ON HOLOMORPHIC K-DIFFERENTIALS ON OPEN RIEMANN SURFACES

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Abstract. Let Σ be a hyperbolic Riemann surface, with a covering map $\pi : \Delta \to \Sigma$, where Δ is the unit disc. Let Λ be a closed subset of Σ such that $\pi^{-1}(\Sigma - \Lambda)$ is connected. Let $k \geq 2$ be an integer. We study spaces of integrable, square-integrable, and bounded holomorphic k-differentials on $\Sigma - \Lambda$ and we obtain, in particular, a description of the kernel of the Poincaré series operator.

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1. Introduction

Construction of automorphic forms via Poincaré series is a classical technique, there is a significant amount of literature on this subject. For example, it is well-known that any holomorphic k-differential (k is an integer, $k \geq 2$) on a compact Riemann surface of genus $g \geq 2$ is obtained from the Poincaré series of a polynomial in z of degree not higher than $k(2g-2)$. There are various descriptions of the kernel of the Poincaré series operator: see, in particular, [K2], [K3], [Lj], [Ma], [Me].

The case $k = 2$ (quadratic differentials) is of special importance in Teichmüller theory and has connections to the Thurston's program. The large variety of references on this subject includes, in particular, [BD1], [BD2], [G], [La], and several McMullen's papers including [Mc].

Our work on this paper was inspired, partially, by the questions discussed in [LM], although we do not present results similar to the results of [LM].

The main results of this paper are stated in Section 3.

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2. Preliminaries

Throughout the paper we shall assume that k is a fixed positive integer, $k \geq 2$, and we shall omit k from notation (including notations for function spaces, norms, inner products, Bergman kernel).

2.1. In this subsection we shall introduce some notation and review basic definitions related to k-differentials on Riemann surfaces.

A *Riemann surface* is a connected 1-dimensional complex manifold. Note that it is a standard definition in complex analysis, but "connected" is sometimes dropped by algebraic geometers.

Let C be a Riemann surface, and let k be a positive integer. A *k-differential* on C is a section of the holomorphic line bundle $(T^{*\prime}C)^{\otimes k}$, where $T^{*\prime}C$ is the holomorphic cotangent bundle on C. Equivalently, a *k-differential* on C is a collection of $\mathbb{C}$-valued functions $\{\phi_\alpha(z_\alpha)\}_{\alpha \in I}$, where z_α is a local complex coordinate on an open set U_α (for an open cover $\{U_\alpha\}_{\alpha \in I}$ of C; I is an index set), and for all $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$, over $U_\alpha \cap U_\beta$

$$\phi_\alpha(z_\alpha) = \phi_\beta(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^k, \tag{1}$$

and the k-differential is usually written as $\phi(z)dz^k$, which is understood as follows: for any $\xi \in C$ choose $\alpha \in I$ such that $\xi \in U_\alpha$, then near ξ $z = z_\alpha$, $\phi = \phi_\alpha$ (this is well-defined because of (1)). Note that dz^k is a conventional notation for $dz^{\otimes k}$.

A k-differential on C is called *holomorphic (resp. meromorphic)* if it is a *holomorphic (resp. meromorphic)* section of $(T^{*\prime}C)^{\otimes k}$, or, equivalently, if all ϕ_α, $\alpha \in I$ are *holomorphic (resp. meromorphic).*

For a Riemann surface $C \subset \Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ a k-differential is (globally) $\phi(z)dz^k$, where z is the coordinate on Δ and ϕ is a $\mathbb{C}$-valued function on C.

The group

$$G = SU(1,1) = \{A \in SL(2,\mathbb{C}) \mid A^t \sigma \bar{A} = \sigma\} =$$
$$\left\{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, a\bar{a} - b\bar{b} = 1\right\}$$

where $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, acts on Δ by Moebius transformations: for $z \in \Delta$, $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G$, $z \mapsto gz = \frac{az+b}{\bar{b}z+\bar{a}}$. Denote $J(g,z) = \frac{d(gz)}{dz} =$

$\frac{1}{(\bar{b}z+\bar{a})^2}$, where $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in G$, $z \in \Delta$. Note the cocycle property: $J(g_1g_2, z) = J(g_1, g_2z)J(g_2, z)$ for $g_1, g_2 \in G$, $z \in \Delta$.

