

Unit 20

Linear Dependence and Independence

The idea of dimension is fairly intuitive. Consider any vector in \mathfrak{R}^m , $(a_1, a_2, a_3, \dots, a_m)$. Each of the m components is independent of the others. That is, choosing a value for a single component in no way affects the available choices of values for the other components, in order to have the choice end up being a vector in \mathfrak{R}^m . For this reason we say that \mathfrak{R}^m has m *degrees of freedom* or equivalently has *dimension* m . Thus, \mathfrak{R}^2 has dimension 2, \mathfrak{R}^3 has dimension 3, and so forth.

Consider now the “subspace” (soon to be defined) of \mathfrak{R}^3 consisting of all vectors (x, y, z) which have $x + y + z = 0$. We may now only choose values for 2 of the components freely and these will determine the value of the third component. For instance, if we choose $x = 1$ and $y = 2$, then we must have $z = -3$, in order to satisfy $x + y + z = 0$. Therefore this subspace is said to have dimension 2.

The rest of this course will be concerned with this idea of dimension and related concepts. In this unit, we lay the foundation with the concepts of linear dependence and independence.

Definition 20.1. A *linear combination* of a set of vectors, $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ in \mathfrak{R}^m is an expression of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$, where each c_i is a scalar.

Example 1. Let $\vec{v}_1 = (1, 0, 1)$, $\vec{v}_2 = (-1, 1, 0)$ and $\vec{v}_3 = (1, 2, 3)$. Express \vec{v}_3 as a linear combination of \vec{v}_1 and \vec{v}_2 .

Solution: We must find scalars c_1 and c_2 so that $\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$.

Using our knowledge of scalar multiplication and addition of vectors, from Unit 13, we set

$$\begin{aligned}(1, 2, 3) &= c_1(1, 0, 1) + c_2(-1, 1, 0) \\ \Rightarrow (1, 2, 3) &= (c_1, 0, c_1) + (-c_2, c_2, 0) \\ \Rightarrow (1, 2, 3) &= (c_1 - c_2, c_2, c_1)\end{aligned}$$

Equating corresponding components we see that what we have is a SLE:

$$\begin{aligned}1 &= c_1 - c_2 \\ 2 &= c_2 \\ 3 &= c_1\end{aligned}$$

Since $c_1 = 3$ and $c_2 = 2$ does satisfy $c_1 - c_2 = 1$, we see that this is the solution to the SLE. Thus, we see that $(1, 2, 3) = 3(1, 0, 1) + 2(-1, 1, 0)$, so we have found the required linear combination to be $\vec{v}_3 = 3\vec{v}_1 + 2\vec{v}_2$.

We now give two definitions for the same thing. Each definition is (of course) equivalent to the other, but sometimes the use of one definition is more suited to a given task than the other.

Definition 20.2. A set of vectors in \mathfrak{R}^m is said to be *linearly dependent* if there is at least one vector in the set which can be expressed as a linear combination of the others. If a set is not linearly dependent then it is said to be *linearly independent*.

Definition 20.3. A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in \mathfrak{R}^m is *linearly dependent* if the vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$ has a solution in which at least one $c_i \neq 0$.

Equivalently, the set is *linearly independent* if the above equation has only the trivial solution, namely $c_1 = c_2 = \dots = c_n = 0$.

From this second definition, we see that a set containing only one vector is linearly dependent if that one vector is the zero vector. That is, we define:

Definition 20.4. A set $S = \{\vec{v}\}$ with $\vec{v} \in \mathfrak{R}^m$ is linearly dependent if $\vec{v} = \vec{0}$ (and linearly independent otherwise).

Note: Linear dependence or independence is a property of a *set* of vectors, **not** of an individual vector. That is, it has to do with the relationship between the vectors in the set. (The definition above describes how this property is applied to a set which contains only one vector.)

Also note: Definition 20.2 says that if S is a set of linearly dependent vectors, then at least one of the vectors in S can be written as a linear combination of the others. This *does not* necessarily mean that *each* vector in the set can be written as a linear combination of the others. For instance, $S = \{(1, 0), (2, 0), (0, 1)\}$ is a set of linearly dependent vectors in \mathfrak{R}^2 , since $(2, 0) = 2(1, 0) + 0(0, 1)$. However, $(0, 1)$ cannot be written as a linear combination of $(1, 0)$ and $(2, 0)$.

Example 2. Determine whether $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, where $\vec{v}_1 = (1, 1, 2, 1)$, $\vec{v}_2 = (0, 2, 1, 1)$, and $\vec{v}_3 = (3, 1, 2, 0)$.

