PART B (50 marks)

6 marks B26. Let $f(x)$ be the function given by

$$f(x) = \begin{cases} \ln(-x) & \text{for } x < 0 \\ x & \text{for } 0 \leq x \leq 1 \\ e^{x-1} & \text{for } x > 1 \end{cases}$$

State the value of the indicated limit, if it exists, in the space provided. Write ∞ or $-\infty$ if appropriate. If a limit does not exist, write DNE.

(a) $\lim_{x \to 0^-} f(x)$ $= \lim_{x \to 0^-} \ln(-x) = -\infty$

(b) $\lim_{x \to 0^+} f(x)$ $= \lim_{x \to 0^+} x = 0$

(c) $\lim_{x \to 0} f(x)$ DNE

(d) $\lim_{x \to 1^-} f(x)$ $= \lim_{x \to 1^-} x = 1$

(e) $\lim_{x \to 1^+} f(x)$ $= \lim_{x \to 1^+} e^{x-1} = e^0 = 1$

(f) $\lim_{x \to 1} f(x)$ $= 1$

NOTE: SHOW ALL YOUR WORK FOR PROBLEM ON THIS PAGE.

3 marks B27. The function

$$f(x) = \begin{cases} 3c - x & \text{if } x < 2 \\ x^2 + c & \text{if } x \geq 2 \end{cases}$$

is continuous at 2. What is the value of c? Justify your answer.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 3c - x = 3c - 2$$

$$f(2) = 2^2 + c = 4 + c$$

$\lim_{x \to 2^-} f(x) = f(2)$ since f is continuous at 2

Thus, $3c - 2 = 4 + c$

$$\Rightarrow 2c = 6$$

$$c = 3$$

NOTE: SHOW ALL YOUR WORK FOR PROBLEMS ON THIS PAGE.

5 *marks* B28. The limit $\lim_{x\to 0} \frac{2^x - 1}{x}$ is the derivative of some function f at some number a.

(*i*) What are $f(x)$ and a?

Answers: $f(x) =$ 2^x OR $2^x - 1$

$a =$ 0

(*ii*) Compute $\lim_{x\to 0} \frac{2^x - 1}{x}$ by evaluating $f'(a)$.

$f'(x) = 2^x \ln 2$

$f'(0) = 2^0 \ln 2 = \ln 2$

Thus, $\lim_{x\to 0} \frac{2^x - 1}{x} = \ln 2$

NOTE: SHOW ALL YOUR WORK FOR PROBLEM ON THIS PAGE.

4 marks B29. Find all values of x in the interval $[0, \pi]$ that satisfy the equation $1 + \cos 2x = \cos x$.

$$1 + \cos 2x = \cos x$$

$$\Rightarrow 1 + 2\cos^2 x - 1 = \cos x$$

$$\Rightarrow 2\cos^2 x = \cos x$$

$$\Rightarrow 2\cos^2 x - \cos x = 0$$

$$\Rightarrow \cos x\,(2\cos x - 1) = 0$$

$$\Rightarrow \cos x = 0 \text{ or } 2\cos x - 1 = 0$$

$$\Rightarrow \cos x = 0 \text{ or } \cos x = 1/2$$

$$\Rightarrow x = \pi/2 \text{ or } x = \pi/3$$

Answers $\frac{\pi}{3}$ and $\pi/2$.

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

6 marks B30. Find an equation of the line tangent to the graph of $\sin(x+y) = y$ at the point $(\pi, 0)$.

Differentiate implicitly to get

$$\cos(x+y)\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$$

At $(\pi, 0)$ we have $\cos\pi\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$

$$\Rightarrow -\left(1+\frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$= -1 - \frac{dy}{dx} = \frac{dy}{dx}$$

$$-1 = 2\frac{dy}{dx}$$

$$\frac{dy}{dx} = -1/2$$

Hence, an equation of the line tangent is

$$y - 0 = -\frac{1}{2}(x-\pi).$$

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

5 *marks* B31. Use the Intermediate Value Theorem to show that there is a root of (i.e. a solution of) the equation $\sin x = 1 - x$ in the interval $\left(0, \frac{\pi}{2}\right)$.