A *hyperbolic Riemann surface* is a Riemann surface $\Sigma = \Gamma\backslash\Delta$, where Γ is a discrete subgroup of G, acting properly discontinuously on Δ (i.e. any $z \in \Delta$ has a neighborhood U_z such that $g(U_z) \cap U_z = \emptyset$ for any $g \in \Gamma$, $g \neq e$, where e is the identity element).

Note: if C is a Riemann surface, and p_1, p_2, p_3 are three distinct points on C, then $C - \{p_1, p_2, p_3\}$ is a hyperbolic Riemann surface. This follows from the uniformization theorem (see, for example, Sections 10-4, 10-6 [A]).

For a subset S of Δ we shall denote by χ_S the characteristic function of S: $\chi_S = 1$ on S and $\chi_S(z) = 0$ for $z \in \Delta - S$.

2.2. In this subsection we shall introduce notations. We shall also review some facts which will be used later - the reference is [K1] (Chapters II, III), see also [E].

Let $\Sigma = \Gamma\backslash\Delta$ be a hyperbolic Riemann surface, and let $\pi : \Delta \to \Sigma$ be the covering map.

The pull-back of a holomorphic (resp., meromorphic) k-differential on Σ to Δ under π is a Γ-invariant holomorphic (resp., meromorphic) k-differential $\varphi(z)dz^k$ on Δ. The Γ-invariance of $\varphi(z)dz^k$ is equivalent to the automorphy law

$$\varphi(gz)J(g, z)^k = \varphi(z) \tag{2}$$

for all $g \in \Gamma, z \in \Delta$. And any Γ-invariant holomorphic (resp., meromorphic) k-differential on Δ defines a holomorphic (resp., meromorphic) k-differential on Σ. The space of holomorphic (resp., meromorphic) k-differentials on Σ is isomorphic, as a complex vector space, to the space of holomorphic (resp., meromorphic) Γ-invariant k-differentials on Δ.

Let $\Lambda \subset \Sigma$ be a closed set such that $\pi^{-1}(\Sigma - \Lambda)$ is connected. Denote $V = \Sigma - \Lambda$, $U = \pi^{-1}(V)$. Clearly $V = \Gamma\backslash U$. Using the uniformization theorem, we conclude that U is a hyperbolic Riemann surface, so $U = \Gamma_0\backslash\Delta$, where Γ_0 is a discrete subgroup of G. Denote by $\pi_0 : \Delta \to U$ the covering map.

The Poincaré metric on U corresponding to the hyperbolic metric $\frac{|dz|}{1-|z|^2}$ on Δ is given by $\frac{|d\xi|}{w(\xi)}$, where $w(\pi_0(z)) = (1 - |z|^2)|\frac{d}{dz}\pi_0(z)|$. The right-hand side depends only on $\pi_0(z)$. Also note: $w(\xi) > 0$. For $g \in \Gamma$ $w(g\xi) = w(\xi)|J(g, \xi)|$ and the area form $\frac{Re(\xi)\wedge Im(\xi)}{w(\xi)^2}$ is Γ-invariant.

Let $\mathcal{D}$ be a Dirichlet fundamental domain for the action of Γ on Δ. Denote $\mathcal{F} = \mathcal{D} \cap U$, this is a fundamental domain for the action of Γ on U.

Let $d\mu$ be the Lebesgue measure on Δ. For Σ, Λ as above we shall define the following spaces of k-differentials (all of them are Banach spaces, isomorphic to function spaces on U, as explained below).

Let $A^{(1)}(V)$ be the (normed) space of holomorphic k-differentials Φ on V such that

$$\int_{\Gamma\backslash U} |\varphi(z)| w(z)^{k-2} d\mu < \infty \tag{3}$$

where $z \in U$, $\varphi(z)dz^k = (\pi|_U)^*\Phi$, and the norm $||\Phi||_1$ is given by (3). The integral (3) can also be written as $\int_{\mathcal{F}} |\varphi(z)| w(z)^{k-2} d\mu$. Let $A^{(1)}_\Gamma(U)$ be the (normed) space of Γ-invariant holomorphic k-differentials $\varphi(z)dz^k$ on U that satisfy (3). $A^{(1)}(V)$ is isomorphic to $A^{(1)}_\Gamma(U)$ and is isomorphic to the space $\mathcal{A}^{(1)}_\Gamma(U)$ of holomorphic functions $\varphi(z)$ on U satisfying (2) (for $g \in \Gamma$, $z \in U$) and (3).

