

# CLASSIFYING SUBCATEGORIES OF MODULES OVER A PID.

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ABSTRACT. Let  $R$  be a commutative ring. A full additive subcategory  $\mathcal{C}$  of  $R$ -modules is *triangulated* if whenever two terms of a short triangulated sequence belong to  $\mathcal{C}$ , then so does the third term. In this note we give a classification of triangulated subcategories of finitely generated modules over a principal ideal domain. As a corollary we show that in the category of finitely generated modules over a PID, thick subcategories (triangulated subcategories closed under direct summands), wide subcategories (abelian subcategories closed under extensions) and Serre subcategories (wide subcategories closed under kernels) coincide and correspond to specialisation closed subsets of  $\text{Spec}(R)$ .

## 1. INTRODUCTION

A full additive subcategory  $\mathcal{C}$  of  $R$ -modules is said to be an *exact* subcategory if it is closed under extensions of  $R$ -modules. These categories were introduced by Quillen in [Qui73] where he has given a more abstract and intrinsic definition of these categories. However, in all examples one encounters, exact categories embed in some abelian category and so it is best to think of them as we defined. It is clear from the work of Quillen [Qui73, Gra76], Waldhausen [Wal85] and other recent model category theorists [DS04] that exact categories play an important role in algebraic  $K$ -theory and Morita theory. So a good understanding of them can be very helpful.

In this note we consider some enriched versions of exact categories: triangulated subcategories (exact subcategories that are closed under kernels of surjective maps and cokernels of injective maps); thick subcategories (triangulated subcategories that are closed under direct summands), wide subcategories (abelian subcategories that are closed under extensions), Serre classes (wide subcategories that are closed under submodules and quotient modules), and torsion theories (Serre classes that are closed under arbitrary direct sums) are some examples of subcategories that have been studied. So one has the following hierarchy of exact subcategories.

Torsion theory  $\Rightarrow$  Serre class  $\Rightarrow$  Wide  $\Rightarrow$  Thick  $\Rightarrow$  Triangulated.

(See [Hov01, section 1] for a good account on these subcategories.) Thus triangulated subcategories are quite general families of subcategories. Note, for instance, that a triangulated subcategory need not be closed under kernels and cokernels. The following simple example should illustrate this point.

*Example 1.1.* The subcategory  $\mathcal{C}$  of all even dimensional rational vector spaces in the category of finite dimensional rational vector spaces is readily seen to be a

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triangulated subcategory. However, it is not closed under kernels. Consider the projection map

$$\begin{array}{ccc} \mathbb{Q}^2 & \longrightarrow & \mathbb{Q}^2 \\ (x, y) & \longmapsto & (x, 0). \end{array}$$

This map has a one dimensional kernel and a one dimensional cokernel, and hence the subcategory  $\mathcal{C}$  is neither closed under kernels nor cokernels.

Since triangulated categories form the most general family in the hierarchy of exact subcategories, if we have a classification of triangulated subcategories, we can use that classification to derive classifications of other families of subcategories. However, classifying triangulated subcategories of modules over a general ring seems to be hopelessly difficult. Even when restricted to the finitely generated modules, we do not know of a recipe for classifying triangulated subcategories which is analogous to the K-theory recipe of Thomason [Tho97] for classifying the triangulated subcategories of a triangulated category. Nevertheless, when  $R$  is a principal ideal domain (PID), in view of the structure theorem for finitely generated  $R$ -modules there is a hope for such a classification. And as we will see, one can give a complete classification of these subcategories over PIDs. We should emphasise that the material in this paper is very simple and should be in the reach of any undergraduate student who is familiar with basic properties of modules over a PID.

The paper is organised as follows. We begin in the next section with a quick recap of some basic facts about modules over PIDs and the notion of length of a module. We then classify the triangulated subcategories of finitely generated modules over a PID in section 3. In the last section we discuss classifications of the aforementioned families of subcategories over PIDs.

