The exponential and logarithm functions

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A formal definition

Definition

We call a function f with domain $\mathbb{R}$ an exponential function if

- it is continuous and
- for each x and y in $\mathbb{R}$, we have $f(x+y) = f(x)f(y)$.

(In words "*the exponential of a sum is the product of the exponentials*".)

Base of an exponential

Definition

Fix an exponential function f. We call the number $a = f(1)$ the **base** of the exponential.

Under this notation,if n is a non-zero natural number, we get

$$f(n) = f(\overbrace{1 + \cdots + 1}^{n \text{ times}}) = \overbrace{f(1) \times \cdots \times f(1)}^{n \text{ times}} = \overbrace{a \times \cdots \times a}^{n \text{ times}} = a^n.$$

Similarly, $f(-n) = (\frac{1}{a})^n = a^{-n}$.
And if m is an integer $f(\frac{m}{n}) = \sqrt[n]{a^m}$.

Proposition

If f is an exponential function then $f(x) \geq 0$ for each $x \in \mathbb{R}$.
In particular the basis of an exponential needs to be a non-negative number.

Exponentials exists and are determined by their basis

Theorem

If a denotes any non-negative real, there exists one and only one exponential function with basis a. It will be denoted $x \mapsto a^x$.

In what follow, we will use notations such as $x \mapsto a^x$, a^x, a^y or b^x.

The exponential with base 0 is the function constant equal to zero, the one with base 1 is the function constant equal to one. These are the only two exponential functions which take constant value. Most comonly used exponential functions are the one of basis e, 2 and 10.

Laws of the exponents and more

Laws of the exponents

If a and b are positive real numbers and if x and y are two real numbers, we have

$$a^{x+y} = a^x a^y, \quad a^{-x} = \frac{1}{a^x}, \quad (a^x)^y = a^{xy}, \quad (ab)^x = a^x b^x.$$

Furthermore we also have

And more

$$a^0 = 1, \quad a^1 = a \text{ and } \quad a^x > 0.$$

Finaly, if p is a integer and q a non-zero natural number we have

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Exercise

Evaluate $16^{-3/4}$. $16^{-3/4} = \frac{1}{8}$

Growth

Proposition

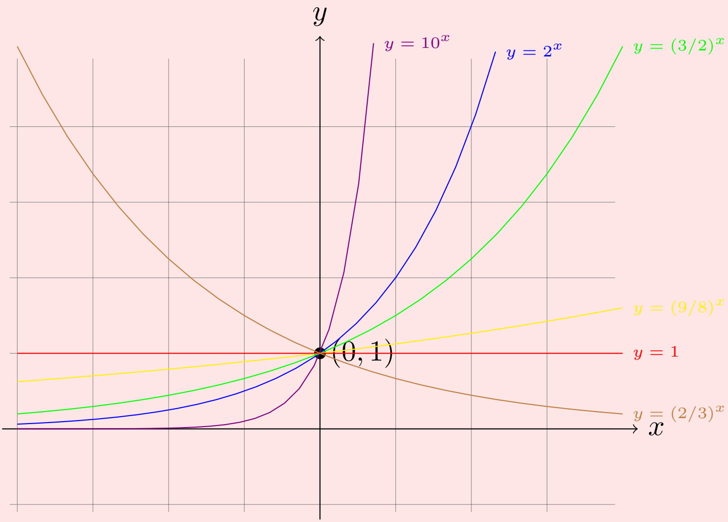
Any function $x \mapsto a^x$ has domain $\mathbb{R}$ and, if $a \neq 0$ nor 1, it has range $(0, +\infty)$.

- **If** $a > 1$
  - *the function $x \mapsto a^x$ is increasing,*
  - *a^x goes to $+\infty$ as x goes to $+\infty$*
  - *and a^x goes to 0 as x goes to $-\infty$.*
- **If** $0 < a < 1$
  - *the function $x \mapsto a^x$ is decreasing,*
  - *a^x goes to 0 as x goes to $+\infty$*
  - *and a^x goes to $+\infty$ as x goes to $-\infty$.*

Proposition

If $a > 1$, $x \mapsto a^x$ grows to $+\infty$ as x goes to $+\infty$ faster than any power function $x \mapsto x^\alpha$.

Some pictures



Exercise

Find the reals a and c such that the graph of the function

$$x \mapsto a^x + c$$

goes through the points $(0, 2)$ and $(3, 9)$.

$$c = 1, a = 2$$

The Euler number

Proposition

If f is an exponential function and f' denotes its derivative then there is a constant c such that

$$f'(x) = cf(x)$$

for all $x \in \mathbb{R}$.

Theorem

For each constant c there is a unique function f such that

$$f'(x) = cf(x)$$

for all $x \in \mathbb{R}$ and $f(0) = 1$.
Furthermore, this function is an exponential function $x \mapsto a^x$.

Definition

The Euler constant e is the base of the unique exponential f satisfying

$$f'(x) = f(x)$$

for all $x \in \mathbb{R}$.

In words ...

In words "$x \mapsto e^x$ *is its own derivative*".
In particular, the slope of the graph of $x \mapsto e^x$ at $(0, 1)$ is $e^0 = 1$.

Theorem

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

e is an irrational number, with value $2.718281828459045235 \cdots$.