Let $A^{(2)}(V)$ be the space of holomorphic k-differentials Φ on V such that

$$\int_{\Gamma\backslash U} |\varphi(z)|^2 w(z)^{2k-2} d\mu < \infty \tag{4}$$

where $z \in U$, $\varphi(z)dz^k = (\pi|_U)^*\Phi$. The integral (4) can also be written as $\int_{\mathcal{F}} |\varphi(z)|^2 w(z)^{2k-2} d\mu$. $A^{(2)}(V)$ is a normed space, with the norm

$$||\Phi||_2 = (\int_{\Gamma\backslash U} |\varphi(z)|^2 w(z)^{2k-2} d\mu)^{1/2}.$$

Let $A^{(2)}_\Gamma(U)$ be the space of Γ-invariant holomorphic k-differentials $\varphi(z)dz^k$ on U that satisfy (4). $A^{(2)}(V)$ is isomorphic to $A^{(2)}_\Gamma(U)$ and is isomorphic to the space $\mathcal{A}^{(2)}_\Gamma(U)$ of holomorphic functions $\varphi(z)$ on U satisfying (2) (for $g \in \Gamma$, $z \in U$) and (4).

Let $B(V)$ be the (normed) space of holomorphic k-differentials Φ on V such that

$$\sup_{z\in\mathcal{F}} |\varphi(z)| w(z)^k < \infty \tag{5}$$

where $z \in U$, $\varphi(z)dz^k = (\pi|_U)^*\Phi$, and the norm $||\Phi||_\infty$ is given by (5). Let $B_\Gamma(U)$ be the space of Γ-invariant holomorphic k-differentials $\varphi(z)dz^k$ on U that satisfy (5). $B(V)$ is isomorphic to $B_\Gamma(U)$ and is isomorphic to the space $\mathcal{B}_\Gamma(U)$ of holomorphic functions $\varphi(z)$ on U satisfying (2) (for $g \in \Gamma$, $z \in U$) and (5).

The spaces $A^{(1)}(\Sigma)$, $A^{(2)}(\Sigma)$, $B(\Sigma)$ are defined by setting $\Lambda = \emptyset$ (i.e. $V = \Sigma$) in the definitions above.

The space $A^{(1)}(U)$ (resp. $A^{(2)}(U)$, $B(U)$) is defined as the normed space of holomorphic k-differentials $\varphi(z)dz^k$ on U such that

$$||\varphi(z)dz^k||_1 = \int_U |\varphi(z)|w(z)^{k-2}d\mu < \infty \tag{6}$$

(resp.

$$||\varphi(z)dz^k||_2 = (\int_U |\varphi(z)|^2 w(z)^{2k-2}d\mu)^{1/2} < \infty, \tag{7}$$

$$||\varphi(z)dz^k||_\infty = \sup_{z\in U} |\varphi(z)|w(z)^k < \infty). \tag{8}$$

The space $A^{(1)}(U)$ (resp. $A^{(2)}(U)$, $B(U)$) is isomorphic to the space $\mathcal{A}^{(1)}(U)$ (resp. $\mathcal{A}^{(2)}(U)$, $\mathcal{B}(U)$) of holomorphic functions $\varphi(z)$ on U satisfying (6) (resp. (7), (8)). The isomorphism is given by the map $\varphi(z) \mapsto \varphi(z)dz^k$.

The space $A^{(1)}(\Delta)$ is defined as the normed space of holomorphic k-differentials $\varphi(z)dz^k$ on Δ such that

$$||\varphi(z)dz^k||_1 = \int_\Delta |\varphi(z)|w(z)^{k-2}d\mu < \infty. \tag{9}$$

It is isomorphic (via $\varphi(z)dz^k \mapsto \varphi(z)$) to the space $\mathcal{A}^{(1)}(\Delta)$ of holomorphic functions $\varphi(z)$ on Δ satisfying (9).

The spaces $A^{(1)}(.)$, $A^{(2)}(.)$, $B(.)$, defined above, are Banach spaces. The Petersson inner product

$$\langle \Phi, \Psi \rangle = \int_{\Gamma\backslash U} \varphi(z)\overline{\psi(z)}w(z)^{2k-2}d\mu,$$

where $(\pi|_U)^*\Phi$ and $(\pi|_U)^*\Psi$ are $\varphi(z)dz^k$ and $\psi(z)dz^k$, $\Phi \in A^{(1)}(V)$, $\Psi \in B(V)$, establishes an antilinear isomorphism between $B(V)$ and $A^{(1)}(V)^*$.

