_*(P),$$

which we denote by $P(\tau, \gamma)_\bullet$. Finally set

$$L(P, \tau, \gamma) = \sum \alpha_i (\mathrm{rk}\mathcal{A}_i \deg(\mathcal{A}) - \mathrm{rk}\mathcal{A} \deg(A_i)).$$

Definition 2.1. We say that P is *semistable* if $L(P, \tau, \gamma) \geq 0$ for every one parameter subgroup γ and reduction of structure group τ.

Note that since we are over a curve this definition is equivalent to the definition given in [Sch02]. Further, since the α_i are positive, the trivial principal bundle is semistable.

Given a positive rational number, δ, there is a related notion of δ-semistability that we do not recall here as we do not need it. However, let us note the following result.

Theorem 2.2. *Fix a Hilbert polynomial F. There is a positive rational number δ_0 such that for every $\delta > \delta_0$ and every principal bundle P with $\rho_*(P)$ having Hilbert polynomial F, the following are equivalent:*

(1) *P is semistable.*
(2) *P is δ-semistable*

Proof. See [GLSS08, Theorem 5.4.1]. □

In [Sch02] and [GLSS08], moduli spaces for δ-semistable bundles are constructed which depend upon an extra parameter. We will always assume that the parameter is chosen so the above theorem applies.

2.3. **The Parameter Space.** The first step towards constructing a moduli space for semistable bundles is to construct a parameter space for them. Consider the family $\mathfrak{B}$ of vector bundles of rank n and trivial determinant that admit a reduction of structure group to a semistable G-bundle. It is shown that $\mathfrak{B}$ is bounded in [Sch02, Remark 3.7].

Using boundedness, we can find an integer N so that for all $m \geq N$ and every $\mathcal{F} \in \mathfrak{B}$ we have

- $\mathrm{H}^i(X, \mathcal{F}(m)) = 0$ for $i > 0$
- $\mathcal{F}(m)$ is generated by global sections.

Set $W = \mathrm{H}^0(X, \mathcal{F}(N))$ for some chosen $\mathcal{F} \in \mathfrak{B}$. We have a quot scheme $\mathfrak{Q}$ of quotients of $\mathcal{O}_X(-N) \otimes W$ of degree 0 and rank $n = \dim \rho$. We have on $\mathfrak{Q} \times X$ a universal quotient

$$W \otimes \pi^* \mathcal{O}_X(-N) \to \mathcal{F}^{\mathrm{univ}},$$

where π is the projection $\mathfrak{Q} \times X \to X$. There is an open subset $U \subset \mathfrak{Q}$ parameterising locally free quotients. In [Sch02, page 1197-1198], a scheme $\mathfrak{N}'$ and a map

$$\mathfrak{N}' \to U$$

is constructed. The fibre of the scheme $\mathfrak{N}'$ over the quotient

$$W \otimes \mathcal{O}_X(-N) \to \mathcal{F}$$

parameterises the collection of algebra homomorphisms

$$\tau : \mathrm{Sym}^*(\mathcal{O}_X^n \otimes \mathcal{F}^\vee)^G \to \mathcal{O}_X.$$

The object of interest in [Sch02] is

$$\mathfrak{N}' /\!\!/ \mathrm{GL}(W).$$

There is an open subscheme $\mathfrak{N}$ parameterising those algebra homomorphisms

$$\tau : \mathrm{Sym}^*(\mathcal{O}_X^n \otimes \mathcal{F}^\vee)^G \to \mathcal{O}_X.$$

that come from reductions of structure group. We are interested in the quotient

$$\mathrm{M}(\rho, G) = \mathfrak{N} /\!\!/ \mathrm{GL}(W).$$

3. A Little Deformation Theory

3.1. **Deformations of Principal Bundles.** We begin by recalling some facts from the deformation theory of principal bundles. Proofs can be found in [Ill71], [Ill72a] and [Ill72b]. We do not need the full force of the results in the cited references as the group scheme $G \times X \to X$ is smooth.

For a G-bundle $P \to X$ we denote by $\mathrm{ad}(P)$ the adjoint bundle which is defined by

$$\mathrm{ad}(P) = P \times_{G,\mathrm{ad}} \mathrm{Lie}(G).$$

Consider an extension of local Artinian k-algebras

$$0 \to I \to A' \to A \to 0$$

with $I^2 = 0$.

Theorem 3.1. *Let P_A be a G-bundle on X_A. Then this bundle extends to $X_{A'}$. The collection of all such lifts is a torsor under*

$$\mathrm{H}^1(X_A, \mathrm{ad}(P_A) \otimes I).$$

Proof. See [Ill72b, Theorem 2.6] and [Ill72b, Remark 2.6.1]. Note that H^2 vanishes as we are on a curve. □

Next we need some facts about flat deformations of coherent sheaves.