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## 2. PRINCIPAL IDEAL DOMAINS

In this short section we recall some well-known facts about PIDs. We begin with the structure theorem for finitely generated (f.g.) modules. Let  $R$  be a PID. If  $M$  is a finitely generated  $R$ -module, then

$$M \cong F(M) \oplus \left( \bigoplus_{p \in \text{Spec}(R)} E_p(M) \right),$$

where  $F$  is a free  $R$ -module of finite rank and  $E_p(M)$  denotes the  $R$ -submodule of  $M$  which consists of all elements that are annihilated by some power of  $p$ . Further, for each prime element  $p$  that occurs in the above sum,  $E_p(M)$  decomposes as

$$E_p(M) \cong R/(p^{l_1}) \oplus R/(p^{l_2}) \oplus \cdots \oplus R/(p^{l_s}),$$

with uniquely determined sequence  $1 \leq l_1 \leq l_2 \leq \cdots \leq l_s$ .

Now we recall the notion of the length of a module. For an  $R$ -module  $M$ , a chain

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_r = 0$$

is called a composition series if each  $M_i/M_{i+1}$  is a simple module (one that does not have any non-trivial submodules). The length  $l(M)$  of a module  $M$  that is both Artinian and Noetherian is defined to be the length of any composition series of  $M$ . The fact that this is well-defined is part of the Jordan-Holder theorem. If

$0 \rightarrow N \rightarrow M \rightarrow K \rightarrow 0$  is a short triangulated sequence of  $R$ -modules of finite length, then we have  $l(M) = l(N) + l(K)$ . In other words, the length function  $l(\cdot)$  is an Euler characteristic function on the category of modules of finite length. Finally by rank of a finitely generated module we will mean the rank of its free part, and similarly by  $p$ -length of a module, we will mean the  $p$ -length of its  $p$ -torsion part.

### 3. CLASSIFICATION OF TRIANGULATED SUBCATEGORIES OVER A PID.

Henceforth  $R$  will denote a principal ideal domain unless otherwise stated. We now classify all the triangulated subcategories in the category  $\mathcal{A}$  of finitely generated modules over  $R$ . We first set up some notations. We will denote by  $\mathcal{B}$  the category of finite torsion  $R$ -modules. For every prime ideal  $p$  in  $R$ , define Euler characteristic functions  $\chi_0$  and  $\chi_p$  on  $\mathcal{A}$  and  $\mathcal{B}$  respectively as follows:

$$\chi_p(X) := \begin{cases} \text{length}_p(X \otimes_R R_{(p)}) & \text{when } p \neq 0 \\ \dim_{R_0}(X \otimes_R R_0) & \text{when } p = 0 \end{cases}$$

These Euler characteristic functions define some full subcategories of  $\mathcal{A}$  and  $\mathcal{B}$  as follows. For each positive integer  $k$ , define  $I_k$  as

$$I_k := \{X \in \mathcal{A} : \chi_0(X) \equiv 0 \pmod{k}\},$$

and for each subgroup  $H$  of  $\bigoplus_{p \in \text{MaxSpec}(R)} \mathbb{Z}$  that is generated by elements all of whose components are non-negative, define  $J_H$  as

$$J_H := \{X \in \mathcal{B} : \bigoplus_{p \in \text{MaxSpec}(R)} \chi_p(X) \in H\}.$$

Since  $\chi_p$  is an Euler characteristic function, it is clear that all these subcategories are triangulated subcategories. We now show that these are all the triangulated subcategories in  $\mathcal{A}$ . We begin with two lemmas which will help streamline the proof.

**Lemma 3.1.** (Descending lemma) *Let  $\mathcal{C}$  be a triangulated subcategory of  $\mathcal{A}$  and let  $M$  be a module in  $\mathcal{C}$  with  $p$ -length  $r$ . Then there exists another module  $N$  in  $\mathcal{C}$  whose  $p$ -torsion is  $(R/p)^r$  and whose  $q$ -torsion is identical with that of  $M$  for all primes  $q \neq p$ .*

*Proof.* Let  $M \cong \bigoplus_{i=1}^k R/p^{r_i} \oplus L$ , with  $\sum r_i = r$  and  $L$   $p$ -torsion free. We want to generate modules with lower highest  $p$ -order (In the module  $M$ , the highest  $p$ -order is  $\max\{r_i\}$ .) by keeping the  $p$ -length of the module constant. This is done until we get a module with highest  $p$ -order = 1, or equivalently, until we get  $\bigoplus_{i=1}^r R/p \oplus L$ . For better clarity, we break the proof into two steps.