The Petersson inner product

$$\langle \varphi(z)dz^k, \psi(z)dz^k \rangle = \int_U \varphi(z)\overline{\psi(z)}w(z)^{2k-2}d\mu,$$

$\varphi(z)dz^k \in A^{(1)}(U)$, $\psi(z)dz^k \in B(U)$ establishes an antilinear isomorphism between $B(U)$ and $A^{(1)}(U)^*$ (Theorem III.2.1 [K1]).

$A^{(2)}(V)$ is a Hilbert space, with the inner product

$$\langle \Phi, \Psi \rangle = \int_{\Gamma\backslash U} \varphi(z)\overline{\psi(z)}w(z)^{2k-2}d\mu \tag{10}$$

where $\Phi, \Psi \in A^{(2)}(V)$, $\varphi(z)dz^k = (\pi|_U)^*\Phi$, $\psi(z)dz^k = (\pi|_U)^*\Psi$.

$A^{(2)}(U)$ is a Hilbert space, with the inner product

$$\langle \varphi(z)dz^k, \psi(z)dz^k \rangle = \int_U \varphi(z)\overline{\psi(z)}w(z)^{2k-2}d\mu.$$

The space $A^{(2)}(U)$ admits a reproducing kernel $K : U \times U \to \mathbb{C}$. It has, in particular, the following properties (Theorem III.3.1 [K1]):

- $K(z,\xi)$ is holomorphic in z, antiholomorphic in ξ,
- $K(z,\xi) = \overline{K(\xi,z)}$,
- as a function of z (with fixed ξ) it belongs to $\mathcal{A}^{(1)}(U)$ and to $\mathcal{A}^{(2)}(U)$,
- for any automorphism τ of U and $z,\xi \in U$ $K(\tau(z),\tau(\xi))J(\tau,z)^k\overline{J(\tau,\xi)}^k = K(z,\xi)$,
- for any function f in $\mathcal{A}^{(1)}(U)$ or in $\mathcal{A}^{(2)}(U)$ or in $\mathcal{B}(U)$ we have:

  $$f(z) = \int_U K(z,\xi)f(\xi)w(\xi)^{2k-2}d\mu$$

  for any $z \in U$.

Also the operator β defined formally by

$$(\beta f)(z) = \int_U K(z,\xi)f(\xi)w(\xi)^{2k-2}d\mu$$

is a bounded linear projection from $L^1(U, w(z)^{k-2}d\mu)$ (resp. $L^2(U, w(z)^{2k-2}d\mu)$, $L^\infty(U, \sup_{z\in U}|.|w(z)^k)$ onto $\mathcal{A}^{(1)}(U)$ (resp. $\mathcal{A}^{(1)}(U)$, $\mathcal{B}(U)$) - see Theorem III.3.2 [K1].

We shall denote by $L^1_\Gamma(U, w(z)^{k-2}d\mu)$ the subspace of $L^1(U, w(z)^{k-2}d\mu)$ that consists of functions satisfying (2) for $g \in \Gamma$, $z \in U$.

The *Poincaré series* for a function $f : U \to \mathbb{C}$ is formally defined as $\sum_{g\in\Gamma} f(gz)J(g,z)^k$. We shall need the following statement.

Theorem 2.1. *Suppose k is a positive integer, $k \geq 2$.*

(i) For any $\varphi(z)dz^k \in A^{(1)}(U)$ the Poincaré series

$$\theta(\varphi)(z) = \sum_{g\in\Gamma} \varphi(gz)J(g,z)^k,$$

converges absolutely and uniformly on compact sets.

(ii) (Theorem III.3.3 [K1]*) The Poincaré series map*

$$\Theta : A^{(1)}(U) \to A^{(1)}_\Gamma(U)$$

$$\varphi(z)dz^k \mapsto \Theta(\varphi(z)dz^k) = \theta(\varphi)(z)dz^k,$$

is a surjective bounded linear operator with the norm $||\Theta|| \leq 1$.

The proof of (i) is in the Appendix, it is similar to the proof in IX.3.3.1. [S].

3. Main results

Let $\Theta_0 : A^{(1)}(\Delta) \to A^{(1)}(U)$ be the Poincaré series map corresponding to π_0.

We start by stating two observations.