Theorem 3.2. *Fix a sheaf $\mathcal{E}$ on X and consider a flat family of coherent sheaves $\mathcal{F}_A$ on X_A that fit into an exact sequence*

$$0 \to \mathcal{K}_A \to \mathcal{E} \otimes_k A \to \mathcal{F}_A \to 0$$

Denote by $\mathcal{F}$ and $\mathcal{K}$ the restriction to the closed point of A. If

$$\operatorname{Ext}^1(\mathcal{K}, \mathcal{F}) = 0$$

then we may extend the exact sequence to $X_{A'}$ so that $\mathcal{F}_A$ extends to a flat family. Furthermore, the collection of all lifts is a torsor under

$$\operatorname{Hom}(\mathcal{K}_A, \mathcal{F}_A \otimes I).$$

Proof. See [Ser86]. □

3.2. The Smoothness of $\mathfrak{N}$.

Proposition 3.3. *Let p be a point of $\mathfrak{N}$ then $\mathfrak{N}$ is smooth at p.*

Proof. Let p correspond to the pair $(q : \mathcal{O}_X(-N) \otimes W \to \mathcal{F}, \sigma)$. Let n be the rank of $\mathcal{F}$ and let P be the bundle corresponding to σ. So P is defined by a cartesian diagram

$$\begin{array}{ccc} \underline{\operatorname{Isom}}(\mathcal{O}_X^n, \mathcal{F}) & \longrightarrow & \underline{\operatorname{Isom}}(\mathcal{O}_X^n, \mathcal{F})/G \\ \uparrow & & \uparrow{\scriptstyle \sigma} \\ P & \longrightarrow & X \end{array}$$

Consider an extension of local Artinian k-algebras

$$0 \to I \to A' \to A \to 0$$

with $I^2 = 0$ and extensions $q_A, \mathcal{F}_A, P_A, \sigma_A$ of the above data to X_A. We need to extend q_A and P_A to $X_{A'}$ in a compatible way. Note that $\operatorname{Ext}^1(\mathcal{K}, \mathcal{F}) = 0$ since

$$\operatorname{Ext}^1(\mathcal{O}_X(-N) \otimes W, \mathcal{F}) \to \operatorname{Ext}^1(\mathcal{K}, \mathcal{F}) \to \operatorname{Ext}^2(\mathcal{F}, \mathcal{F})$$

is exact, $\operatorname{Ext}^1(\mathcal{O}_X(-N) \otimes W, \mathcal{F}) \cong H^1(X, \mathcal{F}(N)) \otimes W^\vee = 0$ by the choice of the quot scheme in 2.3 and $\operatorname{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$ since X is a curve. Then, by the last proposition, $\mathcal{F}_A$ extends to a flat family. Let S be the set of extensions of $\mathcal{F}_A$ to $X_{A'}$ and T the set of extensions of P_A to $X_{A'}$. We have a map given by extension of structure group

$$\psi : T \to S.$$

We need to choose $q_{A'}$ and $P_{A'}$ so that $\psi(P_{A'}) = \mathcal{F}_{A'}$. The set S is a torsor under $H^1(X_A, \operatorname{ad}(\mathcal{F}) \otimes I) = \operatorname{Ext}^1(\mathcal{F}_A, \mathcal{F}_A \otimes I)$. By the choice of N in 2.3, we see that $\operatorname{Ext}^1(\mathcal{O}_X(-N) \otimes W, \mathcal{F}_A \otimes I) \cong H^1(X, \mathcal{F}_A(N) \otimes I) \otimes W^\vee = 0$ so that there is a surjection

$$e : \operatorname{Hom}(\mathcal{K}_A, \mathcal{F}_A \otimes I) \to \operatorname{Ext}^1(\mathcal{F}_A, \mathcal{F}_A \otimes I)$$

arising from the exact sequence

$$0 \to \mathcal{K}_A \to \mathcal{O}_{X_A}(-N) \otimes W \to \mathcal{F}_A \to 0.$$

Furthermore, $\mathrm{Hom}(\mathcal{K}_A, \mathcal{F}_A \otimes I)$ classifies extensions of q_A to $X_{A'}$. One checks if we have two extensions q_1 and q_2 then the classes in S of $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy

$$e(q_2 - q_1) + \mathcal{F}_1 = \mathcal{F}_2 \quad \text{in } S,$$

where + sign denotes the torsor action described in 3.1. We have

$$\mathrm{H}^1(X_A, \mathrm{ad}(P_A) \otimes I) \subseteq \mathrm{H}^1(X_A, \mathrm{ad}(\mathcal{F}_A) \otimes I) = \mathrm{Ext}^1(\mathcal{F}_A, \mathcal{F}_A \otimes I).$$

Choose a lift $(q_{A'}, \mathcal{F}_{A'})$ of $(q_A, \mathcal{F}_A)$ and a lift $P_{A'}$ of P_A. We can hence find an element x of $\mathrm{Hom}(\mathcal{K}_A, \mathcal{F}_A \otimes I)$ so that

$$e(x) + \mathcal{F}_{A'} = \psi(P_{A'}).$$

□

3.3. **The Slice Theorem.** Consider a quotient of the form

$$0 \to \mathcal{K} \to \mathcal{O}_X(-N) \otimes W \to \mathcal{O}_X^n \to 0$$

and the canonical section $\sigma : X \to X \times \mathrm{GL}_n/G$. This pair gives a point 0 of $\mathfrak{N}$ which corresponds to the trivial G bundle $G \times X \to X$ and we wish to apply Luna's étale slice theorem to the action of $\mathrm{GL}(W)$ on $\mathfrak{N}$ near 0.

