# Reduction in the Rationality Problem for Multiplicative Invariant Fields

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**Abstract**

For a faithful $\mathbf{Z}G$ lattice $A$ and a field $K$ on which the group $G$ acts by field automorphisms, let $R$ be the normal subgroup generated by the elements of $G$ which act trivially on $K$ and act as reflections on $A$. We prove that the rationality of the multiplicative invariant field $K(A)^G$ over $K(A^R)^G$ is equivalent to the rationality of $K(A)^{\Omega_G}$ over $K(A^R)^{\Omega_G}$ where $\Omega_G$ is a particular subgroup of $G$ such that $G/R \cong \Omega_G$. We then use this reduction result to prove that $K(A)^G$ is rational over $K$ where $G$ is the automorphism group of a crystallographic root system $\Psi$, $G$ acts trivially on $K$ and $A$ is any lattice on the space $\mathbf{Q}\Psi$.

**Keywords:** rationality, multiplicative invariant fields, automorphism group of a root system

## 1 Introduction

Let $G$ be a finite group, $A$ a faithful $\mathbf{Z}G$ lattice, $K$ a field on which $G$ acts by field automorphisms and $K[A]$ the group ring of $A$ written multiplicatively with $K$-basis $\{e(a)|a \in A\}$. Then there is a natural action of $G$ on $K[A]^\times (\cong K^\times \oplus A$ as abelian groups) which extends the action of $G$ on $K$ and $A$. Hence we obtain an action of $G$ on the group ring $K[A]$ and its quotient field $K(A)$. The ring of invariants $K[A]^G$ is a *multiplicative invariant ring* and the field $K(A)^G$ is a *multiplicative invariant field*.

Multiplicative invariant fields were first studied by Noether [10] in her work on the inverse Galois problem. She showed that a finite group $G$ is a Galois group of a given number field $F$ if $F(G) \equiv F(\mathbf{Z}G)^G$ is rational (i.e. purely transcendental) over $F$ where here $G$ acts trivially on the field $F$. Although counterexamples to the rationality of $F(G)$ over $F$ were later found by Swan [16] and Voskresenskii [18], the problem of determining whether a given multiplicative invariant field $K(A)^G$ is rational over $K^G$ is still open.

This problem has since been generalized by others (eg. Saltman) to ask whether a given twisted multiplicative invariant field $K_\gamma(A)^G$ is (stably) rational over $K^G$. Here $\gamma : G \to \mathrm{Hom}(A, K^\times)$ is a 1-cocycle which defines the action of $G$ on $K_\gamma[A]^\times$ via the isomorphism $H^1(G, \mathrm{Hom}(A, K^\times)) \cong \mathrm{Ext}^1_G(A, K^\times)$. The invariants of the quotient field of the group algebra with this action is then denoted by $K_\gamma(A)^G$. Saltman was able to use (twisted) multiplicative invariant fields to prove that certain $K(G)'s$ were non-rational over $K$ where here $K$ is an algebraically closed field [11, 12, 14, 15]. He also showed that the invariant fields of reductive groups can be expressed as twisted multiplicative invariant fields for their Weyl groups [13] and described solutions to the Galois embedding problem in terms of twisted multiplicative invariant fields [15].

From a geometric perspective, multiplicative invariant fields $K(A)^G$ in which $G$ acts faithfully on both the field and the lattice arise as the function fields of tori over $K$ and hence are called *fields of tori invariants.* Endo, Miyata [4, 3], Swan [16] and Voskresenskii [16, 17] studied the rationality properties of fields of tori invariants. They were able to find necessary and sufficient conditions to determine when a field of tori invariants $K(M)^G$ is stably rational over $K^G$ and when two fields of tori invariants $K(M)^G$ and $K(N)^G$ are stably equivalent over $K^G$.

In this paper, we first prove a reduction result for multiplicative invariant fields $K(A)^G$. Let $R$ be the normal subgroup of $G$ generated by the set of elements of $G$ which act as reflections on $A$ and which act trivially on $K$. We prove in Theorem 5.3 that $K(A)^G$ is rational over $K(A^R)^G$ if and only if $K(A)^{\Omega_G}$ is rational over $K(A^R)^{\Omega_G}$ where $\Omega_G$ is a particular subgroup of $G$ satisfying $G \cong R \rtimes \Omega_G$. We then use this reduction result to prove in Theorem 7.7 that $K(A)^G$ is rational over $K$ where $G$ is the automorphism group of a crystallographic root system $\Psi$ acting naturally on $V = \mathbf{Q}\Psi$, $A$ is a lattice on $V$ and $G$ acts trivially on $K$. By contrast, it is interesting to note the following recent result of Cortella and Kunyavskii [1]: Let $G$ be an automorphism group of an irreducible root system $\Psi$ acting naturally on $V = \mathbf{Q}\Psi$ and faithfully on the field $K$ and let $A$ be either the root or weight lattice of $\Psi$. Then $K(A)^G$ is not rational (or even stably rational) over $K^G$ except in the following cases: $\mathrm{rank}(A) \leq 2$; $A$ is the root lattice of type $A_{2k}$; $A$ is the weight lattice of type $B_n$; or $A$ is the root lattice of type $C_n$.

Here is a brief outline of the paper: In the second section, we review some standard definitions and results about reflections and crystallographic root systems that will be used in the subsequent sections. In Section 3, we associate a crystallographic root system to a faithful $\mathbf{Z}G$-lattice $A$ in order to construct our subgroup $\Omega_G$ and to prepare us for Section 4 in which we discuss Farkas' results on the rationality of multiplicative invariant fields for reflection lattices and Section 5 in which we prove our reduction result. In Section 6, we examine the action of an automorphism group $G$ of a crystallographic root system on a full $\mathbf{Z}G$ lattice $A$ of the rational vector space spanned by the root system. This prepares us for Section 7 in which we prove the rationality of $K(A)^G$ over $K$

where $G$ acts trivially on the field $K$.

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## 2 Crystallographic Root Systems

We summarize here some definitions, notation, and results about reflections and crystallographic root systems which will be used in the exposition that follows. For more details, refer to [7, Ch. III] or [8, Appendix]. The following definition of reflection is not completely standard.

**Definition 2.1.** A *reflection* on a finite-dimensional $\mathbf{Q}$-vector space $E$ is an element $s \in \mathrm{GL(E)}$ of finite order such that the image of $s-1$ on $E$,

$$\mathrm{Im}_E(s-1) = \{v \in E | v = (s-1)x \text{ for some } x \in E\},$$

has dimension 1 over $\mathbf{Q}$.

**Remark 2.2.** Note that a reflection $s$ as defined above must have order 2 and must satisfy $\mathrm{Ker}_E(s+1) = \mathrm{Im}_E(s-1)$. We call a reflection $s$ a reflection relative to $0 \neq \alpha \in E$ if $s\alpha = -\alpha$. In this case, $\{\alpha\}$ is a basis of the subspace $\mathrm{Ker}_E(s+1) = \mathrm{Im}_E(s-1)$. A non-zero vector $\alpha$ does not uniquely determine a reflection relative to $\alpha$. However, if $\Phi$ is a finite set of non-zero vectors spanning $E$, then the group generated by the reflections which stabilize $\Phi$ is finite. This means that if $\tau$ is a reflection relative to $\alpha \in \Phi$ which maps $\Phi$ into itself, then $\tau$ is uniquely determined by $\alpha$.

**Definition 2.3.** A *crystallographic root system* in the finite-dimensional $\mathbf{Q}$-vector space $E$ is a subset $\Phi$ of $E$ satisfying:
(R1) $\Phi$ is finite, spans $E$ and does not contain 0.
(R2) If $\alpha \in \Phi$, the only multiples of $\alpha$ in $\Phi$ are $\pm\alpha$.
(R3) If $\alpha \in \Phi$, there exists a reflection $\tau_\alpha$ relative to $\alpha$ which leaves $\Phi$ stable.
(R4) If $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle = \tau_\alpha(\beta) - \beta$ is an integral multiple of $\alpha$.

The dimension $n$ of the vector space $E$ is called the *rank* of $\Phi$.

If $\Phi'$ is a crystallographic root system in $E'$, then $\Phi'$ is said to be isomorphic to $\Phi$ if there exists an isomorphism of vector spaces from $E'$ to $E$ which maps $\Phi'$ to $\Phi$ and which preserves the integers defined in (R4).

Let $\Phi$ be a crystallographic root system in $E$. Since $\Phi$ spans $E$, and is finite, the group $W(\Phi) \subset \mathrm{GL}(E)$ generated by the $\tau_\alpha$ is also finite and is called the *Weyl group* of $\Phi$. Since $\tau_\alpha$ is uniquely determined by $\alpha$, (R4) is unambiguous and we may define, for any $\alpha \in \Phi$, the $\mathbf{Q}$ linear map

$$\langle \cdot, \alpha \rangle : E \to \mathbf{Q}, \langle x, \alpha \rangle = \tau_\alpha(x) - x$$

There is an inner product $(\cdot,\cdot)$ on $E$ with respect to which $W(\Phi)$ consists of orthogonal transformations. For this inner product, $\langle x, \alpha\rangle = 2\frac{(x,\alpha)}{(\alpha,\alpha)}$, for $x \in V, \alpha \in \Phi$.

A subset $\Delta$ of $\Phi$ is called a *base* if $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a basis of $E$, relative to which each $\alpha \in \Phi$ has a (unique) expression $\alpha = \sum_{i=1}^n c_i\alpha_i$, where the $c_i$ are integers of the same sign. Bases exist [7, 10.1]; $W(\Phi)$ permutes the collection of bases simply transitively, and every root lies in at least one base [7, 10.3]. Elements of a base $\Delta$ are called *simple roots.*

Let $\Delta$ be a base of a root system $\Phi$. Then $W(\Phi)$ is generated by the set $\{\tau_\alpha|\alpha \in \Delta\}$ [7, 10.3].

A root system $\Phi$ is called *irreducible* if it (or equivalently, a base $\Delta$) cannot be partitioned into the union of two proper subsets $A$ and $B$ such that $\langle\alpha, \beta\rangle = 0$ for all $\alpha \in A$ and $\beta \in B$. Every root system is the disjoint union of (uniquely determined) irreducible root systems in certain subspaces of $E$ [7, 10.4]. The irreducible root systems correspond up to isomorphism with the Dynkin diagrams $A_n(n \geq 1), B_n(n \geq 2), C_n(n \geq 3), D_n(n \geq 4), E_6, E_7, E_8, F_4, G_2$ [7, 11.4]. The Dynkin diagrams have nodes corresponding to the simple roots of a base for the irreducible root system and are completely determined by the integers $\langle\alpha, \beta\rangle$, for simple roots $\alpha, \beta$ [7, 11.2].

The automorphism group of a root system $\Phi$, $\mathrm{Aut}(\Phi)$ is the semidirect product of the normal subgroup $W(\Phi)$ and the stabilizer subgroup $\Omega(\Delta)$ of a base $\Delta$ for $\Phi$. If $\Phi$ is irreducible, $\Omega(\Delta)$ corresponds to the group of diagram automorphisms of the associated Dynkin diagram. In the irreducible case, $\Omega(\Delta)$ is trivial except in types $A_n, (n \geq 2)$, $D_n, (n \geq 5)$, $E_6$, in which it is cyclic of order 2; and in type $D_4$, in which it is isomorphic to the symmetric group $S_3$ [7, 12.2].

The *weight lattice* $\Lambda(\Phi)$ associated to the root system $\Phi$ is defined as

$$\Lambda(\Phi) = \{\lambda \in E|\langle\lambda, \alpha\rangle \in \mathbf{Z} \text{ for all } \alpha \in \Phi\}$$

It is a lattice in $E$ of rank $n$. The *root lattice* $\mathbf{Z}\Phi$ is the lattice spanned by $\Phi$. The root lattice is a subgroup of finite index in the weight lattice. If $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a base for $\Phi$, $\Lambda(\Phi)$ has a corresponding basis of *fundamental dominant weights* $\{\omega_1, \ldots, \omega_n\}$ for which $\langle\omega_i, \alpha_j\rangle = \delta_{ij}$ (Kronecker delta). The *fundamental group* $\Lambda(\Phi)/\mathbf{Z}\Phi$ has the following structure for the irreducible types: $A_n : \mathbf{Z}/(n+1)\mathbf{Z}$, $B_n, C_n, E_7 : \mathbf{Z}/2\mathbf{Z}$, $D_n(n \text{ even}) : \mathbf{Z}/2\mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$, $D_n(n \text{ odd}) : \mathbf{Z}/4\mathbf{Z}$, $E_6 : \mathbf{Z}/3\mathbf{Z}$, $E_8, F_4, G_2 : 0$ [7, 13.1].

Given a base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of $\Phi$ and the corresponding fundamental dominant weights $\omega_1, \ldots, \omega_n$, call $\lambda = \sum_{i=1}^n c_i\lambda_i$ *dominant* if all $c_i \in \mathbf{Z}^+$. Each $\lambda \in \Lambda(\Phi)$ is $W(\Phi)$ conjugate to one and only one dominant weight. $E$ is partially ordered by: $\lambda > \mu$ if $\lambda - \mu = \sum_{i=1}^n d_i\alpha_i$, $d_i \in \mathbf{Z}_{\geq 0}$. This ordering depends on the choice of the base $\Delta$. If $\lambda \in \Lambda$ is dominant, then $\lambda > \sigma(\lambda)$ for all $\sigma \in W(\Phi)$ [7, 13.2].

## 3 Suitable Root Systems

Let $G$ be a finite group and let $A$ be a lattice on which $G$ acts faithfully. Set $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$ and let $\Gamma \subset G$ be a $G$-stable set of reflections on $V$ (i.e. $g\Gamma g^{-1} = \Gamma$ for all $g \in G$.) Let $R$ be the (normal) subgroup of $G$ generated by $\Gamma$. In the case in which $R$ is non-trivial, we will associate a type of crystallographic root system to the $\mathbf{Z}G$-lattice $A$. We will examine some of the properties of this kind of root system that will be useful later in our reduction and rationality results. We will also use the root system to construct a subgroup $\Omega_G$ of $G$ which is congruent to $G/R$.