Proposition 3.1. *If Λ is a finite set, then $A^{(2)}(V)$ is isomorphic to $A^{(2)}(\Sigma)$, $B(V)$ is isomorphic to $B(\Sigma)$, $A^{(1)}(V)$ is isomorphic to the space of integrable meromorphic k-differentials on Σ with at most simple poles, all in Λ.*

Proposition 3.2. *Let $\mathbb{P}$ be the set of polynomials in z. The set $\Theta_0(\{p(z)dz^k \mid p \in \mathbb{P}\})$ is dense in $A^{(1)}(U)$. The set $\Theta(\Theta_0(\{p(z)dz^k \mid p \in \mathbb{P}\}))$ is dense in $A^{(1)}(V)$. $A^{(1)}(U)$ and $A^{(1)}(V)$ are separable.*

Our main results are the following two theorems that provide information about the kernel of the Poincaré series map. Theorem 3.3 is analogous to the main theorem in [Lj].

Theorem 3.3. *The set*

$$W = \{\beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi)(z)dz^k \mid g, \gamma \in \Gamma, \phi(z) \in L^1_\Gamma(U, w^{k-2}d\mu)\},$$

is dense in $\ker\Theta$.

Theorem 3.4.

Suppose Γ is infinite. Let P be a subset of $\mathcal{F}$ that has a limit point in $\mathcal{F}$. Let $\mathcal{P}$ be the linear span of the set

$$\{(K(z,p) - \overline{J(g,p)}^k K(z,gp))dz^k \mid p \in P, g \in \Gamma\}.$$

Then $\mathcal{P} \subset \ker\Theta \cap A^{(2)}(U)$ and $\mathcal{P}$ is dense in $A^{(2)}(U)$.

Corollary 3.5. *If Γ is infinite then $\ker\Theta \cap A^{(2)}(U)$ is dense in $A^{(2)}(U)$.*

Remark 3.6. $\mathcal{P}$ is non-trivial. Indeed, suppose $\mathcal{P} = \{0\}$. Then for all $p \in P$, $g \in \Gamma$ we have:

$$0 = K(z,p) - \overline{J(g,p)}^k K(z,gp) = K(z,p) - K(g^{-1}z,p)J(g^{-1},z)^k.$$

Hence $K(z,p)$ as a function of z (with fixed p) belongs to $\mathcal{A}^{(1)}_\Gamma(U)$. But $K(z,p) \in \mathcal{A}^{(1)}(U)$ and $\mathcal{A}^{(1)}_\Gamma(U) \cap \mathcal{A}^{(1)}(U) = \{0\}$ unless Γ is finite (III.2 [K1]).

4. Proofs

4.1. **Proof of Proposition 3.1.** Let Φ be a holomorphic k-differential on V. Then $(\pi|_U)^*\Phi = \phi(z)dz^k$ is a holomorphic k-differential on U. Let z_0 be a point in $\pi^{-1}(\Lambda)$. $\phi(z)$ has an isolated singularity at z_0. If $\Phi \in B(V)$ then $\phi(z)$ is bounded in a small neighborhood of z_0, hence $\phi(z)$ has a removable singularity at z_0. If $\Phi \in A^{(2)}(V)$ then $\phi(z)$ has a removable singularity at z_0. If $\Phi \in A^{(1)}(V)$ then $\phi(z)$ has a removable singularity or a simple pole at z_0. $\square$.

4.2. **Proof of Proposition 3.2.** By Theorem 2.1 Θ and Θ_0 are continuous surjective maps.

For any $f \in \mathcal{A}^{(1)}(\Delta)$ and for any $\epsilon > 0$ there is a polynomial $p(z)$ with rational coefficients such that

$$\int_\Delta |f(z) - p(z)|(1 - |z|^2)^{k-2} d\mu < \epsilon$$

(the proof is analogous to the proof of Corollary 1 in 3.2 [GL], see also the proof of Proposition 1.3 [HKZ]). So $\mathcal{A}^{(1)}(\Delta)$ is separable and $\mathbb{P}$ is dense in $\mathcal{A}^{(1)}(\Delta)$. The statements follow. $\square$

4.3. **Proof of Theorem 3.3.** We shall follow the idea of the proof of the main theorem in [Lj].

Let l be a continuous linear functional on $\ker\Theta$. It will suffice to show that if $l(f) = 0$ for all $f \in W$ then $l(f) = 0$ for all $f \in \ker\Theta$.