Firstly notice that the surjection identifies

$$W \cong \mathrm{H}^0(X, \mathcal{O}_X^n(N)).$$

This in turn gives a direct sum decomposition to W which allows us to identify GL_n with a subgroup of $\mathrm{GL}(W)$.

The discussion of the preceeding subsection allows us to identify the Zariski tangent space of $\mathfrak{N}$ at 0 with the following fibred product in the category of vector spaces

$$\begin{array}{ccc} T_0(\mathfrak{N}) & \longrightarrow & \mathrm{H}^1(X, \mathrm{Lie}(G) \otimes \mathcal{O}_X) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(\mathcal{K}, \mathcal{O}_X^n) & \longrightarrow & \mathrm{H}^1(X, \mathrm{Lie}(\mathrm{GL}_n) \otimes \mathcal{O}_X). \end{array}$$

Note here that

$$H^1(X, \mathrm{ad}(G \times X)) = H^1(\mathrm{Lie}(G) \otimes \mathcal{O}_X) \cong \mathrm{Lie}(G)^g.$$

Using relevant cohomology sequences, we see that the vertical arrows are injective and the horizontal ones surjective. Note also that

$$\mathrm{Hom}(\mathcal{O}_X(-N) \otimes W, \mathcal{O}_X^n) \cong W^\vee \otimes W \cong \mathrm{Lie}(\mathrm{GL}(W))$$

The differential of the orbit map at 0 gives a morphism

$$d\mathrm{orb}_0 : \mathrm{Lie}(\mathrm{GL}(W)) \to T_0(\mathfrak{N}).$$

Composing this with the inclusion into $\mathrm{Hom}(\mathcal{K}, \mathcal{O}_X^n)$ and recalling the definition of W, we may identify it with the natural map

$$\mathrm{Hom}(\mathcal{O}_X(-N) \otimes W, \mathcal{O}_X^n) \to \mathrm{Hom}(\mathcal{K}, \mathcal{O}_X^n).$$

Putting this all together we have

Proposition 3.4. *Let* 0 *be a point of* $\mathfrak{N}$ *corresponding to the trivial* G*-bundle.*

1. *The stabiliser of* 0 *inside* $GL(W)$ *is* G*.*
2. *The normal space to* $\mathrm{orb}(0)$ *can be identified with*

$$\mathrm{H}^1(X, \mathrm{ad}(G \times X)) = \underbrace{\mathrm{Lie}(G) \times \ldots \times \mathrm{Lie}(G)}_{g\ times}.$$

3. *The natural action of the stabiliser on the normal bundle can be identified with the adjoint action.*

Proof. The first two statements are straightforward.

It suffices to prove (3) in the case where $G = \mathrm{GL}_n$, that is the full quot scheme. To see it in this case, let us first spell out the correspondence between $\mathrm{Hom}(\mathcal{K}, \mathcal{O}_X^n)$ and lifts of the extension

$$0 \to \mathcal{K} \to \mathcal{O}_X(-N) \otimes W \to \mathcal{O}_X^n \to 0$$

to the ring of dual numbers. To give a lift of the extension is the same as giving a subsheaf

$$\tilde{\mathcal{K}} \subseteq \mathcal{O}_{X'}(-N) \otimes W,$$

lifting the subsheaf $\mathcal{K}$. One defines the subsheaf corresponding to a homomorphism

$$\phi : \mathcal{K} \to \mathcal{O}_X^n$$

by

$$\tilde{\mathcal{K}} = \{x + \epsilon y \mid \phi(x) = \bar{y}\}$$

where $\bar{y}$ is the image inside $\mathcal{O}_X^n$. A group element $g \in \mathrm{GL}_n$ produces a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{O}_X(-N) \otimes W & \longrightarrow & \mathcal{O}_X^n & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow g & & \downarrow g & & \\ 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{O}_X(-N) \otimes W & \longrightarrow & \mathcal{O}_X^n & \longrightarrow & 0, \end{array}$$

by our identification of GL_n with a subgroup of $\mathrm{GL}(W)$ and the five lemma. Tracing through the identifications above we find that the action on $\mathrm{Hom}(\mathcal{K}, \mathcal{O}_X^n)$ is given by

$$\phi \mapsto g \circ \phi \circ g^{-1}.$$

This induces the adjoint action on the quotient

$$\mathrm{Hom}(\mathcal{K}, \mathcal{O}_X^n) \to \mathrm{H}^1(X, \mathrm{Lie}(\mathrm{GL}_n) \otimes \mathcal{O}_X) \to 0.$$

□

Notation 3.5. We write

$$Z(\mathrm{Lie}(G), g) = \mathrm{Lie}(G) \times \ldots \times \mathrm{Lie}(G).$$

Frequently we will abuse notation and just write Z or $Z(\mathrm{Lie}(G))$ when it is clear from the context what is meant.

Corollary 3.6. *The completion of the local ring of $M(\rho, G)$ at the trivial bundle is isomorphic to $\widehat{\mathcal{O}_{Z/\!\!/G,0}}$.*

Proof. This is just an application of the slice theorem, see [Dré04]. □

4. Geometric Quotients and Azumaya Algebras

We let $\Lambda = k < x_1, \ldots, x_g >$ be a polynomial ring in g noncommuting variables. If R is a k-algebra then an R-point of $Z(\mathrm{Lie}(\mathrm{GL}_n))$ is equivalent to giving a $R \otimes \Lambda$-module structure on R^n.