(a) Assume without loss of generality that  $r_1 = \max\{r_i\}$ . We first build  $R/p \oplus R/p^{r_1-1}$  from  $R/p^{r_1}$ . The following pair of short triangulated sequences will do the job. (The maps are easy to guess, so we will not bother to write them down.)

$$0 \rightarrow R/p^{r_1} \rightarrow R/p^{r_1-1} \oplus R/p^{r_1+1} \rightarrow R/p^{r_1} \rightarrow 0$$

$$0 \rightarrow R/p^{r_1} \rightarrow R/p^{r_1-1} \oplus R/p^{r_1+1} \rightarrow R/p^{r_1-1} \oplus R/p \rightarrow 0$$

(b) Now we build  $R/p \oplus R/p^{r_1-1} \oplus R/p^{r_2} \oplus \cdots \oplus R/p^{r_k} \oplus L$  from  $M = R/p^{r_1} \oplus R/p^{r_2} \oplus \cdots \oplus R/p^{r_k} \oplus L$ . (Note that this decreases the highest  $p$ -order by one.) This can be obtained easily by adding a split short triangulated sequence

$$0 \rightarrow G \rightarrow G \oplus G \rightarrow G \rightarrow 0$$

with  $G = \bigoplus_{i=2}^k R/p^i \oplus L$  to the pair of short triangulated sequences in part (a):

$$\begin{aligned} 0 \rightarrow R/p^{r_1} \oplus G &\rightarrow (R/p^{r_1-1} \oplus G) \oplus (R/p^{r_1+1} \oplus G) \rightarrow R/p^{r_1} \oplus G \rightarrow 0 \\ 0 \rightarrow R/p^{r_1} \oplus G &\rightarrow (R/p^{r_1-1} \oplus G) \oplus (R/p^{r_1+1} \oplus G) \rightarrow R/p^{r_1-1} \oplus R/p \oplus G \rightarrow 0 \end{aligned}$$

A straightforward downward induction will decrease the highest order to 1. In other words, it produces  $\bigoplus_{i=1}^l R/p \oplus L$ , completing the proof of the lemma.  $\square$

**Lemma 3.2.** (Ascending lemma) *Let  $\mathcal{C}$  be a triangulated subcategory of  $\mathcal{A}$ , and let  $M$  be a module in  $\mathcal{C}$  whose  $p$ -torsion part is  $(R/p)^r$ . Now given any partition  $\Sigma r_i$  of  $r$  into non-negative integers, then there exists another module  $N$  in  $\mathcal{C}$  whose  $p$ -torsion is  $\bigoplus R/p^{r_i}$  and whose  $q$ -torsion is identical with that of  $M$  for all primes  $q \neq p$ .*

*Proof.* We have to start with  $(R/p)^r \oplus L$  where  $L$  is  $p$ -torsion free and construct the module  $\bigoplus (R/p^{r_i}) \oplus L$ , where the exponents  $r_i$  are such that  $\Sigma r_i = r$ . Again for clarity, we break the construction into two steps.