Solution: Using the second of the two definitions, we wish to know whether $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ has only the trivial solution.

$$\begin{aligned} \text{Suppose that} \quad & c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \\ \text{Then} \quad & c_1(1, 1, 2, 1) + c_2(0, 2, 1, 1) + c_3(3, 1, 2, 0) = (0, 0, 0, 0) \\ \Rightarrow \quad & (c_1, c_1, 2c_1, c_1) + (0, 2c_2, c_2, c_2) + (3c_3, c_3, 2c_3, 0) = (0, 0, 0, 0) \\ \Rightarrow \quad & (c_1 + 3c_3, c_1 + 2c_2 + c_3, 2c_1 + c_2 + 2c_3, c_1 + c_2) = (0, 0, 0, 0) \end{aligned}$$

Equating corresponding components, we have

$$\begin{aligned} c_1 & + 3c_3 = 0 \\ c_1 + 2c_2 + c_3 & = 0 \\ 2c_1 + c_2 + 2c_3 & = 0 \\ c_1 + c_2 & = 0 \end{aligned}$$

Forming the augmented matrix, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array} \right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

We see that the SLE has the unique solution $c_1 = c_2 = c_3 = 0$. Thus, the homogeneous SLE has only the trivial solution. Since there is no solution in which any of the c_i values is non-zero, we see that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Definition 20.5. An *elementary vector* in \mathfrak{R}^m is a vector in which all of the components are 0 except one, which is 1. If the 1 occurs in the i^{th} component we denote the vector as \vec{e}_i . (That is, we define \vec{e}_i to be the elementary m -vector which has a 1 in component i and 0's in all other components.) Any column of a matrix which is an elementary vector is referred to as an *elementary column*.

Example 3. Write the following vectors:
 (a) \vec{e}_2 in \mathfrak{R}^3 ; (b) \vec{e}_3 in \mathfrak{R}^3 ; (c) \vec{e}_2 in \mathfrak{R}^4 .

Solution:

- (a) $(0, 1, 0) = \vec{e}_2$ in \mathfrak{R}^3 (b) $(0, 0, 1) = \vec{e}_3$ in \mathfrak{R}^3
 (c) $(0, 1, 0, 0) = \vec{e}_2$ in \mathfrak{R}^4

In Example 2, we determined whether a particular set of vectors in \mathfrak{R}^3 was linearly independent. We did this by constructing a homogeneous SLE, and bringing its augmented matrix to RREF, to see whether the system had a unique solution. Notice that if a SLE involving n variables has a unique solution, then the RREF of the coefficient matrix consists of n different elementary columns. This approach gives us the following way to test any set of n vectors in \mathfrak{R}^m for linear independence.

Theorem 20.6. *Test for Linear Independence:*

Given a set $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of n vectors in \mathfrak{R}^m , construct the matrix A whose first column is \vec{v}_1 , second column is \vec{v}_2 , ..., n^{th} column is \vec{v}_n . Then S is linearly independent if and only if the row reduced echelon form of A consists entirely of distinct elementary columns.

Notice that in Example 2, the coefficient matrix of the homogeneous system we constructed had this form, i.e. the i^{th} column was the vector \vec{v}_i for $i = 1, 2, 3$, so the Test for Linear Independence is simply an abbreviated version of the approach used there.

Example 4. Is the set $\{(1, 2, 0, 1), (2, -1, 1, 0), (0, 5, -1, 2)\} = \{v_1, v_2, v_3\}$ linearly independent?

Solution: Using the Test for Linear Independence, we construct the matrix and then place it in RREF. We get the matrix by writing the vectors in the set downwards, as columns in the matrix.

We get:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 0 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that column 3, $(2, -1, 0, 0)$, is not an elementary column. Therefore the set is not linearly independent, and thus is linearly dependent.

Recall the first of our two definitions of linear dependence, Definition 20.2. This says that a set S of n vectors in \mathfrak{R}^m is linearly dependent if one of the vectors in S can be expressed as a linear combination of the other $n - 1$ vectors in the set. The following definitions assist us in referring to this dependence.

Definition 20.7. Consider a vector \vec{x} which can be expressed as a linear combination of some other vectors, $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$. Then we can write $\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_k\vec{u}_k$ for some scalars c_1, c_2, \dots, c_k . We say that the vector \vec{x} is *dependent on* the vectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$, and we refer to the scalars c_1, c_2, \dots, c_k as the *coefficients of dependence*.