Let $f(x) = \sin x - (1-x)$
$= \sin x - 1 + x$

f is continuous on $[0, \pi/2]$

$f(0) = 0 - 1 + 0 = -1$

$f(\pi/2) = \sin \pi/2 - 1 + \pi/2 = 1 - 1 + \pi/2 = \pi/2$

$f(0) < 0 < f(\pi/2)$

Hence, by the Intermediate Value Theorem,
There is a number c between 0 and $\pi/2$
so that $f(c) = 0$

Then $\sin c - 1 + c = 0$

so $\sin c = 1 - c$

i.e., c is a root of $\sin x = 1 - x$ in the interval $(0, \pi/2)$.

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

5 marks B32. Find $\frac{dy}{dx}$ if $y = \cos^3[\sin^{-1}(e^x)]$. Do not simplify your answer.

Let $u = e^x$, $v = \sin^{-1} u$, $w = \cos v$. Then $y = w^3$

$$\frac{du}{dx} = e^x$$

$$\frac{dv}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\frac{dw}{dv} = -\sin v$$

$$\frac{dy}{dw} = 3w^2$$

$$\frac{dy}{dx} = 3w^2(-\sin v)\frac{1}{\sqrt{1-u^2}}\, e^x$$

$$= \frac{-3e^x \cos^2(\sin^{-1} e^x)\sin(\sin^{-1} e^x)}{\sqrt{1-e^{2x}}}$$

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEMS ON THIS PAGE.

5 *marks* B33. For the function $y = x \sin x$,

(a) find $\frac{dy}{dx}$;

$$\frac{dy}{dx} = x \cos x + \sin x$$

(b) find $\frac{d^2y}{dx^2}$;

$$\frac{d^2y}{dx^2} = x(-\sin x) + \cos x + \cos x$$

$$= -x \sin x + 2 \cos x$$

(c) find $y''\left(\frac{\pi}{6}\right)$, i.e. $\frac{d^2y}{dx^2}$ evaluated at $x = \frac{\pi}{6}$.

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{6}} = -\frac{\pi}{6} \sin\frac{\pi}{6} + 2\cos\pi/6$$

$$= -\frac{\pi}{6}\left(\frac{1}{2}\right) + 2\,\frac{\sqrt{3}}{2}$$

$$= -\frac{\pi}{12} + \sqrt{3}$$

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

5 *marks* B34. Determine $\lim\limits_{x \to 0^+} [\ln(\sin 2x) - \ln(\tan x)]$.

$$\ln(\sin 2x) - \ln(\tan x)$$

$$= \ln \frac{\sin 2x}{\tan x}$$

$$= \ln \frac{2 \sin x \cos x}{\frac{\sin x}{\cos x}}$$

$$= \ln [2 \cos^2 x]$$

$$\lim_{x \to 0^+} [\ln(\sin 2x) - \ln(\tan x)]$$

$$= \lim_{x \to 0^+} \ln 2 \cos^2 x$$

$$= \ln[2 (\cos^2 0)]$$

$$= \ln[2 (1)^2]$$

$$= \ln 2$$

NOTE: SHOW ALL YOUR WORK FOR THE PROBLEM ON THIS PAGE.

6 marks B35. Determine $\lim_{x\to 0}\left(\frac{|3x+1|-|1-3x|}{x}\right)$.

For x near 0, $|3x+1| = 3x+1$

and $|1-3x| = 1-3x$

so $|3x+1| - |1-3x| = 3x+1-(1-3x)$

$= 6x$

Thus, $\lim_{x\to 0}\left(\frac{|3x+1|-|1-3x|}{x}\right)$

$= \lim_{x\to 0} \frac{6x}{x}$

$= 6$