By the Hahn-Banach theorem l can be extended to $\tilde{l} \in (A^{(1)}(U))^*$ such that $||\tilde{l}|| = ||l||$. By Theorem 3.2.1 [K1] there is $\psi \in \mathcal{B}(U)$ such that

$$\tilde{l}(f) = \int_U f(z)\overline{\psi(z)} w(z)^{2k-2} d\mu$$

for all $f \in \mathcal{A}^{(1)}(U)$.

Now let us prove two lemmas.

Lemma 4.1. *If $g, \gamma \in \Gamma$, $\phi \in L^1_\Gamma(U, w^{k-2}d\mu)$ then*

$$\beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi)(z)dz^k \in \ker\Theta.$$

Proof of Lemma 4.1. $\phi \in L^1_\Gamma(U, w^{k-2}d\mu)$, therefore $\beta(\chi_{g\mathcal{F}}\phi), \beta(\chi_{\gamma\mathcal{F}}\phi) \in \mathcal{A}^{(1)}(U)$, so $\beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi)(z)dz^k \in A^{(1)}(U)$. We get:

$$\beta(\chi_{g\mathcal{F}}\phi)(z) = \int_{g\mathcal{F}} \phi(\xi) K(z,\xi) w(\xi)^{2k-2} d\mu(\xi),$$

$$\beta(\chi_{\gamma\mathcal{F}}\phi)(z) = \int_{\gamma\mathcal{F}} \phi(\eta) K(z,\eta) w(\eta)^{2k-2} d\mu(\eta) =$$

$$\int_{g\mathcal{F}} \phi(\gamma g^{-1}\xi)K(z,\gamma g^{-1}\xi)w(\gamma g^{-1}\xi)^{2k-2}d\mu(\gamma g^{-1}\xi) =$$

$$\int_{g\mathcal{F}} \phi(\xi)J(\gamma g^{-1},\xi)^{-k}K(z,\gamma g^{-1}\xi)w(\xi)^{2k-2}|J(\gamma g^{-1},\xi)|^{2k}d\mu(\xi) =$$

$$\int_{g\mathcal{F}} \phi(\xi)K(z,\gamma g^{-1}\xi)w(\xi)^{2k-2}\overline{J(\gamma g^{-1},\xi)}^{k}d\mu(\xi),$$

so

$$\theta(\beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi))(z) =$$

$$\sum_{h\in\Gamma}\int_{g\mathcal{F}} \phi(\xi)(K(hz,\xi) - K(hz,\gamma g^{-1}\xi)\overline{J(\gamma g^{-1},\xi)}^{k})w(\xi)^{2k-2}d\mu(\xi)J(h,z)^k$$

which is zero because

$$\sum_{h\in\Gamma} K(hz,\gamma g^{-1}\xi)\overline{J(\gamma g^{-1},\xi)}^{k}J(h,z)^k =$$

$$\sum_{h\in\Gamma} K(g\gamma^{-1}hz,\xi)J(g\gamma^{-1},hz)^kJ(h,z)^k =$$

$$\sum_{h\in\Gamma} K(g\gamma^{-1}hz,\xi)J(g\gamma^{-1}h,z)^k = \sum_{\alpha\in\Gamma} K(\alpha z,\xi)J(\alpha,z)^k$$

where $\alpha = g\gamma^{-1}h$. $\square$.

Lemma 4.2. *If $\psi \in \mathcal{B}(U)$ and for all $\phi \in L^1_\Gamma(U, w^{k-2}d\mu)$*

$$\int_{\mathcal{F}} \phi(z)\overline{\psi(z)}w(z)^{2k-2}d\mu = 0$$

then $\psi(z)$ is identically zero.

Proof of Lemma 4.2. Define $\psi_1 \in L^\infty_\Gamma(U, \sup_{z\in\mathcal{F}}|.|w(z)^k)$ by setting $\psi_1 = \psi$ on $\mathcal{F}$ and $\psi_1(hz) = \psi_1(z)J(h,z)^{-k}$ for all $z \in \mathcal{F}$, $h \in \Gamma$. Then ψ_1 is zero (because $L^\infty_\Gamma(U, \sup_{z\in\mathcal{F}}|.|w(z)^k)$ is the dual of $L^1_\Gamma(U, w^{k-2}d\mu)$ [K1]). Therefore $\psi|_{\mathcal{F}} = 0$ and, since ψ is holomorphic, it must be identically zero. $\square$.

Lemma 4.1 shows that $W \subset \ker\Theta$.