When the Test for Linear Independence determines that a particular set S of vectors is not linearly independent, i.e., is linearly dependent, it does so

Solution: We construct the matrix as in the test for linear independence, and transform it to RREF:

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 1 \\ 2 & 1 & 1 & 3 & 4 \\ 3 & 2 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Notice that column 3 of the row reduced matrix, $(1, -1, 0)$, is not an elementary column. Notice also that in this matrix, column 3 is equal to one times column 1 plus (-1) times column 2, i.e., $(1, -1, 0) = 1(1, 0, 0) + (-1)(0, 1, 0)$. We see that the entries in the non-elementary column 3 are the coefficients of dependence with which this vector can be expressed as a linear combination of the vectors preceding it in the RREF matrix.

What the theorem is telling us is that the third column of the *original* matrix is the *same* linear combination of the vectors which were transformed into these elementary vectors. That is, just as it is true in the RREF matrix that column 3 is equal to 1 times the first column plus -1 times the second column (using columns of the RREF matrix), in the original matrix we also have $C3 = C1 - C2$.

We can, of course, verify this:

$$\vec{v}_1 - \vec{v}_2 = (1, 2, 3) - (1, 1, 2) = (1 - 1, 2 - 1, 3 - 2) = (0, 1, 1) = \vec{v}_3$$

Thus, we see that \vec{v}_3 is dependent on \vec{v}_1 and \vec{v}_2 , and the entries in column 3 of the RREF matrix give the coefficients of dependency. And since we can express \vec{v}_3 as a linear combination of \vec{v}_1 and \vec{v}_2 , we see that the set S is linearly dependent.

Look also at column 5 of the RREF matrix. This column is not an elementary column, so the vector \vec{v}_5 can be written as a linear combination of some of the other vectors. Which ones? We look for the vectors (i.e., columns) containing leading 1's in the RREF, because these are elementary columns. We see that row 1's leading 1 is in column 1, row 2's leading 1 is in column 2 and row 3's leading 1 is in column 4.

In the RREF matrix we have: $(2, -3, 1) = 2(1, 0, 0) - 3(0, 1, 0) + 1(0, 0, 1)$ so that column 5 = $2 \times$ (column 1) + $(-3) \times$ (column) 2 + $1 \times$ (column 4).

This tells us that this relationship also holds true in the original matrix. That is, we have $\vec{v}_5 = 2\vec{v}_1 - 3\vec{v}_2 + \vec{v}_4$. (Again, we can easily confirm that this is true.) Thus, we see that \vec{v}_5 is dependent on \vec{v}_1 , \vec{v}_2 and \vec{v}_4 , and that the entries in column 5 of the RREF matrix give us the coefficients of dependence, i.e., tell us how to write \vec{v}_5 as a linear combination of these vectors.

The following two theorems give us other ways to recognize linear dependence or independence in some cases.

Theorem 20.9. *If S is a set of n vectors from \mathbb{R}^m with $n > m$ then S must be linearly dependent.*

Proof: *Since $n > m$ then the matrix A formed with the vectors from S as columns has more columns than rows, so the RREF of A cannot have all columns being distinct elementary columns. Therefore, from the Test for Linear Independence, we see that S is not linearly independent, and hence is linearly dependent.*

Theorem 20.10. *Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_m\}$ be a set of m vectors in \mathbb{R}^m and let A be the square matrix $[\vec{v}_1 | \vec{v}_2 | \vec{v}_3 | \dots | \vec{v}_m]$. Then S is linearly independent if and only if $\det A \neq 0$.*

Proof: *We must prove this in both directions.*

First, suppose that S is a linearly independent set. Then the RREF of A contains m distinct elementary columns (by the Test for Linear Independence). But A is an $m \times m$ matrix, so this means that the RREF of A is I . Thus, $\det A \neq 0$ (by Theorem 19.5).

Now, suppose that $\det A \neq 0$. Then this tells us that the RREF of A must be I (again, by Theorem 19.5) and so the RREF of A contains m distinct elementary columns. But then the Test for Linear Independence tells us that this means that S is linearly independent.

Therefore S is linearly independent if and only if $\det A \neq 0$.

Example 6. Is $S = \{(1, 0, 1), (2, 1, -1), (5, -1, 0), (-2, 1, 1)\}$ linearly dependent or linearly independent?

Solution: We have a set of $n = 4$ vectors in \mathfrak{R}^3 (i.e., \mathfrak{R}^m with $m = 3$). Since $4 > 3$ (i.e., $n > m$), then S is linearly dependent, by Theorem 20.9.

Example 7. Is $S = \{(1, 1, 1), (1, 0, 1), (0, 2, 0)\}$ linearly dependent or linearly independent?

Solution: We form the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$.

Then expanding along column 3 gives $\det A = 2(-1)^{2+3} \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$.
Since $\det A = 0$, S is linearly dependent, by Theorem 20.10.