About some exercises

September 30, 2009

First, a general remark concerning questions where some trigonometric ratio is asked to you.

When you use a triangle figure, you compute usually the absolute value of this ratio. Don't forget to also **check the sign** ($+$ or $-$) of this ratio by looking at the position of the angle in the different quadrants.

1 What is $\arcsin(\cos 5)$?

By definition, this is the number α in $[-\pi/2, \pi/2]$ for which $\sin\alpha = \cos 5$.

By the equality $\cos x = \sin(x + \pi/2)$ (if you have drawn a picture of the trigonometric circle or of the graphs of sine and cosine, you can see this equality).

Thus $\sin\alpha = \sin(5 + \pi/2)$.

We also know that the equality $\sin x = \sin y$ is equivalent to the condition :

- **either** y is of the form $x + 2k\pi$ for some integer k
- **or** y is of the form $\pi - x + 2k\pi$ for some integer k (see your pictures).

Now the number $5 + \pi/2 - 2\pi$ satifies the first condition and we can check that it is between $-\pi/2$ and $\pi/2$ (recall that π is about 3 and $\pi/2$ about 1.5).

$$\text{Therefore } \sin^{-1}(\cos(5)) = 5 - 3\pi/2.$$

(NB: we could have seen directly from the picture that $\cos x = \sin y$ is equivalent to the condition:

- **either** y is of the form $x + \pi/2 + 2k\pi$ for some integer k
- **or** y is of the form $-x + \pi/2 + 2k\pi$ for some integer k.

But this could be a little bit too much to see for a single look.)

2 Simplify the expression $\cos(2\tan^{-1} x)$.

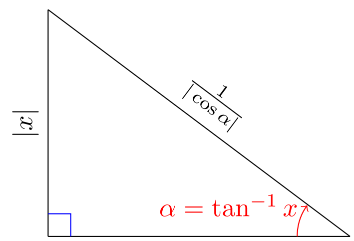
First, we know

$$\begin{aligned}\cos(2\tan^{-1} x) &= \cos^2(\tan^{-1} x) - \sin^2(\tan^{-1} x)\\ &= 1 - 2\sin^2(\tan^{-1} x)\\ &= 2\cos^2(\tan^{-1} x) - 1\end{aligned}$$

(I wrote all the formulæ I know; I don't know which one will be more convenient to use.)

It remains to simplify one of those expressions :

- We can do that by looking at a triangle



  thus by Pythagoras theorem $\frac{1}{\cos^2\alpha} = 1 + x^2$ and

$$\cos^2(\alpha) = \frac{1}{1+x^2}$$

  and

$$\cos(2\tan^{-1} x) = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}.$$

- Or by trying by brute force to find (for instance) $\sin^2(\tan^{-1} x))$.

  We have

$$\begin{aligned}\sin^2(\tan^{-1} x)) &= (\tan(\tan^{-1} x))^2 \cdot \cos^2(\tan^{-1} x))\\ &= x^2\cos^2(\tan^{-1} x))\\ &= x^2(1 - \sin^2(\tan^{-1} x)))\end{aligned}$$

  (the strategy was "try to replace object you don't know by those you know (as $\tan(\tan^{-1} x))$) or those you are interested in (as $\sin^2(\tan^{-1} x))$.)

  Thus by grouping the $\sin^2(\tan^{-1} x))$ terms to the left, we get the equality $(1 + x^2)\sin^2(\tan^{-1} x)) = x^2$ and therefore

$$\sin^2(\tan^{-1} x) = \frac{x^2}{1+x^2}$$

  and

$$\cos(2\tan^{-1} x) = 1 - \frac{2x^2}{1+x^2} = \frac{1-x^2}{1+x^2}.$$