**Remark 3.1.** Suppose $s$ is a reflection on an $n$-dimensional $\mathbf{Q}$-vector space $V$ which also stabilizes a lattice $A$ on $V$. Then $\mathrm{Im}_A(s-1) \subset \mathrm{Ker}_A(s+1)$ are cyclic subgroups of $A$ with $H^1(\langle s\rangle, A) \cong \mathrm{Ker}_A(s+1)/\mathrm{Im}_A(s-1) \cong 0$ or $\mathbf{Z}/2\mathbf{Z}$. In fact, $H^1(\langle s\rangle, A)$ completely determines the isomorphism class of $A$ as an $\mathbf{Z}\langle s\rangle$ lattice [2, 34.31].

$$H^1(\langle s\rangle, A) \cong \mathbf{Z}/2\mathbf{Z} \Leftrightarrow A \cong \mathbf{Z}^- \oplus \mathbf{Z}^{n-1}$$
$$H^1(\langle s\rangle, A) = 0 \Leftrightarrow A \cong \mathbf{Z}\langle s\rangle \oplus \mathbf{Z}^{n-2}$$

where $\mathbf{Z}^- = \mathbf{Z}x$ is the rank 1 $\mathbf{Z}\langle s\rangle$ lattice on which $s$ acts by $sx = -x$.

**Definition 3.2.** Let $e_R = \frac{1}{|R|}\sum_{r\in R} r \in \mathbf{Q}R \subset \mathbf{Q}G$ be the trivial idempotent in $\mathbf{Q}G$ associated with $R$ and let $\pi$ be the linear map $1 - e_R : V \to V$.

**Remark 3.3.** The following definition corresponds to Farkas' notion of a rooting section for a lattice in [6]. Our approach here is more easily adapted to the $G$-equivariant properties that we require.

**Definition 3.4.** Let $A$ be a faithful $\mathbf{Z}G$ lattice on $V$, let $\Gamma$ be a subset of $G$ which is $G$-invariant and consists of reflections acting on $V$ and let $R$ be the (normal) subgroup of $G$ generated by $\Gamma$. A crystallographic root system $\Phi$ for the $\mathbf{Q}$ space $\pi(V)$ with weight lattice $\Lambda(\Phi)$ is called *suitable* for $A$ and $\Gamma$ iff
(i) $\Phi \subset A$
(ii) $\pi(A) \subset \Lambda(\Phi)$
(iii) $R$ is isomorphic to the Weyl group $W(\Phi)$ under the natural restriction map $GL(V) \to GL(\pi(V))$.

A suitable root system $\Phi$ for $A$ and $\Gamma$ is called *$G$-stable* if $G$ stabilizes the finite subset $\Phi$ of $V$.

**Remark 3.5.** If $\Phi$ is a suitable root system for $A$ and $\Gamma$, there is a bijection between reflections in $\Gamma$ and roots in $\Phi$. More explicitly, for $s \in \Gamma$, $\mathrm{Ker}_V(s+1) = \mathrm{Ker}_{\pi V}(s+1)$ is 1-dimensional so there exists $\alpha \in \Phi$ such that $\mathrm{Ker}_{\mathbf{Z}\Phi}(s+1) = \mathrm{Ker}_{\pi V}(s+1) \cap \mathbf{Z}\Phi = \mathbf{Z}\alpha$. Now $s|_{\pi V}$ is the unique reflection on $\pi V$ with $\mathrm{Ker}_{\pi V}(s+1) = \mathbf{Q}\alpha$. Since $s$ acts trivially on $\mathrm{Ker}(\pi) = V^R$, we see that $s$ is the unique reflection in $\Gamma$ with $\mathrm{Ker}_V(s+1) = \mathbf{Q}\alpha$. We denote $s$ by $s_\alpha$ and define $\langle\cdot,\alpha\rangle : V \to \mathbf{Q}$ to be the $\mathbf{Q}$-linear map satisfying $s_\alpha(v) = v - \langle v,\alpha\rangle\alpha$ for $v \in V$ and $\alpha \in \Phi$. Note that the map $\langle\cdot,\alpha\rangle$ is an extension of the definition in the last section to the case in which the root system $\Phi$ does not span the vector space $V$. That is, $\langle\cdot,\alpha\rangle$ restricted to $\mathbf{Q}\Phi = \pi(V)$ coincides with the definition given in Section 2. If $V = \mathbf{Q}\Phi$, then $V^R = 0$, $\pi$ is the identity map, and $s_\alpha$ is the map $\tau_\alpha$ from Definition 2.3.

**Lemma 3.6.** *Let $\Phi$ be a $G$-stable suitable root system for $A$ and $\Gamma$. Then*
*(a) $V = \mathbf{Q}\Phi \oplus V^R$ is a $\mathbf{Q}G$ decomposition of $V$.*
*(b) For $g \in G$, $\alpha, \beta \in \Phi$, we have $gs_\alpha g^{-1} = s_{g\alpha}$ and $\langle g\alpha, g\beta\rangle = \langle\alpha,\beta\rangle$.*

**Proof:**
(a) Since $e_R$ is an idempotent in $\mathbf{Q}G$ and $\mathrm{Im}_V(e_R) = V^R$, it is clear that $V = \mathrm{Ker}_V(e_R) \oplus V^R$ is a $\mathbf{Q}G$ decomposition of $V$. But then since $\Phi$ is a root system for $\pi(V) = \mathrm{Im}_V(1-e_R) = \mathrm{Ker}_V(e_R)$, we see that $\mathbf{Q}\Phi = \mathrm{Ker}_V(e_R)$ so that $V = \mathbf{Q}\Phi \oplus V^R$ is a $\mathbf{Q}G$ decomposition of $V$ as required.

(b) For $g \in G, \alpha, \beta \in \Phi$, we have $s_\alpha \in \Gamma$, $gs_\alpha g^{-1} \in \Gamma$ and $g\alpha \in \Phi$. Since

$$\mathbf{Z}g\alpha = g\mathrm{Ker}_{\mathbf{Z}\Phi}(s_\alpha + 1) = \mathrm{Ker}_{\mathbf{Z}\Phi}(gs_\alpha g^{-1} + 1)$$

we see that $s_{g\alpha} = gs_\alpha g^{-1}$. Since

$$s_{g\beta}(g\alpha) = g\alpha - \langle g\alpha, g\beta\rangle g\beta$$
$$gs_\beta g^{-1}(g\alpha) = g\alpha - \langle\alpha,\beta\rangle g\beta$$

and $s_{g\beta} = gs_\beta g^{-1}$, we see that $\langle g\alpha, g\beta\rangle = \langle\alpha,\beta\rangle$ as required. ■

Fix a suitable crystallographic root system $\Phi$ for $A$ and $\Gamma$. Then let $\Delta$ be a base for $\Phi$ and set

$$\Omega_G = \Omega_G(\Delta) = \{g \in G | g(\Delta) = \Delta\}$$

Note that in the effective case (for which $V^R = 0$), this is a subgroup of the automorphism group of the root system $\Phi$.

**Lemma 3.7.** *With the above notation, we have $G \cong R \rtimes \Omega_G$.*

**Proof:** Let $\varphi : G \to GL(\mathbf{Q}\Phi)$ be the homomorphism which defines the action of $G$ on $\mathbf{Q}\Phi$. Note that $R$ maps isomorphically to the Weyl group $W = W(\Phi)$ and that $\Omega_G$ maps surjectively onto

$$\Omega_{\varphi(G)} \equiv \{\varphi(g) \in \varphi(G) | \varphi(g)(\Delta) = \Delta\}$$

For each $g \in G$, $\varphi(g)(\Delta)$ is another base for the root system $\Phi$. Since $W$ acts simply transitively on the set of bases for the root system $\Phi$ [7, 10.3], we see that there exists a unique $w \in W$ with $w(\Delta) = \varphi(g)(\Delta)$ and $w = \varphi(r)$ for a unique $r \in R$. So $\varphi(g^{-1}r) \in \Omega_{\varphi(G)} = \varphi(\Omega_G)$ and since $\mathrm{Ker}(\varphi) \subset \Omega_G$, we have $g^{-1}r \in \Omega_G$ and hence $g \in R\Omega_G$. Let $r \in R \cap \Omega_G$. Then $\varphi(r) \in W \cap \Omega_{\varphi(G)} = 1$ so $r \in \mathrm{Ker}(\varphi) \cap R$. Since $R$ acts faithfully on $\mathbf{Q}\Phi$, we see that $R \cap \Omega_G = 1$, as required. ■

We need to show that suitable root systems for $A$ and $\Gamma$ exist. Let

$$\Phi_A = \{\alpha | \mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha \text{ for some } s \in \Gamma\}$$

Note that $\mathrm{Ker}_{\mathbf{Z}\Phi_A}(s+1) = \mathrm{Ker}_A(s+1)$ for all $s \in \Gamma$. The following Lemma is adapted from Farkas [6, Lemmas 1–3].

**Lemma 3.8.** *$\Phi_A$ is a $G$-stable suitable root system for $A$ and $\Gamma$.*

**Proof:** Note that $\Phi_A$ is $G$-stable since $\Gamma$ is $G$-stable and

$$\mathrm{Ker}_A(gs_\alpha g^{-1}+1) = g\mathrm{Ker}_A(s_\alpha+1) = \mathbf{Z}g\alpha$$

In particular, $\Phi_A$ is $R$-stable. It is also clear that $0 \notin \Phi_A$ and $\Phi_A$ is finite.

Applying $e_R$ to $s_\alpha\alpha = -\alpha$, we get $e_R\alpha = -e_R\alpha$ and hence $e_R\alpha = 0$ so that $\mathbf{Q}\Phi_A \subset \mathrm{Ker}_V(e_R)$. Let $v \in \mathrm{Ker}_V(e_R)^R$. Then $rv = v$ for all $r \in R$ so that $v = e_Rv = 0$. We see that $\mathrm{Ker}_V(e_R)$ is a $\mathbf{Q}R$ space containing $\mathbf{Q}\Phi_A$ with $\mathrm{Ker}_V(e_R)^R = 0$. But then since $\mathbf{Q}\Phi_A$ is a $\mathbf{Q}R$ subspace of $\mathrm{Ker}_V(e_R)$, there exists a $\mathbf{Q}R$ submodule $U$ with $\mathrm{Ker}_V(e_R) = \mathbf{Q}\Phi_A \oplus U$. So for all $x \in U$ and $\alpha \in \Phi_A$, we have

$$x - s_\alpha(x) = \langle x,\alpha\rangle\alpha \in U \cap \mathbf{Q}\Phi_A = 0$$

which shows that $U \subset \mathrm{Ker}_V(e_R)^R = 0$ and hence that $\Phi_A$ spans $\pi(V)$.

To show that $\mathbf{Q}\alpha \cap \Phi_A = \{\pm\alpha\}$, note that for $c\alpha \in \mathbf{Q}\alpha \cap \Phi_A$, we have

$$\mathbf{Q}\alpha = \mathrm{Ker}_V(s_{c\alpha}+1) = \mathrm{Ker}_V(s_\alpha+1)$$

which implies that

$$\mathbf{Z}c\alpha = \mathrm{Ker}_A(s_{c\alpha}+1) = \mathrm{Ker}_A(s_\alpha+1) = \mathbf{Z}\alpha$$

and hence that $c = \pm 1$ as required.

Let $\alpha, \beta \in \Phi_A$, then

$$(s_\alpha - 1)(\beta) = -\langle\beta,\alpha\rangle\alpha \in \mathrm{Im}_A(s_\alpha - 1) \subset \mathrm{Ker}_A(s_\alpha+1) = \mathbf{Z}\alpha$$

so that $\langle\beta,\alpha\rangle \in \mathbf{Z}$.

Hence $\Phi_A$ is a crystallographic root system for $\mathrm{Ker}_V(e_R)$. Thus $R$ maps onto the Weyl group of $\Phi_A$ on $\pi(V) = \mathrm{Ker}_V(e_R)$ under the restriction map $R \to R|_{\pi(V)}$. Since $R$ acts trivially on $V^R$ and faithfully on $V$ the decomposition $V = \mathbf{Q}\Phi_A \oplus V^R$ from Lemma 3.6 shows that the map is an isomorphism.

With our extended definition of $\langle\cdot,\alpha\rangle$ we find:

$$s_\alpha(e_Rv) = e_Rv - \langle e_Rv,\alpha\rangle\alpha$$

which together with $s_\alpha e_R = e_R$ implies that $\langle e_Rv,\alpha\rangle = 0$ and thus $\langle(1-e_R)v,\alpha\rangle = \langle v,\alpha\rangle$ for all $v \in V$.

Now for $a \in A$, $\langle a,\alpha\rangle\alpha = (1-s_\alpha)a \in \mathrm{Im}_A(1-s_\alpha) \subset \mathrm{Ker}_A(s_\alpha+1) = \mathbf{Z}\alpha$ implies $\langle a,\alpha\rangle \in \mathbf{Z}$. Thus $(1-e_R)a \in \mathrm{Ker}(e_R) = \mathbf{Q}\Phi_A$ has $\langle(1-e_R)a,\alpha\rangle = \langle a,\alpha\rangle \in \mathbf{Z}$ which proves $\pi(A) = (1-e_R)A \subset \Lambda(\Phi_A)$. Since $\Phi_A \subset A$ by construction we see that $\Phi_A$ is a suitable root system for $A$. ■

The following basic lemma will be useful in Section 6:

**Lemma 3.9.** *Let $V$ be a finite dimensional reflection space over $\mathbf{Q}$ with fixed root system $\Phi$ whose Weyl group $W(\Phi)$ has reflection set $\Gamma$ and such that $V = \mathbf{Q}\Phi$. If $\Phi'$ is another root system for $V$ with reflection set $\Gamma$, then $\Phi'$ must* *take the form $\Phi' = \{c_\alpha \alpha | \alpha \in \Phi\}, c_\alpha \in \mathbf{Q}^+$. The decomposition of $\Phi$ into irreducible root systems corresponds bijectively to the decomposition of $\Phi'$. The corresponding decompositions of $W(\Phi) = W(\Phi')$ and $V$ are the same for $\Phi$ and for $\Phi'$.*

**Proof:** Let $\beta \in \Phi'$. Then since $\Phi'$ and $\Phi$ share the same reflection set $\Gamma$, $s_\beta = s_\alpha$ for some $\alpha \in \Phi$. Then

$$\mathbf{Q}\beta = \mathrm{Ker}_V(s_\beta + 1) = \mathrm{Ker}_V(s_\alpha + 1) = \mathbf{Q}\alpha$$

shows that $\beta = c_\alpha \alpha$ for some $c_\alpha \in \mathbf{Q}^\times$. Since $\mathbf{Q}\beta \cap \Phi' = \{\pm\beta\}$ and $\mathbf{Q}\alpha \cap \Phi = \{\pm\alpha\}$, we may choose $c_\alpha \in \mathbf{Q}^+$. So $\Phi' = \{c_\alpha \alpha | \alpha \in \Phi\}$ for some $c_\alpha \in \mathbf{Q}^+$.