Fix a vector space V with a nondegenerate bilinear form B that is either symmetric or alternating. The form B determines an anti-automorphism r of $\mathrm{End}_k(V)$ determined by

$$B(v, f(w)) = B(rf(v), w)$$

for all $f \in \mathrm{End}_k(V)$ and $v, w \in V$. We denote the corresponding group of automorphisms of V preserving the form by G. To give an R-point of $Z(\mathrm{Lie}(G))$ is the same as giving a $R \otimes \Lambda$-module structure on $R \otimes V$ such that if we view $x_i \in \mathrm{End}_R(R \otimes V)$ we have $r(x_i) = -x_i$.

4.1. **Stability.** In this subsection we will give a description of the stable locus of $Z(\mathrm{Lie}(G))$ for the adjoint action of G in terms of the functor of points description above. We denote the stable locus by $Z(\mathrm{Lie}(G))^s$. Fix a representation V of G and $v \in V$ and a one parameter subgroup $\lambda : \mathbb{G}_m \to G$. We can decompose $v = \sum_i v_{n_i}$ where $v_{n_i} \neq 0$ has weight n_i. One defines

$$\mu(v, \lambda) = -\min\{n_i\}.$$

Proposition 4.1. *We have $x \in Z(\mathrm{Lie}(\mathrm{GL}_n))^s$ if and only if the corresponding Λ-module is simple.*

Proof. Suppose that $x = (x_1, x_2, \ldots, x_g)$ is not stable. Then there a 1-parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{GL}_n$ with $\mu(x, \lambda) \leq 0$, see [MFK94, page 49]. We may assume that λ is diagonal or even

$$\lambda(t) = \begin{pmatrix} t^{r_1} & & & \\ & t^{r_2} & & \\ & & \ddots & \\ & & & t^{r_n} \end{pmatrix}$$

with $r_1 \geq r_2 \geq \ldots \geq r_n$. Then the fact $\mu(x, \lambda) \leq 0$ translates into the fact that each of the matrices x_i are of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & * \end{pmatrix}$$

where the size of the blocks $*$ is determined by the nonzero integers in the list $r_i - r_j$ with $i > j$. At any rate, one sees that V has a submodule.

When V is not simple, then V has a submodule W. The subgroup of GL_n preserving the flag $W \subseteq V$ is a parabolic subgroup. One easily reverses the above argument to construct a one parameter subgroup with non-positive μ. ☐

We will call a subspace U of a vector space V equipped with a non-degenerate symmetric or alternating quadratic form B *totally isotropic with respect to* B if the restriction of B to U is zero.

Proposition 4.2. *We fix a nondegenerate symmetric or alternating bilinear form B on V and let G be the corresponding automorphism group preserving the form. A point $x \in Z(\mathrm{Lie}(G))$ is in the stable locus if and only if the corresponding $R \otimes \Lambda$-module structure on $R \otimes V$ has no nontrivial submodules that are totally isotropic with respect to B.*

Proof. One uses the proof of the previous proposition and keeps track of what it means for the one parameter subgroup to be inside G. We give more details for the case of Sp_{2n} and leave the case of SO_{2n} to the reader.

Given a one parameter subgroup

$$\lambda(t) = \begin{pmatrix} t^{r_1} & & & \\ & t^{r_2} & & \\ & & \ddots & \\ & & & t^{r_{2n}} \end{pmatrix}$$

with image inside Sp_{2n} (using the standard symplectic form) by swapping basis vectors we may arrange weights so that $r_1 > r_2 > \ldots > r_n$

and $r_i = -r_{i+n}$. Suppose that we have a point $x = (x_1, x_2, \ldots, x_g) \in Z(\mathrm{Lie}(\mathrm{Sp}_{2n}))$ such that $\mu(x, \lambda) \leq 0$. Then if we write x_i as $n \times n$ blocks, that is

$$x_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$$

we must have $C_i = 0$. Hence there is an totally isotropic submodule. The argument is easily reversed using the analogue of Witt's extension theorem for alternating bilinear forms [Jac85, p. 391]. More explicitly, we use the fact that one can extend an totally isotropic subspace to a maximal totally isotropic subspace and all such maximal totally isotropic subspaces are the same up to automorphism. □

Proposition 4.3. *Suppose that* $G = \mathrm{GL}_n,\ \mathrm{Sp}_{2n}$ *or* SO_{2n}. *We have a* G^{ad}*-principal bundle*

$$Z(\mathrm{Lie}(G))^s \to Z(\mathrm{Lie}(G))^s / G^{ad}.$$

Proof. We need to show that the orbit map

$$\Psi : Z(\mathrm{Lie}(G))^s \times G \to Z(\mathrm{Lie}(G))^s \times Z(\mathrm{Lie}(G))^s$$

is a closed immersion. The action on the stable locus is proper by [MFK94, page 55, Corollary 2.5]. We need to show that points have trivial stabilisers. For GL_n this follows from 4.1 . Suppose that $\alpha \in G$ fixes a point x. This amounts to the fact that α induces a Λ-module automorphism of V.