Suppose that $\tilde{l}(f) = 0$ for all $f \in W$. We get: for all $g, \gamma \in \Gamma$, $\phi \in L^1_\Gamma(U, w^{k-2}d\mu)$

$$0 = \int_U \beta(\chi_{g\mathcal{F}}\phi - \chi_{\gamma\mathcal{F}}\phi)(z)\overline{\psi(z)}w(z)^{2k-2}d\mu =$$

$$\int_U\int_U (\chi_{g\mathcal{F}}\phi(\xi) - \chi_{\gamma\mathcal{F}}\phi(\xi))K(z,\xi)w(\xi)^{2k-2}d\mu(\xi)\overline{\psi(z)}w(z)^{2k-2}d\mu(z) =$$

$$\int_U \overline{\int_U \psi(z)K(\xi,z)w(z)^{2k-2}d\mu(z)}(\chi_{g\mathcal{F}}\phi(\xi) - \chi_{\gamma\mathcal{F}}\phi(\xi))w(\xi)^{2k-2}d\mu(\xi) =$$

$$\int_U (\chi_{g\mathcal{F}}\phi(\xi) - \chi_{\gamma\mathcal{F}}\phi(\xi))\overline{\psi(\xi)}w(\xi)^{2k-2}d\mu(\xi) =$$

$$\int_{g\mathcal{F}} \phi(\xi)\overline{\psi(\xi)}w(\xi)^{2k-2}d\mu(\xi) - \int_{\gamma\mathcal{F}} \phi(\eta)\overline{\psi(\eta)}w(\eta)^{2k-2}d\mu(\eta) =$$

$$\int_{\mathcal{F}} (\phi(gz)\overline{\psi(gz)}|J(g,z)|^{2k} - \phi(\gamma z)\overline{\psi(\gamma z)}|J(\gamma,z)|^{2k})w(z)^{2k-2}d\mu(z) =$$

$$\int_{\mathcal{F}} \phi(z)(\overline{\psi(gz)J(g,z)^k} - \overline{\psi(\gamma z)J(\gamma,z)^k})w(z)^{2k-2}d\mu(z).$$

Hence, by Lemma 4.2, with $\xi = gz$,

$$\psi(\xi) = \psi(\gamma g^{-1}\xi)J(\gamma g^{-1},\xi)^k$$

for all $\xi \in U$, $\gamma, g \in \Gamma$, thus $\psi \in \mathcal{B}_\Gamma(U)$. For any $\phi(z)dz^k \in A^{(1)}(U)$

$$\int_{\mathcal{F}} \theta\phi(z)\overline{\psi(z)}w(z)^{2k-2}d\mu = \int_{\mathcal{F}} \sum_{h\in\Gamma} \phi(hz)J(h,z)^k\overline{\psi(z)}w(z)^{2k-2}d\mu =$$

$$\sum_{h\in\Gamma} \int_{h\mathcal{F}} \phi(\xi)\overline{\psi(\xi)}w(\xi)^{2k-2}d\mu(\xi) = \int_U \phi(\xi)\overline{\psi(\xi)}w(\xi)^{2k-2}d\mu(\xi) = \tilde{l}(\phi).$$

Thus $\tilde{l} = 0$, so $l = 0$ on all of $\ker\Theta$, as required.
□

4.4. **Proof of Theorem 3.4.**

First, we note that $K(z,p) - \overline{J(g,p)}^k K(z,gp)$, as a function of z (with fixed $g \in G$, $p \in P$), is in $\mathcal{A}^{(1)}(U)$ and $\mathcal{A}^{(2)}(U)$.

The following calculation shows that it belongs to $\ker\Theta$:

$$\sum_{\gamma\in\Gamma}(K(\gamma z,p) - \overline{J(g,p)}^k K(\gamma z,gp))J(\gamma,z)^k = \sum_{\gamma\in\Gamma} K(\gamma z,p)J(\gamma,z)^k -$$

$$\sum_{\gamma\in\Gamma} \overline{J(g,p)}^k K(g^{-1}\gamma z,p)J(g^{-1},\gamma z)^k \overline{J(g^{-1},gp)}^k J(\gamma,z)^k =$$

$$\sum_{\gamma\in\Gamma} K(\gamma z,p)J(\gamma,z)^k - \sum_{\gamma\in\Gamma} K(g^{-1}\gamma z,p)J(g^{-1}\gamma,z)^k = 0.$$