Suppose $\Phi = \cup_{i=1}^r \Phi_i$ is a decomposition of $\Phi$ into irreducible root systems. Then since $\langle c_\alpha \alpha, c_\beta \beta \rangle = 0$ iff $\langle \alpha, \beta \rangle = 0$ for all $\alpha, \beta \in \Phi$, we see that the decomposition of $\Phi'$ into irreducible root systems is $\Phi' = \cup_{i=1}^r \Phi'_i$ where $\Phi'_i = \{c_\alpha \alpha | \alpha \in \Phi_i\}$. Note that $s_{c_\alpha \alpha} = s_\alpha$ for all $\alpha \in \Phi$. Then the decompositions $W(\Phi') = \prod_{i=1}^r W(\Phi'_i)$ and $V = \oplus_{i=1}^r \mathbf{Q}\Phi'_i$ correspond to those for $\Phi$ since $W(\Phi') = W(\Phi)$, $W(\Phi'_i) = W(\Phi_i)$ and $\mathbf{Q}\Phi'_i = \mathbf{Q}\Phi_i$ for all $i = 1, \ldots, r$. ■

# 4 Farkas' Rationality Result for Reflection Lattices

In this section, we review the results of Farkas from [6, 5]. Let $G$ be a finite group, let $A$ be a lattice on which $G$ acts faithfully, and let $K$ be a field on which $G$ acts by field automorphisms. Let $\Gamma$ be a $G$-stable set of reflections on $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$ which act trivially on the field $K$ and let $R$ be the normal subgroup generated by $\Gamma$.

Let $K[A]$ be the group algebra of $A$ over $K$ with $K$ basis written multiplicatively as $\{e(a) | a \in A\}$. As a ring, $K[A]$ is isomorphic to the Laurent polynomial ring in n variables where n is the rank of $A$. So $K[A]$ is an integral domain, and hence has a quotient field $K(A)$. There is a natural action on the unit group of $K[A]$, $K[A]^\times$ ($\cong K^\times \oplus A$ as abelian groups) which extends the action of $G$ on $K$ and on $A$. Hence we obtain an action of $G$ on $K[A]$ and on $K(A)$. The invariant ring $K[A]^G$ is called a *multiplicative invariant ring* and the invariant field $K(A)^G$ is called a *multiplicative invariant field*

**Notation:** Fix a suitable root system $\Phi$ for $A$ and $\Gamma$ with weight lattice $\Lambda = \Lambda(\Phi)$ and choose a base $\Delta$ for $\pi(\Phi) = \Phi$. Form the dominant weights

$$\Lambda^+ = \{\omega \in \Lambda | \langle \omega, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$$

**Definition 4.1.** For any $\mathbf{Z}G$ lattice $B$ on $V$, set $X(b) = \sum_{r \in R/R_b} r \cdot e(b)$ where $R_b$ is the stabilizer of $b \in B$ in $R$.

**Remark 4.2.** Recall that each $R$ orbit on $\Lambda$ has precisely one element in $\Lambda^+$ [7, 13.2]. Hence the same is true for $R$ orbits on $\pi(A) \subset \Lambda$ so that $\{X(a) | a \in \pi^{-1}(\Lambda^+) \cap A\}$ is a $K$-basis for $K[A]^R$.

**Remark 4.3.** Let $B = e_R A \oplus \Lambda(\Phi)$. Then $B$ is a $\mathbf{Z}G$ lattice with $A \subset B$ and $\pi(B) = \Lambda(\Phi) = \oplus_{i=1}^n \mathbf{Z}\omega_i$ where $\omega_1, \dots, \omega_n$ are the fundamental dominant weights with respect to the given base $\Delta$.

**Lemma 4.4.**
*(a) [6, Lemma 9] Let $b_1, \dots, b_k \in B$ and form $X(b_1), \dots, X(b_k) \in K[B]^R$. Then $X(b_1) \cdots X(b_k) \in K[A]^R$ if and only if $\sum_{i=1}^n b_i \in A$.*
*(b) [6, proof of Theorem 10] Every $a \in A$ can be written as $a = b_0 + \sum_{i=1}^n k_i\omega_i$ with unique $k_i \in \mathbf{Z}$ and $b_0 \in B^R$. If $a \in \pi^{-1}(\Lambda^+) \cap A$, then $k_i \geq 0$.*

**Notation:** For $a \in A$, write $a = b_0 + \sum_{i=1}^n k_i\omega_i$ as in Lemma 4.4(b). Then set $X_a = X(b_0)X(\omega_1)^{k_1} \cdots X(\omega_n)^{k_n}$. Note that for $a \in \pi^{-1}(\Lambda^+) \cap A$, $X_a \in K[A]^R$ by Lemma 4.4(a).

**Proposition 4.5.**
*(a) [6, Corollary 8] $K[B]^R$ is a polynomial ring in $X(\omega_1), \dots, X(\omega_n)$ over $K[B^R]$.*
*(b) [5, Theorem 16] Let $a \in A$. $X_a \in K[A]^R$ is irreducible iff $\pi(a)$ is indecomposable in $\pi(A) \cap \Lambda^+$.*
*(c) [5, Theorem 16] Let $E$ be the multiplicative monoid $\{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$. Then $E$ generates $K[A]^R$ as a $K[A^R]$ module. There are only a finite number of indecomposable elements in $\Lambda^+ \cap \pi(A)$. Call them $\pi(a_1), \dots, \pi(a_N)$. Then $X_{a_1}, \dots, X_{a_N}$ are irreducible in $K[A]^R$ and generate $E$ as a multiplicative monoid.*

**Remark 4.6.** We will give a proof of the following proposition [6, Theorem 10] as the details of the proof will be required in the proof of Theorem 5.3.

**Proposition 4.7.** *$K(A)^R$ is rational over $K$.*

**Proof:**
By Proposition 4.5(a), $K[B]^R$ is a polynomial ring in $X(\omega_1), \dots, X(\omega_n)$ over $K[B^R]$. The multiplicative subgroup $M$ of the field of fractions $K(B)^R$ generated by $B^R$ and $X(\omega_1), \dots, X(\omega_n)$ is a free abelian group of finite rank whose members are linearly independent over $K$. Now

$$\{e(b_0)X(\omega_1)^{k_1} \cdots X(\omega_n)^{k_n} | b_0 \in B^R, k_i \in \mathbf{Z}, k_i \geq 0\}$$

is a $K$-basis of $K[B]^R$. Let $S$ be the multiplicative monoid generated by $X(\omega_1), \dots, X(\omega_n)$. After localizing at $S$, each $X(\omega_i)$ becomes a unit so that

$$\{e(b_0)X(\omega_1)^{k_1} \cdots X(\omega_n)^{k_n} | b_0 \in B^R, k_i \in \mathbf{Z}\}$$

is a $K$-basis for $S^{-1}K[B]^R$. Hence $S^{-1}K[B]^R$ is the group algebra $K[M]$ and so $K(B)^R = K(M)$.

By Proposition 4.5(c), $K[A]^R$ is generated by the multiplicative monoid $E = \{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$ as a $K[A^R]$ module. Since $A^R \subset E$, we see that in fact $E$ generates $K[A]^R$ over $K$. Now localize $K[A]^R$ at the monoid $E$. Consider $E^{-1}K[A]^R \subset E^{-1}K[B]^R$. We first show that $E^{-1}K[B]^R$ is also the group algebra $K[M]$. It suffices to show that $X(\omega_i)$ is a unit for each $i$. Let $d = [B : A]$. Then $d\omega_j \in \pi^{-1}(\Lambda^+) \cap A$. So $X(\omega_j)^d = X_{d\omega_j}$ is invertible in $E^{-1}K[B]^R$ and hence so is $X(\omega_j)$. So $E^{-1}K[B]^R = K[M]$.

Now let $L$ be the subgroup of $M$ generated by $E$. Hence $L$ is also free abelian of finite rank. By Proposition 4.5(c), $\{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$ is a $K$-generating set for $K[A]^R$. Hence $L$ spans $E^{-1}K[A]^R$ over $K$ so that $E^{-1}K[A]^R = K[L]$ is a group algebra. So $K(A)^R \cong K(L)$ as required. ■

# 5 Reduction

Let $G$ be a finite group acting faithfully on a lattice $A$ which also acts on a field $K$. Let $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$, and let $R$ be the normal subgroup of $G$ generated by $\Gamma$, the set of reflections in $V$ which act trivially on $K$. We will use the notation of Section 3.

Choose a suitable crystallographic root system $\Phi$ for $A$ and $\Gamma$. Recall from Lemma 3.6, for $g \in G$, $\alpha \in \Phi$ and $v \in V$, $gs_\alpha g^{-1} = s_{g\alpha}$ and $\langle gv, g\alpha \rangle = \langle v, \alpha \rangle$. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a base for $\Phi$ and let $\Lambda$ be the weight lattice of $\Phi$ with basis of fundamental dominant weights $\{\omega_1, \dots, \omega_n\}$ corresponding to the base $\Delta$. Now set

$$\Omega_G = \Omega_G(\Delta) = \{g \in G | g(\Delta) = \Delta\}$$

Then from Lemma 3.7(a), we recall that $G \cong R \rtimes \Omega_G$.

**Lemma 5.1.** *$\mathbf{Z}\Phi$ and $\Lambda$ are isomorphic $\mathbf{Z}\Omega_G$ permutation lattices.*

**Proof:**

$\Delta = \{\alpha_1, \dots, \alpha_n\}$ is a $\mathbf{Z}$ basis for the root lattice $\mathbf{Z}\Phi$ and the set of fundamental dominant weights $\{\omega_1, \dots, \omega_n\}$ corresponding to $\Delta$ is a $\mathbf{Z}$ basis for $\Lambda$. By definition, $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$ so that for $\lambda \in \Lambda$, $\lambda = \sum_{i=1}^n \langle \lambda, \alpha_i \rangle \omega_i$. Since $\Omega_G$ stabilizes the base $\Delta$, we see that for each $t \in \Omega_G$, there exists $\sigma \in S_n$ such that $t(\alpha_i) = \alpha_{\sigma(i)}$ for $i = 1, \dots, n$. This shows that $\mathbf{Z}\Phi$ is a permutation $\mathbf{Z}\Omega_G$ lattice. Now $\langle t\omega_i, \alpha_{\sigma(j)} \rangle = \langle t\omega_i, t\alpha_j \rangle = \langle \omega_i, \alpha_j \rangle = \delta_{ij} = \delta_{\sigma(i)\sigma(j)} = \langle \omega_{\sigma(i)}, \alpha_{\sigma(j)} \rangle$ for all $j$ implies that $t\omega_i = \sum_{j=1}^n \langle t\omega_i, \alpha_j \rangle \omega_j = \omega_{\sigma(i)}$ for all $i$. This shows that $\mathbf{Z}\Phi \to \Lambda, \alpha_i \mapsto \omega_i$ is a $\mathbf{Z}\Omega_G$ isomorphism. ■

**Remark 5.2.** For any simply laced root system $\Phi$, the weight lattice is the $\mathbf{Z}$-dual of the root lattice. In these cases, Lemma 4.1 follows directly from the fact that for a finite group $H$, a permutation $\mathbf{Z}H$ lattice is isomorphic to its $\mathbf{Z}$-dual. However, if $\Phi$ is not simply laced, the weight lattice is no longer the $\mathbf{Z}$-dual of the root lattice. Lemma 4.1 shows that for any root system, the root lattice and the weight lattice are isomorphic as lattices over the stabilizer subgroup of a base for the root system.

**Theorem 5.3.** *The invariant fields $K(A)^G$ and $K(A)^{\Omega_G}$ are isomorphic under an isomorphism that is the identity on $K(A^R)^G = K(A^R)^{\Omega_G}$. In particular, $K(A)^G$ is rational over $K(A^R)^G$ if and only if $K(A)^{\Omega_G}$ is rational over $K(A^R)^{\Omega_G}$.*

**Proof:** $B = e_R A \oplus \Lambda$ gives a $\mathbf{Z}G$ lattice containing $A$ with $B^R = e_R A$ and $\pi(B) = \Lambda$. Using the notation of the proof of Proposition 4.7, we let $M$ be the multiplicative subgroup of $K(B)^R$ generated by $e(e_R A)$ and $X(\omega_1), \ldots, X(\omega_n)$ (note that $\Lambda \subset B$ and $\pi(\omega_i) = \omega_i$) which is then free abelian and linearly independent over $K$. Recall that each $a \in \pi^{-1}(\Lambda^+) \cap A$ has a unique expression as $a = a_0 + \sum_{i=1} k_i \omega_i$ where $a_0 \in e_R A$ and $k_i \geq 0$. Then $X_a \equiv e(a_0) \prod_{i=1}^n X(\omega_i)^{k_i} \in K[A]^R$. Then $E = \{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$ is a multiplicative monoid with $e(A^R) \subset E$. In Proposition 4.7, we showed that $E^{-1}K[B]^R = K[M]$ and $E^{-1}K[A]^R = K[L]$ where $L$ is the subgroup of $M$ generated by $E$, which is hence also free abelian. Note that $K[M]$ and $K[L]$ are the $K$-subalgebras of $K(B)^R$, respectively $K(A)^R$ generated by $M$ and $L$. They are both group algebras of free abelian groups of finite rank written multiplicatively.

$R$ acts trivially on $K[M] \subset K(B)^R$ and on $K[L] \subset K(A)^R$. We will now show that $\Omega_G \cong G/R$ also acts on $K[M]$ and $K[L]$ inducing an action of $G$ on $K[M]$ and $K[L]$ by inflation.