Let $\lambda \neq 0$ be an eigenvalue of α and consider the λ-eigenspace of α, $K = E_\lambda(\alpha)$. As α preserves the form B we have that K is totally isotropic. Explicitly if $v, w \in K$ then

$$\begin{aligned} B(v, w) &= B(\alpha(v), \alpha(w)) \\ &= \lambda^2 B(v, w). \end{aligned}$$

So $B(v, w) = 0$ and K is totally isotropic. Hence $K = V$. □

4.2. **Nori's Obstruction to Universal Bundles.**

Proposition 4.4. *There exists a split Azumaya algebra with an action of* PGL_n *on* $Z(\mathrm{Lie}(\mathrm{GL}_n))$.

Proof. First let us write down the action of PGL_n on the functor of points of $Z(\mathrm{Lie}(\mathrm{GL}_n))$. Recall that an R-point of $Z(\mathrm{Lie}(\mathrm{GL}_n))$ is the same thing as a $R \otimes \Lambda$-module structure on R^n. An R-point of GL_n acts on R^n giving a different module structure and hence a different map to $Z(\mathrm{Lie}(\mathrm{GL}_n))$. The centre acts trivially hence we really have a PGL_n action. This action lifts in a natural way to the Azumaya algebra $\mathrm{End}_R(R^n)$. □

Corollary 4.5. *Let $U \subseteq Z(\mathrm{Lie}(G))$ be a G^{ad}-stable open subset on which G^{ad} acts freely. Then there exists an Azumaya algebra on U/G.*

Proof. This follows by restricting the algebra in the proposition to U. □

We will denote the Azumaya algebra constructed above by the symbol $\mathcal{A}(\rho, U)$. We will often abuse notation and just write $\mathcal{A}$.

We denote the completion of the local ring at 0 of $Z(\mathrm{Lie}(G))/\!\!/G$ by $\widehat{\mathcal{O}_{Z/\!\!/G,0}}$ and its function field by $\hat{k}_{Z/\!\!/G,0}$. We define $\mathcal{O}^s_{Z/\!\!/G,0}$ by the Cartesian square

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}^s_{Z/\!\!/G,0}) & \longrightarrow & Z^s/G \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathcal{O}_{Z/\!\!/G,0}) & \longrightarrow & Z/\!\!/G. \end{array}$$

The ring $\widehat{\mathcal{O}^s_{Z/\!\!/G,0}}$ is defined by a similar Cartesian square.

The class of $\mathcal{A}$ in $\mathrm{Br}(\widehat{\mathcal{O}^s_{Z/\!\!/G,0}})$ is called *Nori's obstruction.*

The key result is:

Theorem 4.6. *Denote by $M(\rho, G)^0$ the connected component of the moduli space containing the trivial bundle. If a universal bundle exists on some open subset of $M(\rho, G)^0$ then the Brauer class of $\mathcal{A}$ in $Br(\hat{k}_{Z/\!\!/G,0})$ vanishes.*

Proof. Using [Dré04, Lemma 5.1] and 3.4, there is an open neighbourhood V of the trivial bundle in $\mathfrak{N}$ and a G invariant morphism

$$\phi : V \to \mathrm{T}_0(\mathfrak{N}),$$

where 0 is the trivial bundle. By [Sch02, Proposition 5.1], a universal bundle produces a rational section

$$\sigma : \mathrm{M}(\rho, G)^0 \dashrightarrow \mathfrak{N}.$$

Composing $\phi \circ \sigma$ with the projection to the normal space to the orbit

$$\mathrm{T}_0(\mathfrak{N}) \to \mathrm{N}_{\mathrm{orb}(0)}$$

and recalling that the slice theorem tells us that

$$\widehat{\mathcal{O}_{M,0}} \cong \widehat{\mathcal{O}_{Z/\!\!/G,0}}$$

we see that the map

$$Z^s \to Z^s/G^{\mathrm{ad}}$$

has a section over $\hat{k}_{Z/\!\!/G,0}$. Hence our algebra is split. □

5. Some Remarks on Generic Quaternion Algebras

We will recall here some facts on the generic quaternion algebra over the field $k(x,y)$ that will be needed below. A detailed exposition can be found in [Dra83, Sections 10 and 11]. The generic quaternion algebra, denoted $\mathcal{B}_{k(x,y)}$, is the algebra generated by the symbols a and b subject to the relations $a^2 = x$, $b^2 = y$ and $ab = -ba$. It is split by the quadratic extension $k(\sqrt{x}, y)$.

Proposition 5.1. (1) *One can realise $\mathcal{B}_{k(x,y)}$ as the subalgebra of $M_2(k(\sqrt{x}, y))$ generated by the matrices*

$$\begin{pmatrix} \sqrt{x} & 0 \\ 0 & -\sqrt{x} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ y & 0 \end{pmatrix}$$

(2) *Let σ be the generator of $\mathrm{Gal}(k(\sqrt{x}, y)/k(x,y))$. Extend the action of $\mathrm{Gal}(k(\sqrt{x}, y)/k(x,y))$ to $M_2(k(\sqrt{x}, y))$ by defining*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{\sigma} = \begin{pmatrix} \delta^{\sigma} & \frac{\gamma^{\sigma}}{y} \\ \beta^{\sigma} y & \alpha^{\sigma} \end{pmatrix}.$$

Then $\mathcal{B}_{k(x,y)}$ is the subalgebra fixed by $\mathrm{Gal}(k(\sqrt{x}, y)/k(x,y))$.