(a) We first generate  $R/p^{r+1}$  from  $R/p^r \oplus R/p$ . The following pair of short triangulated sequences will do this job.

$$\begin{aligned} 0 \rightarrow R/p \oplus R/p^r &\rightarrow R/p \oplus R/p^{r+1} \oplus R/p^r \rightarrow R/p \oplus R/p^r \rightarrow 0 \\ 0 \rightarrow R/p \oplus R/p^r &\rightarrow R/p \oplus R/p^r \oplus R/p^{r+1} \rightarrow R/p^{r+1} \rightarrow 0 \end{aligned}$$

(b) Now we have to show that  $R/p^{r+1} \oplus G$  can be generated from  $R/p^r \oplus R/p \oplus G$  where  $G$  is any  $R$ -module. Triangulatedly as before we add, to the above pair of short triangulated sequences, a split short triangulated sequence  $0 \rightarrow G \rightarrow G \oplus G \rightarrow G \rightarrow 0$  to get

$$\begin{aligned} 0 \rightarrow R/p \oplus R/p^r \oplus G &\rightarrow R/p \oplus R/p^{r+1} \oplus R/p^r \oplus G \oplus G \rightarrow R/p \oplus R/p^r \oplus G \rightarrow 0 \\ 0 \rightarrow R/p \oplus R/p^r \oplus G &\rightarrow R/p \oplus R/p^r \oplus R/p^{r+1} \oplus G \oplus G \rightarrow R/p^{r+1} \oplus G \rightarrow 0 \end{aligned}$$

Now, as before, a simple induction will complete the proof of the lemma.  $\square$

**Theorem 3.3.** *Let  $R$  be a PID and let  $\mathcal{A}$  denote the category of finitely generated  $R$ -modules. Then a non-zero subcategory  $\mathcal{C}$  is a triangulated subcategory if and only if, either  $\mathcal{C} = I_k$  for some positive integer  $k$ , or  $J_H$  for some subgroup  $H$  of  $\bigoplus_{p \in \text{MaxSpec}(R)} \mathbb{Z}$  that is generated by elements all of whose components are non-negative.*

*Proof.* First of all it is easy to see that the subcategories  $I_k$  and  $J_H$  as defined in the statement of the theorem are triangulated and pair-wise distinct subcategories. So we have to show that every triangulated subcategory  $\mathcal{C}$  is equal to one of these. The proof of this statement divides naturally into two cases.

**Case(i)**  $\mathcal{C}$  contains a module of rank at least one. Pick a module  $M$  of smallest non-zero rank (exists by assumption) and let  $k$  denote the rank of  $M$ . We claim that  $\mathcal{C} = I_k$ . In other words, we have to show that  $\mathcal{C}$  comprises of all modules of the form  $R^{kl} \oplus T$  where  $l$  is a non-negative integer and  $T$  is a torsion module. This will be done by following a series of straightforward reductions. First of all, it suffices to build free modules and torsion modules from  $M$  (because triangulated subcategories are closed under taking direct sums). Then, the triangulated sequence

$$0 \rightarrow \text{Tor}(M) \rightarrow M (= R^k \oplus \text{Tor}(M)) \rightarrow R^k \rightarrow 0$$

gives a further reduction ( $Tor(M)$  denote the torsion submodule of  $M$ ): It suffices to build an arbitrary torsion module out of  $M$ . Now recall from the structure theorem that any torsion module is a direct sum of cyclic modules of the form  $R/p^t$ . Again, triangulated subcategories are closed under taking direct sums, therefore it is enough to produce  $R/p^t$  for any prime  $p$  and any integer  $t$ . The following short triangulated sequence tells us that this is always possible: (here we use the fact that  $k \geq 1$ )

$$0 \rightarrow R \oplus (R^{k-1} \oplus TorM) \xrightarrow{p^k \oplus (id \oplus id)} R \oplus (R^{k-1} \oplus TorM) \rightarrow R/p^t \rightarrow 0.$$

So this completes the first case.

**Case(ii)** *All modules in  $\mathcal{C}$  have rank zero.* Equivalently, this means that  $\mathcal{C}$  consists of torsion  $R$ -modules. Let  $\chi(-) := \bigoplus_{p \in \text{MaxSpec}(R)} \chi_p(-)$  and define  $H$  to be the subgroup of  $\bigoplus_{p \in \text{MaxSpec}(R)} \mathbb{Z}$  generated by the elements  $\chi(X)$  as  $X$  ranges in  $\mathcal{C}$ . We claim that  $\mathcal{C} = J_H$ . So let  $M$  be an  $R$ -module with  $\chi(M)$  belonging to  $H$ . We want to show that  $M$  in  $\mathcal{C}$ . By the ascending lemma it suffices to show that there is a module  $N$  in  $\mathcal{C}$  which is of the form  $\bigoplus_{p \in \text{MaxSpec}(R)} (R/p)^{r_p}$  and has the same  $p$ -lengths as those of  $M$  for all primes  $p$ . By assumption, we know that