Let $t \in \Omega_G$. By the last lemma, we know that $t\omega_i = \omega_{\sigma(i)}$ for some $\sigma \in S_n$. Now

$$\begin{aligned} tX(\omega_i) &= \sum_{r \in R/R_{\omega_i}} tr \cdot e(\omega_i) = \sum_{r \in R/R_{\omega_i}} r^t t \cdot e(\omega_i) = \sum_{r \in R/R_{t\omega_i}} r \cdot e(t\omega_i) \\ &= X(t\omega_i) = X(\omega_{\sigma(i)}) \in K[M] \end{aligned}$$

and for $b_0 \in B^R$, $t \cdot e(b_0) = e(tb_0) \in K[M]$. We may express $b \in B$ uniquely as $b = b_0 + \sum_{i=1}^n k_i \omega_i$ for $b_0 \in B^R$ and $k_i \in \mathbf{Z}$. Note that $M = \{X_b | b \in B\}$ where we recall that $X_b = e(b_0) \prod_{i=1}^n X(\omega_i)^{k_i}$. Then

$$tX_b = t \cdot (e(b_0) \prod_{i=1}^n X(\omega_i)^{k_i}) = e(tb_0) \prod_{i=1}^n X(t\omega_i)^{k_i} = e(tb_0) \prod_{i=1}^n X(\omega_{\sigma(i)})^{k_i} = X_{tb}$$

defines the action of $\Omega_G$ on $K[M]$.

**Claim:** $L = \{X_a | a \in A\}$.

Since $L$ is generated as a group by the multiplicative monoid

$$E = \{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$$

then $z \in L$ can be expressed as $z = \frac{X'_a}{X_{a''}} = X_{a'-a''}$ where $a', a'' \in \pi^{-1}(\Lambda^+) \cap A$.

Conversely, if $a \in A$, then $a = a_0 + \sum_{i=1}^n k_i \omega_i$ where $a_0 \in A_0$ and $k_i \in \mathbf{Z}$. Since $\Lambda/\pi(A)$ is finite, there exists $d \in \mathbf{N}$ such that $d\omega_i \in \pi(A)$ for all $i = 1, \ldots, n$. Choose $x_i \in A$ such that $\pi(x_i) = d\omega_i$ and $m_i$ such that $k_i + dm_i \geq 0$ and set $a' = \sum_{i=1}^n dm_i x_i$. Then $a + a', a' \in \pi^{-1}(\Lambda^+) \cap A$. So

$$X_a = \left( \frac{X_{a+a'}}{X_{a'}} \right) \in L$$

Then for $a \in A$, by the calculation above, we have $tX_a = X_{ta} \in K[L]$ showing that $\Omega_G$ also acts on $K[L]$.

Now $B \to M, b \mapsto X_b$ is an abelian group isomorphism inducing the $K$-algebra isomorphism $\varphi : K[B] \to K[M], e(b) \mapsto X_b$. Note that $\varphi|_{K[B^R]} = id$. We check that $\varphi$ is $\Omega_G$-equivariant where $\Omega_G$ acts on $K[B]$ by $t \cdot e(b) = e(tb)$ and on $K[M]$ by $tX_b = X_{tb}$ for $t \in \Omega_G$ and $b \in B$. Indeed, $\varphi(t \cdot e(b)) = \varphi(e(tb)) = X_{tb} = t\varphi(e(b))$ as required. Note that $\varphi(K[A]) = K[L]$ since $L = \{X_a | a \in A\}$, $\varphi|_{K^\times} = id$ and $\varphi(e(a)) = X_a$. So $K[B]$ and $K[M]$, respectively $K[A]$ and $K[L]$, are isomorphic as $K$-algebras under an $\Omega_G$- equivariant isomorphism which acts as the identity on $K[B^R]$, respectively $K[A^R]$.

So $K(A)^G = (K(A)^R)^{G/R} = (K(L))^{\Omega_G} \overset{\varphi^{-1}}{\cong} K(A)^{\Omega_G}$ as required. Note that the above isomorphism acts as the identity on $K(A^R)^G = K(A^R)^{\Omega_G}$. The final statement of the theorem is a direct consequence of this isomorphism. ■

**Remark 5.4.** A more general version of Theorem 5.3 for twisted multiplicative invariant fields is found in [9].

# 6 The Automorphism Group of a Root System

The automorphism group $G$ of a crystallographic root system $\Psi$ acts naturally on the $\mathbf{Q}$-vector space $V = \mathbf{Q}\Psi$ spanned by the root system. In this section, we will determine the structure of the action of $G$ on a full $\mathbf{Z}G$ lattice $A$ of $V$. In particular, we will determine some information about the full reflection subgroup of the action of $G$ on $V$ and the stabilizer of a base for $\Phi$.

Let $\Psi = \cup_{i=1}^m (\Psi_i)^{l_i}$ be the decomposition of the crystallographic root system $\Psi$ into irreducible crystallographic root systems with $\Psi_i$ distinct. Let $V_i = \mathbf{Q}\Psi_i$ be the $\mathbf{Q}$-vector space spanned by $\Psi$. Then

$$G = \mathrm{Aut}(\Psi) = \prod_{i=1}^m \mathrm{Aut}(\Psi_i^{l_i}) = \prod_{i=1}^m \mathrm{Aut}(\Psi_i)^{l_i} \rtimes S_{l_i}$$

acts diagonally on $V = \oplus_{i=1}^m V_i^{l_i}$ with $\mathrm{Aut}(\Psi_i)^{l_i}$ acting diagonally on $V_i^{l_i}$ and $S_{l_i}$ permuting components of $V_i^{l_i}$.

We first need some technical lemmas about lattices for the wreath product $H^k \rtimes S_k$, where $H$ is any finite group. These will be useful in analyzing the structure of the the root lattice $\mathbf{Z}\Psi$ and the weight lattice $\Lambda(\Psi)$ as $\mathbf{Z}G$ lattices.

**Definition 6.1.** Let $H$ be a finite group and $X$ be a $\mathbf{Z}H$ lattice. We construct a natural $H^k \rtimes S_k$ lattice $\mathrm{Wr}_{H,k}X$ as follows: Let $I_k$ be the $k$ element set $\{1, \ldots, k\}$. Identifying $H^k$ with $\mathrm{Map}(I_k, H)$, $S_k$ acts on $H^k$ as $f^\sigma(i) = f(\sigma^{-1}i)$ and $H^k \rtimes S_k$ is the semidirect product corresponding to this action. Hence elements of $H^k \rtimes S_k$ take the form $f\sigma$ with multiplication

$$(f\sigma)(f'\sigma') = ff'^\sigma\sigma\sigma'$$

and inverses $(f\sigma)^{-1} = (f^{\sigma^{-1}})^{-1}\sigma$. Then if $S_{k-1}$ is the subgroup of $S_k$ which stabilizes $k$ in $I_k$, the map $p_k : H^k \rtimes S_{k-1} \to H$ defined by $f\sigma \mapsto f(k)$ is a epimorphism since $S_{k-1}$ centralizes the last component of $H^k$. We then define the $H^k \rtimes S^k$ lattice $\mathrm{Wr}_{H,k}X$ as $\mathrm{Ind}_{H^k \rtimes S_{k-1}}^{H^k \rtimes S_k} \mathrm{Inf}_H^{H^k \rtimes S_{k-1}} X$ where the inflation is computed with respect to the epimorphism $p_k$.

**Lemma 6.2.**
*(a) For a* $\mathbf{Z}H$ *module* $X$*, the* $\mathbf{Z}H^k \rtimes S_k$ *lattice* $\mathrm{Wr}_{H,k}X$ *can be identified with* $\oplus_{i=1}^k X^{(i)}$ *where* $\sigma \in S_k$ *maps* $X^{(i)}$ *identically onto* $X^{(\sigma(i))}$ *and* $H^k$ *acts componentwise on* $\oplus_{i=1}^k X^{(i)}$ *so that as a lattice for the ith copy of* $H$ *in* $H^k$*,* $X^{(i)}$ *is isomorphic to* $X$*.*
*(b) If* $X$ *is a faithful* $\mathbf{Z}H$ *lattice, then* $\mathrm{Wr}_{H,k}X$ *is a faithful* $H^k \rtimes S_k$ *lattice.*

**Proof:**
(a) $\{(i,k)|i = 1,\ldots,k\}$ is a transversal for $H^k \rtimes S_{k-1}$ in $H^k \rtimes S_k$ where $(i,k) \in S_k$ denotes the transposition if $i \neq k$ and the identity if $i = k$. Then set $X^{(i)} = (i,k) \otimes \mathrm{Inf}_H^{H^k \rtimes S_{k-1}} X$ for $i = 1,\ldots,k$. For $\sigma \in S_k, \sigma(i,k) = (\sigma(i),k)\sigma'$ where $\sigma' \in S_{k-1}$. So

$$\sigma((i,k) \otimes x) = (\sigma(i),k) \otimes \sigma' x = (\sigma(i),k) \otimes x$$

which shows that $\sigma(X^{(i)}) = X^{\sigma(i)}$. For $f \in H^k$, $f(i,k) = (i,k)(f^{(i,k)})$ so that

$$f((i,k) \otimes x) = (i,k) \otimes (f^{(i,k)}(k))x = (i,k) \otimes f(i)x$$

This shows that $H^k$ acts componentwise on $\oplus_{i=1}^k X^{(i)}$ and that the ith copy of $H$ in $H^k$ acts on $X^{(i)}$ in the same way as $H$ acts on $X$.
(b) Let $f\sigma \in H^k \rtimes S_k$ act trivially on $\mathrm{Wr}_{H,k}X$. Then, in particular, $f\sigma X^{(i)} = X^{(i)}$ for all $i = 1,\ldots,k$ so that $X^{(\sigma(i))} = \sigma X^{(i)} = f(i)^{-1}X^{(i)} = X^{(i)}$ for all $i$ implies that $\sigma = 1$. Now $\sum_{i=1}^k x^{(i)} = f\sum_{i=1}^k x^{(i)} = \sum_{i=1}^k f(i)x^{(i)}$ shows that $f(i)$ acts trivially on $X^{(i)}$. Since $X^{(i)}$ is congruent to $X$ as a $\mathbf{Z}H$ lattice, we see that $f(i) = 1$ for all $i$ since the action of $H$ on $X$ is faithful. So $f\sigma = 1$ as required. ■

**Notation:** Let a finite group $H$ act faithfully on a lattice $X$ and hence on the vector space $V = \mathbf{Q}X$. Let $\Gamma(V)$ be the subset of $H$ consisting of reflections acting on $V$, let $R(V)$ be the (normal) subgroup generated by $\Gamma(V)$, let $\Phi(V)$ be a root system on $V$ with reflection set $\Gamma(V)$, let $\Delta(V)$ be a base for $\Phi(V)$ and let $\Omega(\Delta(V))$ be the subgroup of $H$ which stabilizes the base $\Delta(V)$. Note that if there are no reflections in the action of $H$ on $V$ then $\Gamma(V) = \phi$, so that $\Phi(V) = \phi$, $\Delta(V) = \phi$, $R(V) = 1$ and $\Omega(\Delta(V)) = H$.

Given a faithful $\mathbf{Z}H$ lattice $X$ and its associated $\mathbf{Q}H$ space $V = \mathbf{Q}X$, we will be interested in finding the set of reflections $\Gamma(V^k)$ for the action of $H^k \rtimes S_k$ on $V^k \equiv \mathbf{Q}\mathrm{Wr}_{H,k}X$, the subgroup $R(V^k)$ generated by $\Sigma(V^k)$, a root system $\Phi(V^k)$ for $V^k$ with reflection set $\Sigma(V^k)$, a base $\Delta(V^k)$ of $\Phi(V^k)$ and the subgroup $\Omega(\Delta(V^k))$ of $H^k \rtimes S_k$ that stabilizes the base $\Delta(V^k)$. The next lemma will relate those structures with the corresponding structures for $V$.

**Lemma 6.3.** *Let $X$ be a faithful $\mathbf{Z}H$ lattice, and let $V = \mathbf{Q}X$ be the associated $\mathbf{Q}H$ space. For the faithful $\mathbf{Z}H^k \rtimes S_k$ lattice $\mathrm{Wr}_{H,k}X$, we define $V^k = \mathbf{Q}\mathrm{Wr}_{H,k}X$. Then if $H^{(i)}$ denotes the ith copy of $H$ in $H^k$, $V^k = \oplus_{i=1}^k V^{(i)}$ where $V^{(i)} = \mathbf{Q}X^{(i)}$ is a $\mathbf{Q}H^{(i)}$ space. For $i = 1, \ldots, k$, let $v^{(i)} = (i,k)\otimes v \in V^{(i)}$ denote the copy of $v \in V$ in $V^{(i)}$ and for $h \in H$, let $h^{(i)}$ denote the element of $H^k$ that acts as $h$ in $V^{(i)}$ and trivially on $V^{(j)}$, for $j \neq i$.*

*If $\dim(V) > 1$, then we have*

$$\Gamma(V^k) = \cup_{i=1}^k \Gamma(V^{(i)})$$
$$\Phi(V^k) = \cup_{i=1}^k \Phi(V^{(i)})$$
$$\Delta(V^k) = \cup_{i=1}^k \Delta(V^{(i)})$$
$$R(V^k) = R(V)^k$$
$$\Omega(\Delta(V^k)) = \Omega(\Delta(V))^k \rtimes S_k$$

*If $\dim(V) = 1$, then $H = 1$ or $H = \langle h \rangle = C_2$. If $H = 1$, then $\Gamma(V) = \phi$ and if $H = C_2$, then $\Gamma(V) = \{h\}$. Let $\{v\}$ be a basis for $V$. Then $V^{(i)}$ has basis $\{v^{(i)}\}$ for all $i = 1, \ldots, k$. Let $\Gamma(k) = \{(i,j) | i \neq j\}$ be the set of transpositions in $S_k$, let $\Phi(k) = \{v^{(i)} - v^{(j)} | i \neq j\}$ and let $\Delta(k) = \{v^{(i)} - v^{(i+1)} | 1 \leq i \leq j\}$. Then we have*

$$\Gamma(V^k) = \cup_{i=1}^k \{h^{(i)}\} \cup \Gamma(k)$$
$$\Phi(V^k) = \cup_{i=1}^k \{v^{(i)}\} \cup \Phi(k)$$
$$\Delta(V^k) = \cup_{i=1}^k \{v^{(i)}\} \cup \Delta(k)$$
$$R(V^k) = H^k \rtimes S_k$$
$$\Omega(\Delta(V^k)) = 1$$