Proof. The first part is carried out in [Dra83, section 10] and the second part follows from the calculations in [Dra83, pages 77–78] by some linear algebra. □

6. The Case of Sp_{2n}

Consider the field $\hat{L} = k((x,y))$ and the generic quaternion algebra $\mathcal{B} \otimes \hat{L} = \hat{\mathcal{B}}$ over $\hat{L}$. It is split by the field extension $\hat{K} = k((\sqrt{x}, y))$. We will construct an $\hat{L}$-point of $\mathrm{Spec}(\widehat{\mathcal{O}^s_{Z/\!/G,0}})$ that pulls back Nori's obstruction to $\hat{\mathcal{B}} \otimes M_n(\hat{L})$. (Recall the definition of $\mathrm{Spec}(\widehat{\mathcal{O}^s_{Z/\!/G,0}})$ from §4.2.) By [Mil80, IV Corollary 2.6] and 4.6, this will show that no universal bundle exists on the moduli space for $G = \mathrm{Sp}_{2n}$.

6.1. Construction of the $\hat{L}$-point.

Let $R = k[\sqrt{x}, y]$ and set

$$A = \begin{pmatrix} \sqrt{x}D & 0 \\ 0 & -\sqrt{x}D \end{pmatrix} \quad C = \begin{pmatrix} 0 & I_n \\ yI_n & 0 \end{pmatrix}.$$

where $D = \mathrm{diag}(1, 2, \ldots, n)$. We consider R^{2n} with the symplectic form B determined by the matrix

$$\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

and consider $A, C \in \mathrm{End}_R(R^{2n})$. There is an anti-involution r of $M_{2n}(R)$ determined by taking adjoints that is

$$B(r(f)(v), w) = B(v, f(w)).$$

One easily checks that $r(A) = -A$ and $r(C) = -C$.

Let V be a vector space of dimension $2n$ over k with the standard symplectic form B. Recall that the symplectic group Sp_{2n} is defined as $\{g \in GL(V) | B(v, w) = B(gv, gw) \text{ for all } v, w \in V\}$. The centre of Sp_{2n} is $\{\pm I_{2n}\}$. Taking the quotient gives us the projective symplectic group PSp_{2n}. We also have the group of symplectic similitudes GSp_{2n} defined as $\{g \in GL(V) | B(v, w) = \lambda B(gv, gw) \text{ for all } v, w \in V, \lambda \in k^*\}$. The scalar matrices form a subgroup of GSp_{2n} and the quotient is isomorphic to PSp_{2n}.

Consider the $R \otimes \Lambda$-module structure on R^{2n} where the action of x_i is defined by

$$x_i = \begin{cases} A & i = 1 \\ yC & i \geq 2. \end{cases},$$

Note that $x_i \in \mathrm{Lie}(\mathrm{Sp}_{2n})$. In view of the discussion at the start of section 4, this yields an R-point

$$\phi : \mathrm{Spec}(R) \to Z(\mathrm{Lie}(\mathrm{Sp}_{2n})).$$

There is an associated $K = k(\sqrt{x}, y)$-point

$$\phi_0 : \mathrm{Spec}(K) \to Z(\mathrm{Lie}(\mathrm{Sp}_{2n})).$$

There is an action of $\mathbb{Z}/2\mathbb{Z}$ on R sending $\sqrt{x}$ to $-\sqrt{x}$ and fixing y. Viewing PSp_{2n} as a quotient of GSp_{2n} we define an inclusion

$$\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathrm{PSp}_{2n}$$

with image C. Let R_y be the localisation of R obtained by inverting y. We have an R_y point of $Z(\mathrm{Lie}(\mathrm{Sp}_{2n}))$ denoted by $\phi_y$which is obtained from ϕ. With this definition ϕ_y is equivariant for the $\mathbb{Z}/2\mathbb{Z}$-action as

$$CAC^{-1} = -A$$

and C acts trivially by conjugation on the other x_i defined above.

Setting $S = k[x, y]_y$, we obtain a map

$$\bar{\phi}_y : \mathrm{Spec}(S) \to Z(\mathrm{Lie}(\mathrm{Sp}_{2n})) /\!\!/ \mathrm{Sp}_{2n}$$

by passing to quotients.

Proposition 6.1. (1) *The map $\bar{\phi}_y$ induces a morphism*

$$\bar{\phi}_0 : \mathrm{Spec}(k(x, y)) \to Z(\mathrm{Lie}(\mathrm{Sp}_{2n}))^s / \mathrm{PSp}_{2n}.$$

(2) *Abusing notation slightly, we have ${\bar{\phi}_0}^*(\mathcal{A}) = \mathcal{B}_{k(x,y)} \otimes M_n(k(x, y))$.*

Proof. (1) Set $K = k(\sqrt{x}, y)$. We need to show that the $K \otimes \Lambda$-module structure defined above on K^{2n} has no totally isotropic submodules. Suppose that there exists an totally isotropic submodule V. Then V must contain an A eigenvector v as the endomorphism A is semisimple with distinct eigenvalues. But then one can show that $B(v, C(v)) \neq 0$, so that the $K \otimes \Lambda$ submodule generated by v is not totally isotropic.
(2) In 5.1 part 2, the generic quaternion algebra was constructed as a geometric quotient of the matrix algebra $M_2(K)$ with a prescribed $\mathbb{Z}/2\mathbb{Z}$-action.