One-to-one functions and inverses

Definition

A function f is said to be one-to-one (or injective) if no two distincts values in the domain are sent to a same image.
That is if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

Proposition

If f is increasiong then it is one-to-one.
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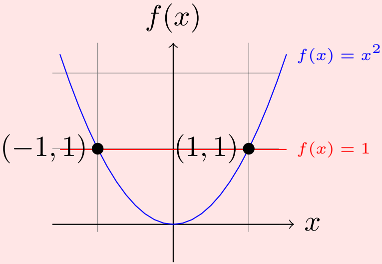
Horizontal test line

A function is one-to-one if no horizontal line itersects its graph more than once.

$x \mapsto x^2$ is not 1-1



Inverse function

Proposition/definition

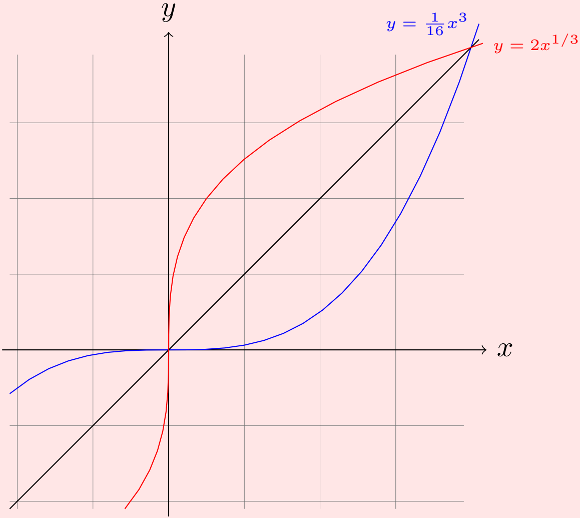
Let f be a function, A its domain and B its range.
Suppose that f is one-to-one.
Then there is a unique function, denoted f^{-1} with domain B satisfying

$$y = f(x) \quad \text{if and only if } x = f^{-1}(y)$$

for every $x \in A$ and $y \in B$.
This function is called the **inverse** of f.

The graph of f^{-1} is the symetric of f's about $y = x$

In picture



The cancelation rule

Let f be a one-to-one function with domain A and range B.

- $f^{-1}(f(x)) = x$ for each $x \in A$.
- $f(f^{-1}(x)) = x$ for each $x \in B$.

The function $x \mapsto \sqrt[3]{x}$ is the inverse of $x \mapsto x^3$.
Although $x \mapsto x^2$ is not one-to-one, its **restriction** to $(0, +\infty)$ is and the inverse of this restriction is $x \mapsto \sqrt{x}$.

Definition

If a is neither 0 nor 1, the exponential function $x \mapsto a^x$ is one-to-one, with domain $\mathbb{R}$ and image $(0, +\infty)$.
We call its inverse the logarithm function with basis a and it is denoted

$$x \mapsto \log_a(x).$$

Thus	for every real x	we have	$\log_a(a^x) = x$
And	for every real $x > 0$	we have	$a^{log_a(x)} = x.$

Definition

The logarithm with base e is called the Neperian (or natural) logarithm.
It is denoted $x \mapsto \ln(x)$.

Exercise

What is $log_{10}(1000)$? What is $\log_{16}(2)$? What is $\ln(1)$?

$\log_{10}(1000) = 3$, $\log_{16}(2) = \frac{1}{3}$, $\ln(1) = 0$.

Laws of the logarithms

The laws

Let a,x and y be three positive real numbers and β be any real number then

$$\log_a(xy) = \log_a(x) + \log_a(y), \quad \log_a(\tfrac{1}{x}) = -\log_a(x),$$
$$\log_a(x^\beta) = \beta \log_a(x) \text{ and } \quad \log_a(1) = 0.$$

In words "*the logarithm of a product is the sum of the logarithms*".

Proposition

If a,b and x are three positive real numbers then

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

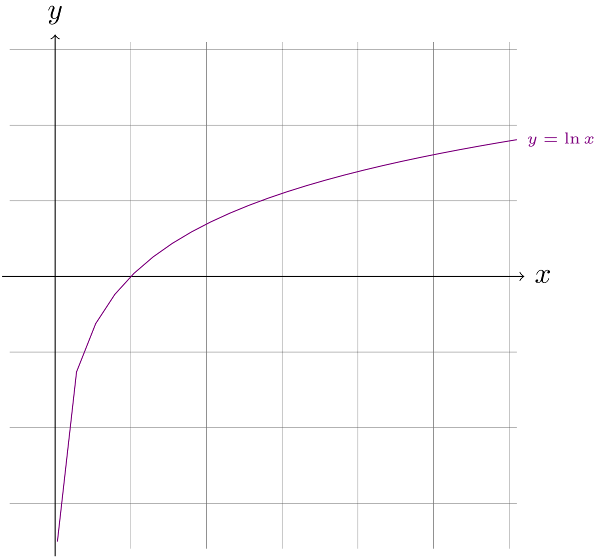
$$\log_a(x) = \log_a(b^{log_b(x)}) = \log_b(x)\log_a(b).$$

Similarly,we have

Proposition

$$a^x = e^{x \ln a}$$

The graph of the neperian logarithm



Derivation

The derivation rules for logarithm and exponential functions are :

f	f'
$x \mapsto e^x$	$x \mapsto e^x$
$x \mapsto a^x$	$x \mapsto \ln(a) \cdot a^x$
$x \mapsto \ln x$	$x \mapsto \frac{1}{x}$
$x \mapsto \log_a x$	$x \mapsto \frac{1}{\ln(a) \cdot x}$