**Proof:** Suppose $\dim(V) > 1$. Let $s \in H^k \rtimes S_k$ be a reflection not in $H^k$. Then $s = r\sigma$ where $r \in H^k$ and $1 \neq \sigma \in S_k$. Then $\sigma(j) \neq j$ for some j. Since $V$ has dimension larger than 1, there exist two linearly independent elements $x^{(j)}, y^{(j)}$ in $V^{(j)}$, the $j$th copy of $V$. But then $\mathrm{Im}_{V^k}(r\sigma - 1)$ contains $rx^{(\sigma(j))} - x^{(j)}$ and $ry^{(\sigma(j))} - y^{(j)}$. If $rx^{(\sigma(j))} - x^{(j)} = c(ry^{(\sigma(j))} - y^{(j)})$ for some $0 \neq c \in \mathbf{Q}$, then $r\sigma(x^{(j)} - cy^{(j)}) = x^{(j)} - cy^{(j)}$ and so $x^{(\sigma(j))} - cy^{(\sigma(j))} = r^{-1}(x^{(j)} - cy^{(j)}) \in V^{(\sigma(j))} \cap V^{(j)} = 0$ implies a contradiction. So the reflections in $H^k \rtimes S_k$ are those in $H^k$. Now, let $r \in H^k$. Since $\mathrm{Im}_{V^k}(r-1) = \oplus_{i=1}^k \mathrm{Im}_{V^{(i)}}(r-1)$, $r$ is a reflection precisely if it acts as a reflection in some $V^{(i)}$ and acts trivially on all other copies of $V$. Hence the set of reflections of $H^k \rtimes S_k$ on $V^k$ is $\Gamma(V^k) = \cup_{i=1}^k \Gamma(V^{(i)})$, $\Phi(V^k) = \cup_{i=1}^k \Phi(V^{(i)})$ is a root system for $V^k$ with reflection set $\Gamma(V^k)$, $\Delta(V^k) = \cup_{i=1}^k \Delta(V^{(i)})$ is a base for $\Phi(V^k)$ and the reflection subgroup of $H^k \rtimes S_k$ on $V^k$ is indeed $R(V)^k$. Since $\Omega(\Delta(V^k))$ stabilizes $\Delta(V^k)$ and since $\sigma \in S_k$ sends $\Delta(V^{(i)})$ identically to $\Delta(V^{(\sigma(i))})$, we see that $S_k$ stabilizes $\Delta(V^k)$. As no reflection stabilizes $\Delta(V^k)$, we conclude that $\Omega(\Delta(V)) \rtimes S_k$ is the stabilizer subgroup with respect to $\Delta(V^k)$.

If $H \neq 1$ and $\dim(V) = 1$, then $H = \langle h \rangle \cong C_2$ as $V$ is a faithful $\mathbf{Q}H$ module. Then clearly, the reflection set of $H$ acting on $V$ is $\Gamma(V) = \{h\}$, a root system for $V$ with reflection set $\Gamma(V)$ is $\Phi(V) = \{v\}$, and the base for $\Phi(V)$ is $\Delta(V) = \Phi(V) = \{v\}$, and $H$ is the reflection subgroup on $V$. So $H^k$ is a reflection subgroup on $V^k$. Moreover, the transpositions of $S_k$ act as reflections on $V^k$ are the only reflections in $S_k$ acting on $V^k$. Since they generate $S_k$, $S_k$ is a reflection subgroup acting on $V^k$. For $i \neq j$, the transposition $(i,j)$ acts as a reflection in $v^{(i)} - v^{(j)}$. This shows that the set of reflections of $H^k \rtimes S_k$ acting on $V^k$, the associated root system, and its base are as stated above. The reflection subgroup is all of $H^k \rtimes S_k$ in this case and the stabilizer subgroup is trivial. ■

For an irreducible crystallographic root system $\Phi$ on $V$, the next lemma examines the action of $\mathrm{Aut}(\Phi^l) = \mathrm{Aut}(\Phi)^l \rtimes S_l$ on the root lattice and weight lattice of the crystallographic root system on $V^l$, $\Phi^l$, consisting of a disjoint union of $l$ copies of $\Phi$.

**Lemma 6.4.** *Let $\Phi$ be an irreducible crystallographic root system of rank $n$ on a vector space $V$. Let $\Phi^l$ denote the crystallographic root system formed by a disjoint union of $l$ copies of $\Phi$ on $V^l$.*
*(a) There exists an irreducible crystallographic root system $\hat{\Phi}$ of rank $n$ on $V$ such that $\Phi \subset \widehat{\Phi}$, $\mathrm{Aut}(\Phi) = \mathrm{Aut}(\widehat{\Phi})$ and the full reflection subgroup of $\mathrm{Aut}(\Phi)$ is $W(\widehat{\Phi})$. In particular, for $\Phi \neq A_2, A_3, D_n$, we may take $\widehat{\Phi} = \Phi$. For the remaining cases:*

$$\widehat{A_2} = G_2, \widehat{A_3} = B_3, \widehat{D_4} = F_4, \widehat{D_n} = B_n, n \geq 5,$$

*we have $\mathrm{Aut}(\Phi) = W(\widehat{\Phi})$.*
*(b) If $H$ is a subgroup of an automorphism group for an irreducible root system $\Phi$ with weight lattice $\Lambda(\Phi)$ then we have*

$$\Lambda(\Phi)^l \cong \mathrm{Wr_{H,l}}\Lambda(\Phi) \qquad \mathbf{Z}\Phi^l \cong \mathrm{Wr_{H,l}}\mathbf{Z}\Phi$$

*as $H^l \rtimes S_l$ lattices.*
*(c) Let $V = \mathbf{Q}\Phi = \mathbf{Q}\widehat{\Phi}$ and let $\Delta$ be a base for $\widehat{\Phi}$. Then $\mathrm{Aut}(A_1^l) = \mathrm{Aut}(B_l) = W(B_l)$ is a reflection group on $V^l$. If $\Phi \neq A_1$, then $\mathrm{Aut}(\Phi^l) = \mathrm{Aut}(\widehat{\Phi}^l)$, the reflection subgroup of $\mathrm{Aut}(\Phi^l)$ is $W(\widehat{\Phi}^l)$ and the stabilizer subgroup for the base $\Delta^l$ is $T^l \rtimes S_l$ where $T$ is the diagram automorphism group for $\widehat{\Phi}$ with respect to $\Delta$.*

**Proof:** We refer to [7, 12.1–12.2] for relevant facts about automorphism groups and the construction of irreducible root systems.

(a) For $\Phi \neq A_n, n \geq 2, D_n, E_6$, we have $\mathrm{Aut}(\Phi) = W(\Phi)$ so the result is trivial.

For $\Phi = A_n, n \geq 4; E_6$, note that $[\mathrm{Aut}(\Phi) : W(\Phi)] = 2$. Suppose $\mathrm{Aut}(\Phi)$ were a reflection group. Then there would exist an irreducible root system of the same rank with Weyl group of the same cardinality as $\mathrm{Aut}(\Phi)$. Examining the orders of irreducible Weyl groups [7, p. 66], we see that this cannot hold. By contradiction, $\mathrm{Aut}(\Phi)$ has reflection subgroup $W(\Phi)$ in these cases.

For the remaining cases, we need to look more closely at the realization of each irreducible root system. Following [7, 12.1], for any $k$, let $\mathbf{Q}^k$ have standard basis $\{\epsilon_1, \ldots, \epsilon_k\}$ with the usual inner product $(\cdot,\cdot)$. Let $I_k = \oplus_{i=1}^k \mathbf{Z}\epsilon_i$ be a lattice on $\mathbf{Q}^k$. To construct $A_n$, let $E_n$ be the subspace of $\mathbf{Q}^{n+1}$ orthogonal to $\epsilon_1 + \cdots + \epsilon_{n+1}$, and let $I'_n = I_n \cap E_n$. Then

$$A_n = \{\alpha \in I'_n | (\alpha,\alpha) = 2\} = \{\epsilon_i - \epsilon_j | i \neq j\}$$

is a root system on $E_n$ which has base $\{\alpha_i = \epsilon_i - \epsilon_{i+1} | i = 1, \ldots, n\}$. The diagram automorphism group is of order 2 and is generated by $t_n$ which maps $\alpha_i$ to $\alpha_{n+1-i}$ for all $i$.

Note that the root system $G_2$ is realized as

$$G_2 = \{\alpha \in I'_2 | (\alpha,\alpha) = 2 \text{ or } 6\} = \{\epsilon_i - \epsilon_j | i \neq j\} \cup \{\pm(2\epsilon_i - \epsilon_j - \epsilon_k) | \{i,j,k\} = \{1,2,3\}\}$$

So $A_2 \subset G_2$ , and the diagram automorphism group is generated by $s_{\epsilon_1 - 2\epsilon_2 + \epsilon_3} \in W(G_2) = \mathrm{Aut}(G_2)$.

The irreducible root system $D_n$ is realized by

$$D_n = \{\alpha \in I_n | (\alpha,\alpha) = 2\} = \{\pm\epsilon_i \pm \epsilon_j | i \neq j\}$$

and has base given by

$$\{\alpha_i = \epsilon_i - \epsilon_{i+1} | i = 1, \ldots, n-1\} \cup \{\alpha_n = \epsilon_{n-1} - \epsilon_n\}$$

For $n \neq 4$, its diagram automorphism group is generated by $t_n$ which interchanges $\alpha_{n-1}$ and $\alpha_n$ and fixes the other elements of the base. Note that for $n = 3$, $A_3$ can be replaced by its isomorphic copy $D_3$. So for $n \neq 4$, the root system $D_n$ is a subset of the root system $B_n$ which can be realized as

$$B_n = \{\pm\epsilon_i \pm \epsilon_j | i \neq j\} \cup \{\pm\epsilon_i | i = 1, \ldots, n\}$$

Also note that the generator of diagram automorphism group for $D_n, n \neq 4$ can be expressed as the reflection $s_{(\epsilon_{n-1} - \epsilon_n) - (\epsilon_{n-1} + \epsilon_n)} = s_{2\epsilon_n} = s_{\epsilon_n}$ which is in $W(B_n) = \mathrm{Aut}(B_n)$.

The irreducible root system $D_4$ is a subset of the irreducible root system $F_4$ which is realized as

$$F_4 = \{\pm\epsilon_i | i = 1, \ldots, 4\} \cup \{\pm\epsilon_i \pm \epsilon_j | 1 \leq i \leq j \leq 4\} \cup \{\pm\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}$$

on the space $\mathbf{Q}^4$. The diagram automorphism group of $D_4$ is generated by the elements $\sigma_{13}$ and $\sigma_{14}$ where $\sigma_{ij}$ interchanges $\alpha_i$ and $\alpha_j$ and fixes the other elements of the base. Then $\sigma_{13} = s_{(\epsilon_1 - \epsilon_2) - (\epsilon_3 - \epsilon_4)} = s_{\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)}$ and $\sigma_{14} = s_{(\epsilon_1 - \epsilon_2) - (\epsilon_3 + \epsilon_4)} = s_{\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)}$ where both are in $W(F_4) = \mathrm{Aut}(F_4)$.

So for $\Phi = A_2, A_3, D_n, n \geq 4$, we have shown that $\Phi$ is a subset of an irreducible root system $\widehat{\Phi}$, and that $\mathrm{Aut}(\Phi)$ is a subgroup of $\mathrm{Aut}(\widehat{\Phi}) = W(\widehat{\Phi})$. It turns out that in each of these cases $|\mathrm{Aut}(\Phi)| = |W(\widehat{\Phi})|$ so that $\mathrm{Aut}(\Phi) = \mathrm{Aut}(\widehat{\Phi}) = W(\widehat{\Phi})$ as required.

(b) follows immediately from Lemma 6.2(a) and the description of the action of $\mathrm{Aut}(\Phi)^l \rtimes S_l$ on $\Lambda(\Phi)^l$ and $(\mathbf{Z}\Phi)^l$.

(c) Note that $A_1$ is the only irreducible root system of size 1. If $\Phi = A_1$, we note that we may embed $A_1^l$ into $B_l$ with base $\{\epsilon_1, \ldots, \epsilon_l\}$ so that $W(A_1)^l \leq W(B_l)$. Since the stabilizer of a base for $A_1^l$ is generated by the subset $\{s_{\epsilon_i - \epsilon_j} | i < j\}$ of $W(B_l)$, we find that $\mathrm{Aut}(A_1^l) = W(B_l)$. Since $\mathrm{Aut}(\Phi^l) = \mathrm{Aut}(\Phi)^l \rtimes S_l$, the rest of the statement follows from applying parts (a) and (b) and Lemma 6.3 to the $\mathrm{Aut}(\Phi)$ lattice $X = \mathbf{Z}\Phi$. ■

Next we will determine the reflection subgroup of the action of the automorphism group $G$ of the crystallographic root system $\Psi$. We first need some additional notation and definitions.

**Notation:** Let $\Psi = \cup_{i=1}^m \Psi_i^{l_i}$ be the decomposition of $\Psi$ into a disjoint union of distinct irreducible root systems. Let $V_i = \mathbf{Q}\Psi_i$ for each $i = 1, \ldots, k$. Then recall that

$$G = \mathrm{Aut}(\Psi) = \prod_{i=1}^m \mathrm{Aut}(\Psi_i)^{l_i} \rtimes S_{l_i}$$

acts diagonally on $V = \mathbf{Q}\Psi = \oplus_{i=1}^m V_i^{l_i}$ where $\mathrm{Aut}(\Psi_i)^{l_i}$ acts diagonally on $V_i^{l_i}$ and $S_{l_i}$ permutes the isomorphic copies of $V_i$ in $V_i^{l_i}$. Let $\Gamma_0$ be the ($G$-stable) set of reflections in $W(\Psi) = \prod_{i=1}^m W(\Psi_i)^{l_i}$ and let $\Gamma$ be the set of all reflections in $G$. Let $R$ be the group generated by $\Gamma$. Note that $R$ is the full reflection subgroup of $G$.