There is a $\mathbb{Z}/2\mathbb{Z}$-action on $M_2(K) \otimes M_n(K)$ where $\mathbb{Z}/2\mathbb{Z}$ acts on the second factor via its Galois action on K. Note that

$$C \begin{pmatrix} X & Y \\ W & Z \end{pmatrix} C^{-1} = \begin{pmatrix} Z & y^{-1}W \\ yY & X \end{pmatrix}$$

The discussion above shows that this is the pullback of an action of $\mathbb{Z}/2\mathbb{Z}$ on $Z(\mathrm{Lie}(\mathrm{Sp}_{2n}))$ via ϕ_0. The result follows via taking quotients. □

Corollary 6.2. *There exists a map*

$$\hat{\phi} : \mathrm{Spec}(k((x, y))) \to \mathrm{Spec}(\widehat{\mathcal{O}^s_{Z/\!/G,0}}).$$

such that $\hat{\phi}^* \mathcal{A} = \mathcal{B}_{k(x,y)} \otimes M_n(k(x, y))$.

Proof. The map ϕ sends 0 to 0 hence there is a map on completions of local rings. Therefore we obtain a map

$$\phi_c : \mathrm{Spec}(k((\sqrt{x}, y))) \to \widehat{\mathcal{O}_{Z,0}}.$$

In view of the above proposition it suffices to show that it descends to a morphism

$$\mathrm{Spec}(k((x, y))) \to \widehat{\mathcal{O}_{Z/\!/G,0}}.$$

The fact that ϕ sends 0 to 0 gives a commutative diagram of local rings and local homomorphisms

$$\begin{array}{ccc} k((\sqrt{x}, y)) & \longleftarrow & \mathcal{O}_{Z,0} \\ \uparrow & & \uparrow \\ k((x, y)) & \longleftarrow & \mathcal{O}_{Z/\!/G,0} \end{array}$$

with ϕ_c induced by completion from the top line. The image of the maximal ideal of $\mathcal{O}_{Z/\!/G,0}$ must be contained in the $\mathcal{O}_{Z/\!/G,0}$ submodule of $k((x, y))$ generated by (x, y). But the field $k((x, y))$ is complete with respect to the induced topology and hence we obtain our map. □

Corollary 6.3. *There is no universal bundle over the generic point of $M(\rho, \mathrm{Sp}_{2n})$.*

Proof. The algebra $\mathfrak{B}$ is not split by [Dra83, Corollary 4, page 82]. It follows the algebra $\mathcal{A}$ is not split over $\widehat{\mathcal{O}^s_{Z/\!\!/G,0}}$. The ring $\widehat{\mathcal{O}^s_{Z/\!\!/G,0}}$ is regular as the map

$$Z^s \to Z^s/\mathrm{PSp}_{2n}$$

is a principal bundle and hence Z^s/PSp_{2n} is smooth. By [Mil80, IV Corollary 2.6], the algebra does not split over the generic point of $\widehat{\mathcal{O}^s_{Z/\!\!/G,0}}$. The required result follows from 4.6. □

7. The Case of SO_{2n}

Set $R = k[\sqrt{x}, y]$. We consider on R^{2n} the symmetric form given by the matrix

$$\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

We set SO_{2n} to be the orthogonal group preserving the associated symmetric form B. We define an $R \otimes \Lambda$-module structure on R^{2n} where the action is given by the following matrices :

$$x_1 = \begin{pmatrix} \sqrt{x}D & 0 \\ 0 & -\sqrt{x}D \end{pmatrix}$$

and for $i > 1$

$$x_i = \begin{pmatrix} \sqrt{x}A & yZ \\ y^2 Z & -\sqrt{x}A \end{pmatrix}.$$

In the above $D = \mathrm{diag}(1, 2, \ldots, n)$, $A = (a_{ij})$, and $Z = (z_{ij})$ are given by

$$a_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ or } j = 1 \\ 0 & \text{otherwise} \end{cases} \qquad z_{ij} = \begin{cases} -1 & \text{if } j = 1, i \neq 1 \\ 1 & \text{if } i = 1, j \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $x_i \in \mathrm{Lie}(\mathrm{SO}_{2n})$. This gives an R-point $\phi : \mathrm{Spec}(R) \to Z(\mathrm{Lie}(\mathrm{SO}_{2n}))$ as described at the start of section 4. The definition seems a little arbitrary but it is this way to ensure that it is easy to show that our point hits the stable locus of $Z(\mathrm{Lie}(\mathrm{SO}_{2n}))$.

Denote by PSO_{2n} the group $\mathrm{SO}_{2n}/Z(\mathrm{SO}_{2n})$, that is the adjoint form of SO_{2n}. We denote by GSO_{2n} the group of linear transformations that preserve the standard bilinear form up to a scalar. Finally we define PGSO_{2n} to be

$$\mathrm{GSO}_{2n}/Z(\mathrm{GSO}_{2n}).$$

There is a natural identification $\mathrm{PSO}_{2n} \cong \mathrm{PGSO}_{2n}$ coming from the fact that scalar matrices are central.