Finally, let us assume that $f(z)dz^k \in A^{(2)}(U)$ and

$$\int_U f(z)\overline{(K(z,p) - \overline{J(g,p)}^k K(z,gp))}w(z)^{2k-2}d\mu = 0$$

for all $p \in P$, $g \in \Gamma$. We need to show that f is identically zero. We have:

$$f(p) = \int_U f(z)K(p,z)w(z)^{2k-2}d\mu =$$

$$\int_U f(z)J(g,p)^k K(gp,z))w(z)^{2k-2}d\mu(z) = f(gp)J(g,p)^k$$

for all $p \in P$, $g \in \Gamma$. Therefore the holomorphic function $f(p) - f(gp)J(g,p)^k$ is zero on P, hence it's identically zero, so $f \in \mathcal{A}_\Gamma^{(2)}(U) \cap \mathcal{A}^{(2)}(U)$ which is $\{0\}$ (III.2 [K1]).□

Appendix: **Proof of Theorem 2.1(i)**

Let $z_0 \in U$, let $r > 0$ be sufficiently small so that the closed disc $\bar{B}(z_0;r) = \{z \in \mathbb{C} | \ |z - z_0 \leq r\}$ is in $\mathcal{F}$. Let us show that the series $\theta(\varphi)$ converges absolutely at z_0. By the Mean Value Theorem for a function f holomorphic on an open set containing $\bar{B}(z_0;r)$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})dt,$$

so

$$\int_{\bar{B}(z_0;r)} f(z)dxdy = \int_0^r \int_0^{2\pi} f(z_0 + \rho e^{it})\rho dt d\rho =$$

$$2\pi \int_0^r f(z_0)\rho d\rho = f(z_0)\pi r^2,$$

hence $f(z_0) = \frac{1}{\pi r^2}\int_{\bar{B}(z_0;r)} f(z)dxdy$. Let $m = \min_{z \in \bar{B}(z_0;r)} w(z)^{k-2}$. Note: $m > 0$. We have: for $\gamma \in \Gamma$

$$|\varphi(\gamma z_0)J(\gamma, z_0)^k| = |\frac{1}{\pi r^2}\int_{\bar{B}(z_0;r)} \varphi(\gamma z)J(\gamma,z)^k dxdy| \leq$$

$$\frac{1}{\pi r^2}\int_{\bar{B}(z_0;r)} |\varphi(\gamma z)J(\gamma,z)^k| dxdy \leq$$

$$\frac{1}{m\pi r^2}\int_{\bar{B}(z_0;r)} |\varphi(\gamma z)J(\gamma,z)^k| w(z)^{k-2}dxdy.$$

We get:

$$\sum_{\gamma\in\Gamma} |\varphi(\gamma z_0)J(\gamma,z_0)^k| \leq \frac{1}{m\pi r^2}\sum_{\gamma\in\Gamma}\int_{\bar{B}(z_0;r)} |\varphi(\gamma z)J(\gamma,z)^k| w(z)^{k-2}dxdy =$$

$$\frac{1}{m\pi r^2}\sum_{\gamma\in\Gamma}\int_{\gamma(\bar{B}(z_0;r))} |\varphi(\eta)| w(\eta)^{k-2} dRe(\eta) dIm(\eta) \leq$$

$$\frac{1}{m\pi r^2}\int_U |\varphi(\eta)|w(\eta)^{k-2}dRe(\eta)dIm(\eta) < \infty.$$

Now let's choose a compact set $K \subset U$ and show that the series converges uniformly on K.

Let $r > 0$ be sufficiently small, so that the disc of radius r centered at any point of K is contained in a compact set K' and $K' \subset U$. Let q be the number of elements of Γ such that $\gamma K' \cap K' \neq \emptyset$. For $0 < \epsilon < 1$ denote by C_ϵ the closed disc centered at 0 of radius $1-\epsilon$. There are at most finitely many $\gamma \in \Gamma$ such that $\gamma K' \cap C_\epsilon \neq \emptyset$ (p. 219 [S]). Write Σ'_ϵ to denote the sum over all other $\gamma \in \Gamma$. As above, we obtain, for $z \in K$:

$$\Sigma'_\epsilon |\varphi(\gamma z)J(\gamma, z)^k| \leq \frac{q}{m_K \pi r^2}\int_{U-C_\epsilon} |\varphi(z)|w(z)^{k-2}dxdy,$$

where $m_K = \min_{\xi \in K'} w(\xi)^{k-2}$. As $\epsilon \to 1$ the right hand side goes to zero. This proves uniform convergence. □

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