**Definition 6.5.** For an arbitrary crystallographic root system $\Phi$ where $\Phi = \cup_{i=1}^m \Phi_i^{l_i}$ is a disjoint union of irreducible root systems with the $\Phi_i$ distinct, we define $\overline{\Phi} = \cup_{i=1}^m \overline{\Phi_i}^{k_i}$ where $\overline{\Phi_i} = \widehat{\Phi_i}$ and $k_i = l_i$ if $\Phi_i \neq A_1$ whereas $\overline{\Phi_i} = B_{l_i}$ and $k_i = 1$ if $\Phi_i = A_1$. Note that $\mathrm{Aut}(\Phi_i^{l_i}) = \mathrm{Aut}(\overline{\Phi_i}^{k_i})$ by the previous lemma and that $\overline{\Phi} = \cup_{i=1}^m \overline{\Phi_i}^{k_i}$ is a decomposition of $\overline{\Phi}$ into irreducible root systems although the $\overline{\Phi_i}$ may not be distinct. For this reason, it is not necessarily true that $\mathrm{Aut}(\Phi) = \mathrm{Aut}(\overline{\Phi})$.

**Lemma 6.6.** *The full reflection subgroup $R$ of $G = \mathrm{Aut}(\Psi) = \prod_{i=1}^m \mathrm{Aut}(\Psi_i)^{l_i}$ acting on $V = \mathbf{Q}\Psi$ is $R = W(\overline{\Psi}) = \prod_{i=1}^m W(\overline{\Psi_i}^{k_i})$. Let $\Pi_i$ be a base for $\overline{\Psi_i}$ so that $\Pi = \cup_{i=1}^m \Pi_i^{k_i}$ is a base for $\overline{\Psi}$. The stabilizer of the base $\Pi$ in $G$ is $\Omega \equiv \Omega_G(\Pi) = \prod_{i=1}^m \Omega_i$, where*

$$\Omega_i = \Omega(\Pi_i^{k_i}) = \{g \in \mathrm{Aut}(\overline{\Psi_i}^{k_i}) | g(\Pi_i^{k_i}) = \Pi_i^{k_i}\} = T_i^{k_i} \rtimes S_{k_i}$$

*where $T_i = \{g \in \mathrm{Aut}(\overline{\Psi_i}) | g(\Pi_i) = \Pi_i\}$ is the diagram automorphism group of $\overline{\Psi_i}$ with respect to $\Pi_i$.*

**Proof:** Since $\mathrm{Aut}(\Psi) = \prod_{i=1}^m \mathrm{Aut}(\Psi_i^{l_i})$ acts diagonally on $V = \oplus_{i=1}^m V_i^{l_i}$, an element of $s \in \mathrm{Aut}(\Psi)$ has $\mathrm{Im}_V(s-1) = \oplus_{i=1}^m \mathrm{Im}_{V_i^{l_i}}(s-1)$ so that it can act as a reflection on $V$ iff it is a reflection in $\mathrm{Aut}(\Psi_i^{l_i}) = \mathrm{Aut}(\overline{\Psi_i}^{k_i})$ for some $i$.

So the reflection subgroup of $\mathrm{Aut}(\Psi)$ is $R = \prod_{i=1}^{m} R_i$ where $R_i$ is the reflection subgroup of $\mathrm{Aut}(\Psi_i^{l_i}) = \mathrm{Aut}(\overline{\Psi_i}^{k_i})$. Hence $R_i = W(\overline{\Psi_i}^{k_i})$ by Lemma 6.4.

The stabilizer of the base $\Pi$ in $G$ is

$$\begin{aligned}\Omega_G(\Pi) &= \{g \in G | g\Pi = \Pi\} \\ &= \prod_{i=1}^{m} \{g \in \mathrm{Aut}(\overline{\Psi_i}^{k_i}) | g\Pi_i^{k_i} = \Pi_i^{k_i}\} \\ &= \prod_{i=1}^{m} \Omega_i\end{aligned}$$

where the $\Omega_i$ are as described above. ■

The last lemma in this section translates the information about $\mathrm{Aut}(\Psi)$ into information about a suitable root system for the $\mathbf{Z}G-$ lattice $A$.

**Notation:** For a $\mathbf{Z}G$ lattice $A$ on $V$ and a $G$-stable set of reflections $S$, set

$$\Phi_{A,S} = \{\alpha \in A | \mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha \text{ for some } s \in S\}$$

We write $\Phi_A$ for $\Phi_{A,\Gamma}$ where $\Gamma$ is the set of reflections in $G$ acting on $V$.

**Lemma 6.7.** *Let $\Psi$ be a crystallographic root system with $\Psi = \cup_{i=1}^{m} \Psi_i^{l_i}$ its decomposition into distinct irreducibles. Then let $G$ be the automorphism group of $\Psi$, let $V$ be the vector space $\mathbf{Q}\Psi$, and let $\Gamma$ be the subset of $G$ consisting of all the reflections acting on $V$. For a $\mathbf{Z}G$ lattice $A$ on $V$, the $G$-stable suitable root system for $A$ and $\Gamma$, $\Phi_A$ can be expressed as a disjoint union of irreducible crystallographic root systems*

$$\Phi_A = \cup_{i=1}^{m} \Phi_i^{k_i}$$

*In this decomposition, $\Phi_i$ can be any irreducible root system except for $A_n, n = 2, 3$ and $D_n, n \geq 4$. Also, if $\Phi_i^{k_i} = A_1^{k_i}$ for some $i$, the multiplicity $k_i$ must be 1. The irreducible root systems $\Phi_i$ may not be distinct. The group $G$ can be expressed as $G = \prod_{i=1}^{m} \mathrm{Aut}(\Phi_i^{k_i})$. Moreover, $G$ acts diagonally on $V = \oplus_{i=1}^{m} V_i^{k_i}$ where $V_i = \mathbf{Q}\Phi_i$. A base $\Delta_A$ for $\Phi_A$ may be expressed as $\Delta_A = \cup_{i=1}^{m} \Delta_i^{k_i}$ where $\Delta_i$ is a base for $\Phi_i$. The full reflection subgroup of $G$ acting on $V$ is $R = \prod_{i=1}^{m} R_i$ with $R_i = W(\Phi_i)^{k_i}$ and the stabilizer of the base $\Delta_A$ is $\Omega = \prod_{i=1}^{m} \Omega_i$ where*

$$\Omega_i = \{g \in \mathrm{Aut}(\Phi_i^{k_i}) | g(\Delta_i^{k_i}) = \Delta_i^{k_i}\} = T_i^{k_i} \rtimes S_{k_i}$$

*and $T_i = \{g \in \mathrm{Aut}(\Phi_i) | g(\Delta_i) = \Delta_i\}$ is the stabilizer subgroup of the base $\Delta_i$ or the diagram automorphism group of $\Phi_i$ with respect to $\Delta_i$. $G = \prod_{i=1}^{m} \mathrm{Aut}(\Phi_i^{k_i})$ also acts diagonally on the root lattice $\mathbf{Z}\Phi_A = \oplus_{i=1}^{m} \mathbf{Z}\Phi_i^{k_i}$ and on the weight lattice $\Lambda(\Phi_A) = \oplus_{i=1}^{m} \Lambda(\Phi_i^{k_i})$. Moreover,*

$$\mathbf{Z}\Phi_A \subset A \subset \Lambda(\Phi_A)$$

**Proof:**

Since $\Phi_A$ and $\overline{\Psi}$ have the same reflection set $\Gamma$, the set of reflections in $R = W(\overline{\Psi})$, we find by Lemma 3.9 that $\Phi_A = \{c_\alpha \alpha | \alpha \in \overline{\Psi}\}$ for some $c_\alpha \in \mathbf{Q}^+$. Since both $\overline{\Psi}$ and $\Phi_A$ are $G$-stable, we see that $c_{g\alpha} = c_\alpha$ for all $g \in G$ and $\alpha \in \overline{\Psi}$.

Recall that $\Gamma_0$ is the set of reflections in $W(\Psi)$. Since $\Gamma_0$ is contained in $\Gamma$, $\Phi_{A,\Gamma_0}$ is a subset of $\Phi_{A,\Gamma} \equiv \Phi_A$. Since $\Psi$ and $\Phi_{A,\Gamma_0}$ both have reflection set $\Gamma_0$, we have $\Phi_{A,\Gamma_0} = \{c_\alpha \alpha | \alpha \in \Psi\}$ (where the $c_\alpha$'s match with those above for $\alpha \in \Psi$) so that $\mathrm{Aut}(\Phi_{A,\Gamma_0}) = \mathrm{Aut}(\Psi) = G$. Set $\Sigma_i = \{c_\alpha \alpha | \alpha \in \Psi_i\}$, $i = 1, \ldots, m$. By Lemma 3.9, $\Phi_{A,\Gamma_0} = \cup_{i=1}^m \Sigma_i^{l_i}$ is the decomposition of $\Phi_{A,\Gamma_0}$ into irreducibles with $\Sigma_i$ distinct. Note that $\overline{\Sigma_i^{k_i}} = \{c_\alpha \alpha | \alpha \in \overline{\Psi_i^{k_i}}\}$ so that $\Phi_A = \cup_{i=1}^m \overline{\Sigma_i^{k_i}}$ and $\Delta_A = \cup_{i=1}^m \Delta_i^{k_i}$. So we have

$$G = \mathrm{Aut}(\Psi) = \mathrm{Aut}(\Phi_{A,\Gamma_0}) = \prod_{i=1}^m \mathrm{Aut}(\overline{\Sigma_i^{k_i}})$$

The reflection subgroup of $G$ acting on $V$ is

$$R = W(\Phi_A) = \prod_{i=1}^m W(\overline{\Sigma_i^{k_i}}) = \prod_{i=1}^m W(\overline{\Psi_i^{k_i}}) = W(\overline{\Psi})$$

and the stabilizer of the base $\Delta_A$ is

$$\Omega = \Omega_G(\Delta_A) = \prod_{i=1}^m \Omega_i$$

where

$$\begin{aligned}\Omega_i &= \{g \in \mathrm{Aut}(\overline{\Psi_i^{k_i}}) | g(\Pi_i^{k_i}) = \Pi_i^{k_i}\} \\ &= \{g \in \mathrm{Aut}(\overline{\Sigma_i^{k_i}}) | g(\Delta_i^{k_i}) = \Delta_i^{k_i}\} = T_i^{k_i} \rtimes S_{k_i}\end{aligned}$$

where $T_i$ is the stabilizer of $\Delta_i$ in $\overline{\Sigma_i}$. $G = \prod_{i=1}^m \mathrm{Aut}(\overline{\Sigma_i^{k_i}})$, $R = \prod_{i=1}^m R_i$ and $\Omega = \prod_{i=1}^m \Omega_i$ all act diagonally on $V = \oplus_{i=1}^m V_i^{k_i} = \oplus_{i=1}^m \mathbf{Q}\overline{\Sigma_i^{k_i}}$ and hence act diagonally on $\mathbf{Z}\Phi_A = \oplus_{i=1}^m \mathbf{Z}\overline{\Sigma_i^{k_i}}$, $\Lambda(\Phi_A) = \oplus_{i=1}^m \Lambda(\overline{\Sigma_i^{k_i}})$. The fact that $\mathbf{Z}\Phi_A \subset A \subset \Lambda(\Phi_A)$ follows from Lemma 3.8 and the fact that $V^R = 0$.

To obtain the statement of the Lemma, set $\Phi_i = \overline{\Sigma_i}$. Then $\Phi_i$ can be any irreducible root system except for one of type $A_2, A_3$, or $D_n, n \geq 4$. To show this, assume that some $\Phi_i$ is of type $A_2, A_3$, or $D_n, n \geq 4$. Then, by Lemma 6.4(a), $\mathrm{Aut}(\Phi_i)$ would be itself generated by reflections on $\mathbf{Q}\Phi_i$. Since $W(\Phi_i)$ is a proper subgroup of $\mathrm{Aut}(\Phi_i)$ in each case, $W(\Phi_i)$ is not the full reflection subgroup of $\mathrm{Aut}(\Phi_i)$, contradicting the fact that $\Phi_i = \overline{\Sigma_i}$ for some irreducible root system $\Sigma_i$. If $\Phi_i$ is not of type $A_2, A_3$ or $D_n, n \geq 4$, we have $\overline{\Phi_i} = \Phi_i$. This shows that any other type of irreducible could be represented in the decomposition. Applying Lemma 6.6, it can be similarly shown that a component of the type $\Phi_i^{k_i} = A_1^{k_i}$ must have $k_i = 1$. ■

## 7 Rationality Result

In this last section, we will use Theorem 5.3 as well as Farkas' rationality result and a result of Endo and Miyata on fields of tori invariants to prove that $K(A)^G$ is rational over $K$ where $G$ is the automorphism group of the crystallographic root system $\Psi$, $A$ is a full $\mathbf{Z}G$ lattice on $\mathbf{Q}\Psi$ and where $G$ acts trivially on the field $K$. We will adopt the notation of Lemma 6.7. In the decomposition of $\Phi_A = \cup_{i=1}^m \Phi_i^{k_i}$ of this Lemma, we will refer to $\Phi_i^{k_i}$ as the $i$th component of $\Phi_A$.

**Remark 7.1.** The next lemma will be useful for finding compatible decompositions of $\mathbf{Z}\Phi_A$ and $\Lambda(\Phi_A)$ as $\mathbf{Z}\Omega$ lattices. It was suggested by the referee in order to simplify the proof of Lemma 7.3.