This identification gives a map

$$\psi : \mathbb{Z}/2\mathbb{Z} \to \mathrm{PSO}_{2n}$$

with image

$$\begin{pmatrix} 0 & I_n \\ yI_n & 0 \end{pmatrix} \overset{\text{defn}}{=} C$$

Its scheme theoretic image is a closed subgroup $H < \mathrm{PSO}_{2n}$ defined over $k(y)$.

Denote by σ the automorphism of R fixing y and sending $\sqrt{x}$ to $-\sqrt{x}$. Note that

$$Cx_iC^{-1} = x_i^\sigma.$$

Hence if we set $R_y = k[\sqrt{x}, y]_y$ we obtain an induced R_y point that induces a map

$$\mathrm{Spec}(k[x,y]_y) \to Z(\mathrm{Lie}(\mathrm{SO}_{2n}))/\!\!/\mathrm{SO}_{2n}.$$

Proposition 7.1. (1) *There is an induced morphism*

$$\bar{\phi}_0 : \mathrm{Spec}(k(x,y)) \to Z(\mathrm{Lie}(\mathrm{SO}_{2n}))^s/\mathrm{PSO}_{2n}.$$

(2) $\bar{\phi}_0^*\mathcal{A} = \mathcal{B}_{k(x,y)} \otimes M_n(k(x,y))$.

Proof. (1) We need to show that our map ϕ sends the generic point to the stable locus. By 4.2 this amounts to showing that there are no totally isotropic submodule. Suppose that we have an totally isotropic submodule $V \subseteq k(x,y)^{2n}$. As x_1 is semisimple with distinct eigenvalues, V must contain an x_1 eigenvector. We may as well assume that is $e_i \in k(x,y)^{2n}$ where $[e_i]_j = \delta_{ij}$. But then $B(x_2(e_i), x_2(e_i)) \neq 0$ so that the $K \otimes \Lambda$ module generated by e_i is not totally isotropic.
(2) This is similar to the analogous result for the symplectic group. The algebra $\mathcal{A}$ is a quotient of the trivial Azumaya algebra of rank $4n^2$ over Z where SO_{2n} acts on this algebra by conjugation. Pulling this action back via ψ and ϕ one obtains the action described in 5.1. □

Corollary 7.2. *There exists a map*

$$\hat{\phi} : \mathrm{Spec}(k((x,y))) \to \mathrm{Spec}(\widehat{\mathcal{O}^s_{Z/\!\!/G,0}}).$$

such that $\hat{\phi}^*\mathcal{A} = \mathcal{B}_{k(x,y)} \otimes M_n(k(x,y))$.

Proof. As for the case of the symplectic group. □

Corollary 7.3. *There is no universal bundle on the generic point of* $M(\rho, \mathrm{SO}_{2n})^0$.

Proof. This follows from the above discussion and 4.6. □

References

[BBNN06] V. Balaji, I. Biswas, D. S. Nagaraj, and P. E. Newstead. Universal families on moduli spaces of principal bundles on curves. *Int. Math. Res. Not.*, pages Art. ID 80641, 16, 2006.

[Dra83] P. K. Draxl. *Skew fields*, volume 81 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1983.

[Dré04] Jean-Marc Drézet. Luna's slice theorem and applications. In *Algebraic group actions and quotients*, pages 39–89. Hindawi Publ. Corp., Cairo, 2004.

[GLSS08] T. L. Gómez, A. Langer, A. H. W. Schmitt, and I. Sols. Moduli spaces for principal bundles in arbitrary characteristic. *Adv. Math.*, 219(4):1177–1245, 2008.

[Ill71] Luc Illusie. *Complexe cotangent et déformations. I.* Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin, 1971.

[Ill72a] Luc Illusie. *Complexe cotangent et déformations. II.* Lecture Notes in Mathematics, Vol. 283. Springer-Verlag, Berlin, 1972.

[Ill72b] Luc Illusie. Cotangent complex and deformations of torsors and group schemes. In *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N.S., 1971)*, pages 159–189. Lecture Notes in Math., Vol. 274. Springer, Berlin, 1972.

[Jac85] Nathan Jacobson. *Basic algebra. I.* W. H. Freeman and Company, New York, second edition, 1985.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[Ram96a] A. Ramanathan. Moduli for principal bundles over algebraic curves. I. *Proc. Indian Acad. Sci. Math. Sci.*, 106(3):301–328, 1996.

[Ram96b] A. Ramanathan. Moduli for principal bundles over algebraic curves. II. *Proc. Indian Acad. Sci. Math. Sci.*, 106(4):421–449, 1996.

[Sch02] Alexander H. W. Schmitt. Singular principal bundles over higher-dimensional manifolds and their moduli spaces. *Int. Math. Res. Not.*, (23):1183–1209, 2002.

[Ser86] Edoardo Sernesi. *Topics on families of projective schemes*, volume 73 of *Queen's Papers in Pure and Applied Mathematics*. Queen's University, Kingston, ON, 1986.

[Ses82] C. S. Seshadri. *Fibrés vectoriels sur les courbes algébriques*, volume 96 of *Astérisque*. Société Mathématique de France, Paris, 1982. Notes written by J.-M. Drezet from a course at the École Normale Supérieure, June 1980.