$$\chi(M) = \sum a_i \chi(A_i) - \sum b_i \chi(B_i),$$

for some modules  $A_i$  and  $B_i$  in  $\mathcal{C}$  and some positive coefficients  $a_i$  and  $b_i$ . By the descending lemma  $A_i$  and  $B_i$  can be chosen such that

$$A_i = \bigoplus_{p \in \text{MaxSpec}(R)} (R/p)^{\alpha_{p_i}},$$

$$B_i = \bigoplus_{p \in \text{MaxSpec}(R)} (R/p)^{\beta_{p_i}}.$$

where  $\alpha_i$  and  $\beta_i$  are non-negative integers. Note that for each  $p$ ,  $\sum b_i \beta_{p_i} - a_i \alpha_{p_i}$  is the  $p$ -length of the module  $M$  and therefore is a non-negative integer. So there is an inclusion  $\bigoplus (B_i)^{b_i} \hookrightarrow \bigoplus (A_i)^{a_i}$  whose cokernel is evidently the desired candidate for  $N$ .  $\square$

#### 4. CLASSIFICATIONS OF OTHER FAMILIES OF SUBCATEGORIES

Recall the definitions of thick, wide and Serre subcategories from the introduction. It turns out that these three families coincide in  $\mathcal{A}$ , the category of finitely generated modules over a PID. This will be shown directly from the classification of triangulated subcategories of  $\mathcal{A}$  that is obtained in the previous section.

**4.1. Thick subcategories.** To get a classification of the non-zero thick subcategories of  $\mathcal{A}$  we have to find out which of the  $I_k$ s and the  $J_H$ s are closed under direct summands. One can easily check that  $I_k$  is closed under direct summands if and only if  $k = 1$  and  $J_H$  is so precisely when  $H$  is spanned by the unit vectors of the form  $e_p$  which have zeros everywhere except in the one spot (corresponding to the prime  $p$ ) where there is a one. In other words, the only thick subcategories of  $\mathcal{A}$  are  $\mathcal{A}$  itself and the subcategories of  $S$ -torsion modules where  $S$  is a subset of maximal primes in  $R$ .

**4.2. Wide subcategories.** To get a classification of the non-zero wide subcategories, we look for those thick subcategories that are closed under kernels and cokernels. It is clear that both  $\mathcal{A}$  and the category of  $S$ -torsion modules (where  $S$  is a subset of maximal primes) are both abelian and closed under extensions and therefore wide.

**4.3. Serre subcategories.** It is easy to see that both  $\mathcal{A}$  and the category of  $S$ -torsion modules ( $S$  is a subset of maximal primes) are closed under subobjects and quotient objects. So these are the only Serre subcategories in  $\mathcal{A}$ .

Thus we have show that in  $\mathcal{A}$  the thick subcategories, wide subcategories and Serre subcategories all coincide.

It is also clear that these subcategories are in 1-1 correspondence with the specialisation closed subsets (subsets that are a union of closed subsets in the Zariski topology) of  $\text{Spec}(R)$  because a non-trivial specialisation closed subset of a  $\text{Spec}(R)$  when  $R$  is a PID is precisely a subset of maximal primes, the trivial ones being  $\text{Spec}(R)$  and the empty set which correspond respectively to the categories  $\mathcal{A}$  and the trivial category.

*Remark 4.1.* There are some interesting classifications of some of these subcategories in the literature. For example, the wide subcategories of finitely presented modules over a regular coherent ring have been classified by Hovey [Hov01], and the thick subcategories of finite dimensional representations over finite  $p$ -groups have been classified by Benson-Carlson-Rickard [BCR97].

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