**Lemma 7.2.**
*(a) For a finite group $H$, let $N$ be a pure free $\mathbf{Z}H$ sublattice of a $\mathbf{Z}H$ lattice $M$. Then $N$ is a direct summand of $M$.*
*(b) Let $M' \subset M$ be isomorphic permutation $\mathbf{Z}C_2$ lattices of the same rank and let $N$ be a $\mathbf{Z}C_2$ sublattice which is a direct summand of of both $M'$ and $M$. Then there exist isomorphic permutation $\mathbf{Z}C_2$ lattices $L' \subset L$ such that*

$$M' \cong N \oplus L' \qquad M \cong N \oplus L$$

*as $\mathbf{Z}C_2$ lattices.*

**Proof:**
(a) For a $\mathbf{Z}H$ lattice $X$, let $X^*$ denote its $\mathbf{Z}$-dual $\mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})$. Taking $\mathbf{Z}$-duals of the $\mathbf{Z}H$ exact sequence of lattices

$$0 \to N \to M \to M/N \to 0$$

we obtain a $\mathbf{Z}H$ exact sequence of lattices

$$0 \to (M/N)^* \to M^* \to N^* \to 0$$

with $N^* \cong N$ a free $\mathbf{Z}G$ lattice [2, (10.29)]. Hence the latter sequence splits. Taking $\mathbf{Z}$-duals again and recalling that $X^{**} \cong X$ [2, (10.26)(b)] for all $\mathbf{Z}H$ lattices, we obtain the desired result.
(b) By hypothesis, there exists a $\mathbf{Z}C_2$ lattice $L$ with $M \cong N \oplus L$ and hence for the $\mathbf{Z}C_2$ lattice $L' = L \cap M'$, we have $M' \cong N \oplus L'$. Now $L' \subset L$ are $\mathbf{Z}C_2$ lattices of the same rank. So, for the generator $t$ of $C_2$, $\mathrm{rank}(\mathrm{Ker}_L(t-1)) = \mathrm{rank}(\mathrm{Ker}_{L'}(t-1))$ and $\mathrm{rank}(\mathrm{Ker}_L(t+1)) = \mathrm{rank}(\mathrm{Ker}_{L'}(t+1))$. Moreover, since each is a direct summand of a permutation lattice, each is coflasque. Now, recall that any $\mathbf{Z}C_2$ lattice is isomorphic to a direct summand of copies of the lattices $\mathbf{Z}$, $\mathbf{Z}C_2$ and $\mathbf{Z}^-$ [2, (34.31)]. Supppose

$$L \cong \mathbf{Z}^m \oplus (\mathbf{Z}C_2)^n \oplus (\mathbf{Z}^-)^r \qquad L' \cong \mathbf{Z}^{m'} \oplus (\mathbf{Z}C_2)^{n'} \oplus (\mathbf{Z}^-)^{r'}$$

for some non-negative integers $m, m', n, n', r, r'$. As both $L$ and $L'$ are coflasque, we find that $r = r' = 0$. The conditions on ranks given above then show that

$m+n = m'+n'$ and $n = n'$ respectively, so that $L$ and $L'$ are indeed isomorphic permutation lattices. ∎

By Lemma 5.1, $\mathbf{Z}\Phi_A$ and $\Lambda(\Phi_A)$ are isomorphic permutation lattices for $\Omega$. The following lemma refines this description:

**Lemma 7.3.**
*(a) For each $i$, the diagram automorphism group of $\Phi_i$ with respect to $\Delta_i$, $T_i$, is either trivial or isomorphic to $C_2$. There exist $\mathbf{Z}T_i$ permutation lattices $N_i, L_i'$ and $L_i$ such that $L_i' \subset L_i$ are both faithful $\mathbf{Z}T_i$ lattices and such that there exist decompositions $\mathbf{Z}\Phi_i = N_i \oplus L_i'$ and $\Lambda(\Phi_i) = N_i \oplus L_i$. In each case, we have $L_i' \cong L_i$. If $T_i = 1$, then $L_i = \mathbf{Z}$. If $T_i \cong C_2$, then $L_i \cong \mathbf{Z}T_i$ or $L_i \cong \mathbf{Z}T_i \oplus \mathbf{Z}$.*
*(b) For each $i$, there exist $\mathbf{Z}\Omega_i$ permutation lattices $P_i, Q_i'$ and $Q_i$ such that $\Omega_i$ acts faithfully on both $Q_i'$ and $Q_i$, $Q_i' \subset Q_i$, and there exist decompositions $\mathbf{Z}\Phi_i^{k_i} = P_i \oplus Q_i'$ and $\Lambda(\Phi_i^{k_i}) = P_i \oplus Q_i$. In each case, we have $Q_i' \cong Q_i$. If $\Omega_i = S_{k_i}$, then $Q_i \cong \mathrm{Ind}_{S_{k_i-1}}^{S_{k_i}}\mathbf{Z}$. If $\Omega_i = T_i^{k_i} \rtimes S_{k_i}$ where $T_i \neq 1$, then $Q_i \cong \mathrm{Ind}_{T^{k_i-1}\rtimes S_{k_i-1}}^{T^{k_i}\rtimes S_{k_i}}\mathbf{Z}$ or $Q_i \cong \mathrm{Ind}_{T^{k_i-1}\rtimes S_{k_i-1}}^{T^{k_i}\rtimes S_{k_i}}\mathbf{Z} \oplus \mathrm{Ind}_{T^{k_i}\rtimes S_{k_i-1}}^{T^{k_i}\rtimes S_{k_i}}\mathbf{Z}$.*
*(c) $P_A = \oplus_{i=1}^m P_i$, $Q_A = \oplus_{i=1}^m Q_i$ and $Q_A' = \oplus_{i=1}^m Q_i'$ are $\mathbf{Z}\Omega$ permutation lattices such that $\mathbf{Z}\Phi_A = P_A \oplus Q_A'$ and $\Lambda(\Phi_A) = P_A \oplus Q_A$ where $\Omega$ acts faithfully on $Q_A$ and $Q_A' \subset Q_A$.*

**Proof:** Since $\Omega = \prod_{i=1}^m \Omega_i$ acts componentwise on $\Lambda(\Phi_A) = \oplus_{i=1}^m \Lambda(\Phi_i^{k_i})$ and $\mathbf{Z}\Phi_A = \oplus_{i=1}^m \mathbf{Z}\Phi_i^{k_i}$, we see that (c) follows immediately from (b).

To simplify notation in the proofs of (a) and (b), we will suppress the subscript $i$ in referring to the ith component of the decomposition of $\Phi_A$.
(a) By [7, p. 66], the only irreducible root systems with non-trivial diagram automorphism groups are of types $A_n$ for $n \geq 2$, $E_6$ or $D_n, n \geq 4$. But in Lemma 6.7, we noted that $(D_n)^k$, $(A_2)^k$, $(A_3)^k$ cannot occur as components of $\Phi_A$. So, for each component $\Phi^k$ of $\Phi_A$ such that $\Phi$ has non-trivial diagram automorphism group $T$, $T$ is cyclic of order 2. So we will divide the proof of (a) into two cases.
**Case 1:** $T = 1$

To show (a), we need only find a suitable $\mathbf{Z}$ decomposition. Since $\Lambda(\Phi)/\mathbf{Z}\Phi$ is cyclic, we may choose a $\mathbf{Z}$-basis $x_1, \ldots, x_n$ of $\Lambda(\Phi)$ such that $x_1, \ldots, x_{n-1}, dx_n$ is a $\mathbf{Z}$-basis of $\mathbf{Z}\Phi$. Then we may take $N = \oplus_{j=1}^{n-1}\mathbf{Z}x_j \cong \mathbf{Z}^{n-1}$, $L = \mathbf{Z}x_n$ and $L' = dL$. So $L' \subset L$ and both are rank 1 trivial lattices. Clearly $\mathbf{Z}\Phi = N \oplus L'$ and $\Lambda(\Phi) = N \oplus L$.
**Case 2:** $T \neq 1$.

As noted above, $T = \langle t \rangle$ is a cyclic group of order 2. By Lemma 7.2 (b) and the fact that $\mathbf{Z}\Phi$ and $\Lambda(\Phi)$ are both isomorphic $\mathbf{Z}T$ lattices, to show (a) it suffices to find a $\mathbf{Z}T$ sublattice $N$ which is a direct summand of both $\mathbf{Z}\Phi$ and $\Lambda(\Phi)$ in the two subcases: $\Phi$ of type $A_n$ for $n \geq 4$ and $\Phi$ of type $E_6$.
**Case 2a:** $\Phi$ has type $A_n$ for $n \geq 4$.

By [7, p. 59], the base for the root system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ can be expressed in terms of the fundamental dominant weights $\{\omega_1, \ldots, \omega_n\}$ as

$$\begin{aligned}\alpha_1 &= 2\omega_1 - \omega_2 \\ \alpha_i &= -\omega_{i-1} + 2\omega_i - \omega_{i+1} \text{ for } 1 < i < n \\ \alpha_n &= -\omega_{n-1} + 2\omega_n\end{aligned}$$

The $T$-action is given by $t\omega_i = \omega_{n+1-i}$ and $t\alpha_i = \alpha_{n+1-i}$.

Since $\{\omega_n, \alpha_2, \ldots, \alpha_n\}$ is also a $\mathbf{Z}$ basis for $\Lambda(\Phi)$, we can take

$$N = \begin{cases} \oplus_{i=2}^{n-1} \mathbf{Z}\alpha_i \cong (\mathbf{Z}T)^{n/2-1} & \text{if } n \text{ is even} \\ \oplus_{i=2, i\neq(n+1)/2}^{n-1} \mathbf{Z}\alpha_i \cong (\mathbf{Z}T)^{(n-1)/2-1} & \text{if } n \text{ is odd} \end{cases}$$

Then $N$ is a pure free sublattice of both $\mathbf{Z}\Phi$ and $\Lambda(\Phi)$ so that by Lemma 7.2, it is a direct summand of each lattice. The structure of $\mathbf{Z}\Phi/N \cong L'$ as a $\mathbf{Z}T$ lattice is easy to determine. From this structure and Lemma 7.2, we hence obtain a decomposition of $\mathbf{Z}\Phi$ and $\Lambda(\Phi)$ as in (a) in which

$$L \cong L' \cong \begin{cases} \mathbf{Z}T & \text{if } n \text{ is even} \\ \mathbf{Z}T \oplus \mathbf{Z} & \text{if } n \text{ is odd} \end{cases}$$

**Case 2b**: $\Phi$ has type $E_6$

By [7, p. 59], $\Delta = \{\alpha_1, \ldots, \alpha_6\}$ can be expressed in terms of the corresponding basis of fundamental dominant weights as follows:

$$\begin{aligned}\alpha_1 &= 2\omega_1 - \omega_3 \\ \alpha_2 &= 2\omega_2 - \omega_4 \\ \alpha_3 &= -\omega_1 + 2\omega_3 - \omega_4 \\ \alpha_4 &= -\omega_2 - \omega_3 + 2\omega_4 - \omega_5 \\ \alpha_5 &= -\omega_4 + 2\omega_5 - \omega_6 \\ \alpha_6 &= -\omega_5 + 2\omega_6\end{aligned}$$

Note that $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \omega_2, \omega_3\}$ is an alternate $\mathbf{Z}$- basis for $\Lambda(\Phi)$ since

$$\begin{aligned}\omega_4 &= -\alpha_2 + 2\omega_2 \\ \omega_1 &= -\alpha_3 - \omega_4 + 2\omega_3 \\ \omega_5 &= -\alpha_4 - \omega_2 - \omega_3 + 2\omega_4 \\ \omega_6 &= -\alpha_5 - \omega_4 + 2\omega_5\end{aligned}$$

Let $N = \mathbf{Z}\alpha_2 \oplus \mathbf{Z}\alpha_3 \oplus \mathbf{Z}\alpha_4 \oplus \mathbf{Z}\alpha_5$. Note that $t$ acts on $\Lambda(\Phi)$ and $\mathbf{Z}\Phi$ by

$$\begin{aligned} t\alpha_1 &= \alpha_6 & t\omega_1 &= \omega_6 \\ t\alpha_2 &= \alpha_2 & t\omega_2 &= \omega_2 \\ t\alpha_3 &= \alpha_5 & t\omega_3 &= \omega_5 \\ t\alpha_4 &= \alpha_4 & t\omega_4 &= \omega_4 \\ t\alpha_5 &= \alpha_3 & t\omega_5 &= \omega_3 \\ t\alpha_6 &= \alpha_1 & t\omega_6 &= \omega_1 \end{aligned}$$

So $N \cong \mathbf{Z}T \oplus \mathbf{Z}^2$, $\mathbf{Z}\Phi/N = \mathbf{Z}\overline{\alpha}_1 \oplus \mathbf{Z}\overline{\alpha}_6 \cong \mathbf{Z}T$ and $\mathbf{Z}\Phi/N \rightarrowtail \Lambda(\Phi)/N \twoheadrightarrow \Lambda(\Phi)/\mathbf{Z}\Phi$ with $\Lambda(\Phi)/\mathbf{Z}\Phi$ of order 3 implies $H^1(T, \Lambda(\Phi)/N) = 0$ hence $\Lambda(\Phi)/N \cong \mathbf{Z}T$ (as $T$ does not act trivially on $\Lambda(\Phi)/N$). So the $\mathbf{Z}T$ exact sequence $N \rightarrowtail \Lambda(\Phi) \twoheadrightarrow \Lambda(\Phi)/N$ splits as required. Hence by Lemma 7.2 there exist $\mathbf{Z}T$ lattices $L$ and $L'$ with $\Lambda(\Phi) = N \oplus L, \mathbf{Z}\Phi = N \oplus L'$ and $L' \cong L \cong \mathbf{Z}T$ as required.

(b) By Lemma 6.4(b), we see that $\Lambda(\Phi)^k \cong \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \Lambda(\Phi)$ and $\mathbf{Z}\Phi^k \cong \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \mathbf{Z}\Phi$ as $\Omega = T^k \rtimes S_k$ lattices. Since induction and inflation preserve direct sums and send permutation lattices to permutation lattices, we see that by Lemma 6.2, that $\mathrm{Wr_{T,k}X}$ is a (faithful) permutation $\Omega = T^k \rtimes S_k$ lattice if $X$ is a (faithful) permutation $T$ lattice. So to obtain the result of (b), we can take $P = \mathrm{Wr_{T,k}N}$, $Q' = \mathrm{Wr_{T,k}L'}$ and $Q = \mathrm{Wr_{T,k}L}$ as the required $\mathbf{Z}\Omega$ permutation lattices. Then $Q' \subset Q$, $\Omega = T^k \rtimes S_k$ acts faithfully on both $Q'$ and $Q$, $Q' \cong Q$ and $\mathbf{Z}\Phi^k = P \oplus Q'$ and $\Omega(\Phi^k) = P \oplus Q$ are the required decompositions.

We need only determine $P, Q', Q$ as $\mathbf{Z}\Omega$ lattices. Since $\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \mathbf{Z} \cong \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$, $\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \mathbf{Z}T \cong \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$, $P \cong \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} N$, and $Q \cong Q' \cong \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L$ we see from the proof of (a) that

$$P \cong \begin{cases} (\mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z})^{\frac{n-1}{2}-1}, & \Phi = A_n, n \text{ odd} \\ (\mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z})^{\frac{n}{2}-1}, & \Phi = A_n, n \text{ even} \\ \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z} \oplus (\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z})^2, & \Phi = E_6 \\ (\mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z})^{n-1}, & \text{otherwise} \end{cases}$$

$$Q \cong Q' \cong \begin{cases} \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}, & \Phi = A_n, n \text{ even}, E_6 \\ \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z} \oplus \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}, & \Phi = A_n, n \text{ odd} \\ \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}, & \text{otherwise} \end{cases}$$

so that (b) is proved. ∎

**Definition 7.4.** Let $\Phi^k$ be a component in the decomposition of $\Phi_A$. Let $Q' \cong Q$ be the $\mathbf{Z}\Omega$ direct summands of $\mathbf{Z}\Phi^k$ defined in Lemma 7.3. Let $T$ be the group of diagram automorphisms for a base $\Delta$ of $\Phi$. Recall that $Q' = \mathrm{Wr_{T,k}L'}$ and $Q = \mathrm{Wr_{T,k}L}$ where $L' \subset L$ were faithful $\mathbf{Z}T$ lattices which were direct summands of $\mathbf{Z}\Phi$, resp. $\Lambda(\Phi)$. Then $\Phi^k$ is said to be of type:

I if $\Omega = S_k$, $L' \cong L \cong \mathbf{Z}$, and $Q' \cong Q \cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}$

II if $\Omega = T^k \rtimes S_k$, $L' \cong L \cong \mathbf{Z}T$, and $Q' \cong Q \cong \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$

III if $\Omega = T^k \rtimes S_k$, $L' \cong L \cong \mathbf{Z}T \oplus \mathbf{Z}$ and $Q' \cong Q \cong \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z} \oplus \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$

Observe that by Lemma 7.3, these are all the possibilities for a component $\Phi$ occurring in the decomposition of $\Phi_A$.

For each type of component, we need to find the reflection subgroup $R^\sharp$ of the action of $\Omega$ on $Q$, the stabilizer $\Omega^\sharp$ of a base of an associated root system and the structure of $Q$ as a $\mathbf{Z}\Omega^\sharp$ lattice. We also need a decomposition of $Q$ into permutation $\mathbf{Z}\Omega^\sharp$ lattices similar to that in Lemma 7.3.

**Lemma 7.5.**
*(a) Let $\Phi^k$ be a component of the decomposition of $\Phi_A$. Then with the above notation:*
*$\Phi^k$ of type I: $R^\sharp = S_k$, $\Omega^\sharp = 1$, $\mathrm{Res}_{\Omega^\sharp} Q \cong \mathbf{Z}^k$.*
*$\Phi^k$ of type II: $R^\sharp = T^k$, $\Omega^\sharp = S_k$, $\mathrm{Res}_{\Omega^\sharp} Q \cong \mathrm{Ind}_{S_{k-1}}^{S_k}\mathbf{Z} \oplus \mathrm{Ind}_{S_{k-1}}^{S_k}\mathbf{Z}$*
*$\Phi^k$ of type III: $R^\sharp = T^k$, $\Omega^\sharp = S_k$, $\mathrm{Res}_{\Omega^\sharp} Q \cong \mathrm{Ind}_{S_{k-1}}^{S_k}\mathbf{Z} \oplus \mathrm{Ind}_{S_{k-1}}^{S_k}\mathbf{Z} \oplus \mathrm{Ind}_{S_{k-1}}^{S_k}\mathbf{Z}$*
*(b) For $\Phi^k$ as in (a) of any type, we have $\mathbf{Z}\Omega^\sharp$ permutation lattices $C, D, D'$ such that there exist decompositions*

$$Q = C \oplus D \qquad Q' = C \oplus D'$$

*as $\mathbf{Z}\Omega^\sharp$ lattices where $\Omega^\sharp$ acts faithfully on $D$ and $D' \subset D$. Let $d$ be the order of $\Lambda(\Phi)/\mathbf{Z}\Phi$. Then for $\Phi^k$ of type I, $D \cong \mathbf{Z}^k, D' = dD$ and for $\Phi^k$ of type II,III, $D \cong \mathrm{Ind}_{S_{k-1}}^{S_k}\mathbf{Z}, D' = dD$.*
*(c) For each component $\Phi_i^{k_i}$ in the decomposition of $\Phi_A$, let $R_i^\sharp$, $\Omega_i^\sharp$ be given as in (a) and $C_i, D_i, D_i'$ be given as in (b). Set $C_A = \oplus_{i=1}^m C_i, D_A = \oplus_{i=1}^m D_i, D_A' = \oplus_{i=1}^m D_i'$. Then the reflection group acting on $Q_A$ is $R^\sharp = \prod_{i=1}^m R_i^\sharp$ and the stabilizer of a base of the associated root system is $\Omega^\sharp = \prod_{i=1}^m \Omega_i^\sharp$. Then $C_A$, $D_A, D_A'$ are $\mathbf{Z}\Omega^\sharp$ permutation lattices and there exist decompositions*

$$Q_A = C_A \oplus D_A \qquad Q_A' = C_A \oplus D_A'$$

*of $\mathbf{Z}\Omega^\sharp$ lattices where $\Omega^\sharp$ acts faithfully on $D_A$ and $D_A' \subset D_A$.*

**Proof:**
(a) For $\Phi^k$ of type I, $\Omega = S_k$ and $Q \cong \mathrm{Ind}_{S_{k-1}}^{S_k}\mathbf{Z} \cong \mathbf{Z}[S_k/S_{k-1}]$. Applying Lemma 6.2(c) to the $\mathbf{Z}$ lattice $L'$ of rank 1, we see that $R^\sharp = S_k$ and $\Omega^\sharp = 1$ as required. It is then also clear that $\mathrm{Res}_{\Omega^\sharp} Q' = \mathrm{Res}_{\Omega^\sharp} Q = \mathbf{Z}^k$.

For $\Phi^k$ of type II or III, $\Omega = T^k \rtimes S_k$ where $T \cong C_2$ and $Q' = \mathrm{Wr_{T,k}L'}$, $Q = \mathrm{Wr_{T,k}L}$ where $L' \cong L$ are faithful $\mathbf{Z}T$ lattices. Note that in each type $T$ is generated by reflections on its action on $L' \cong L$, as these are either isomorphic to $\mathbf{Z}T$ or $\mathbf{Z}T \oplus \mathbf{Z}$. Applying Lemma 6.2(c) to the $\mathbf{Z}T$ lattice $L'$ or $L$, we see that $R^\sharp = T^k$ and $\Omega^\sharp = S_k$. Now, since $S_k(T^k \rtimes S_{k-1}) = T^k \rtimes S_k$, we see from the Mackey decomposition [2, p.237] that

$$\begin{aligned}\mathrm{Res}_{S_k}^{T^k \rtimes S_k} Q &= \mathrm{Res}_{S_k}^{T^k \rtimes S_k} \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L \\ &\cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathrm{Res}_{S_{k-1}}^{T^k \rtimes S_{k-1}} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L \\ &\cong (\mathrm{Ind}_{S_{k-1}}^{S_k}\mathbf{Z})^{\mathrm{rank} L}\end{aligned}$$

where the last equality follows from the fact that $S_{k-1}$ is contained in the kernel of the inflation map $T^k \rtimes S_{k-1} \to T$. The same calculation for $L'$ shows that we have a parallel statement for $Q'$. So we have determined the structure of $Q'$ and $Q$ as $\mathbf{Z}\Omega^\sharp$ lattices.

(b) For $\Phi^k$ of type I with $\Omega^\sharp = 1$, we have $Q = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L = \mathrm{Ind}_{S_{k-1}}^{S_k} L \cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}$ and $Q' = dQ$. We may take $C = 0$, $D = Q$, $D' = Q' = dD$ to get the required decompositions.

Let $\Phi^k$ be of type II or III with $\Omega^\sharp = S_k$. Then in Lemma 7.3, we found $\mathbf{Z}T$ lattices $L', L$ with $T \cong C_2$ such that

$$Q = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L \qquad Q' = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L'$$

and $L' \subset L$. Then since restriction, induction and inflation preserve direct sums and permutation lattices, it suffices to find appropriate splittings of $L', L$ as $\mathbf{Z}$-lattices. But $L/L' \cong \Omega/\mathbf{Z}\Phi$ is cyclic of order $d$. Hence there exist lattices $C_0, D_0, D_0'$ where $D_0' = dD_0$ are of rank 1 and $L' = C_0 \oplus D_0', L = C_0 \oplus D_0$. Then we can take $C = \mathrm{Ind}_{S_{k-1}}^{S_k} C_0, D = \mathrm{Ind}_{S_{k-1}}^{S_k} D_0, D' = \mathrm{Ind}_{S_{k-1}}^{S_k} D_0'$ to get the required decompositions.

(c) Since $\Omega = \prod_{i=1}^m \Omega_i$ acts componentwise on $Q_A = \oplus_{i=1}^m Q_i$ and $Q_A' = \oplus_{i=1}^m Q_i'$, the results follow easily from (a) and (b). ■

Before starting on the proof of the rationality result, we state the following result on fields of tori invariants due to Endo and Miyata. We recall that a field of tori invariants is a multiplicative invariant field for which the group acts faithfully on the base field.

**Proposition 7.6.** *[4] Let $P$ be a $\mathbf{Z}G$ permutation lattice. The field of tori invariants $K(P)^G$ is rational over $K^G$.*

We finally have enough information to prove the main theorem of the paper.

**Theorem 7.7.** *Let $\Psi$ be a crystallographic root system for the $\mathbf{Q}$ space $V$. Then $G = \mathrm{Aut}(\Psi)$ acts faithfully on $V$. For any $\mathbf{Z}G$ lattice $A$ on $V$, $K(A)^G$ is rational over $K$ where $G$ acts trivially on $K$.*

**Proof:** By Theorem 5.3, we need only show that $K(A)^\Omega$ is rational over $K$. By Lemma 6.7, we have $\mathbf{Z}\Phi_A \subset A \subset \Lambda(\Phi_A)$. By Lemma 7.3, we can decompose $\Phi_A$ and $\Lambda(\Phi_A)$ as $\mathbf{Z}\Phi_A = P_A \oplus Q_A'$, $\Lambda(\Phi_A) = P_A \oplus Q_A$ where $P_A, Q_A, Q_A'$ are $\mathbf{Z}\Omega$ permutation lattices and $\Omega$ acts faithfully on $Q_A' \subset Q_A$. Take $B_A = Q_A \cap A$. Then $A = P_A \oplus B_A$ is a decomposition of $\mathbf{Z}\Omega$ lattices with $\Omega$ acting faithfully on $B_A$ and $Q_A' \subset B_A \subset Q_A$. Then by Proposition 7.6, $K(A)^\Omega$ is rational over $K(B_A)^\Omega$.

It now suffices to prove the rationality of $K(B_A)^\Omega$ over $K$. In order to do this, we want to apply Theorem 5.3 to $K(B_A)^\Omega$. Recall that $\Omega = \prod_{i=1}^k \Omega_i$ and $Q_A' = \oplus_{i=1}^m Q_i' \subset B_A \subset Q_A = \oplus_{i=1}^m Q_i$ where $Q_i' \cong Q_i$ are $\mathbf{Z}\Omega_i$ direct summands of $\mathbf{Z}\Phi_i^{k_i}$, respectively $\Lambda(\Phi_i^{k_i})$.

By Theorem 5.3, it suffices to show that $K(B_A)^{\Omega^\sharp}$ is rational over $K$. Now, by Lemma 7.5, $Q_A' \subset B_A \subset Q_A$ where there exist permutation $\mathbf{Z}\Omega^\sharp$ lattices $C_A, D_A', D_A$ and decompositions

$$Q_A' = C_A \oplus D_A' \qquad Q_A = C_A \oplus D_A$$

of $\mathbf{Z}\Omega^\sharp$ lattices with $\Omega^\sharp$ acting faithfully on $D_A$ and $D'_A \subset D_A$. Setting $Y_A = B_A \cap D_A$, we see that $B_A = C_A \oplus Y_A$ is a decomposition of $\mathbf{Z}\Omega^\sharp$ lattices with $C_A$ permutation, $D'_A \subset Y_A \subset D_A$ and $\Omega^\sharp$ acting faithfully on $Y_A$. So we may again use Proposition 7.6 to show that $K(B_A)^{\Omega^\sharp}$ is rational over $K(Y_A)^{\Omega^\sharp}$. Now we have reduced the problem to showing that $K(Y_A)^{\Omega^\sharp}$ is rational over $K$. But observe that $\Omega^\sharp \cong \prod_{i=1}^m \Omega_i^\sharp$ acts diagonally on $D_A = \oplus_{i=1}^m D_i$ and $D'_A = \oplus_{i=1}^m D'_i$. From Lemma 7.5, we see that if $i$ corresponds to an component of type II,III, $\Omega_i^\sharp = S_{k_i}$ acts faithfully as a group of reflections on $D_i \cong D'_i \cong \mathrm{Ind}_{S_{k_i-1}}^{S_{k_i}} \mathbf{Z}$ and if $i$ corresponds to an irreducible of type I, $\Omega_i^\sharp = 1$ acts trivially (and also faithfully) on $D_i, D'_i$. If all the irreducibles are of type I, $\Omega^\sharp$ acts trivially on $D'_A$, $D_A$ and hence on $Y_A$, so $K(Y_A)^{\Omega^\sharp} = K(Y_A)$ is clearly rational over $K$. Otherwise, $\Omega^\sharp$ acts faithfully as a group of reflections on $D_A$ and $D'_A$ and hence also on $Y_A$. So Farkas' result (Proposition 4.7) shows that $K(Y_A)^{\Omega^\sharp}$ is rational over $K$ in this case. Hence we finally have proved the rationality of $K(A)^G$ over $K$. ∎

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