Unit 4
The Definite Integral

We know what an indefinite integral is: the general antiderivative of the integrand function. There is a related (although in some ways vastly different) concept, the definite integral, which uses similar-looking notation. The definite integral of a function is defined only for intervals on which the function is continuous. Let's briefly review what that means.

Recall the definition of continuity of a function at a point:

Definition 4.1. A function $f(x)$ is said to be *continuous* at a value c in the domain of f if

$$\lim_{x \to c} f(x) = f(c)$$

Note that this requires that:

1. $f(c)$ must be defined,
2. $\lim\limits_{x \to c} f(x)$ must exist,
   (i.e., the function approaches the same limiting value from both sides at c.)
3. and these two numbers must be equal.

Also recall how we extend this idea of continuity to intervals:

Definition 4.2. A function $f(x)$ is said to be *continuous on a closed interval* $[a, b]$ if f is continuous at c for every value $c \in [a, b]$.

In layman's terms, a function is continous on an interval if you can draw the function without having to lift your pencil off the page.

Discontinuities in a function occur at two kinds of places:

1. places where the function is not defined
   (for instance a place where the function has *denominator* $= 0$,)

2. places where the function suddenly jumps from one value to another.

Consider any function $f(x)$ which is continuous on some interval $[a, b]$. The function may be positive-valued in some parts of the interval and negative-valued in other parts of the interval. That is, the graph of $y = f(x)$ may lie above the x-axis in some places and below the x-axis in others, all within the interval $[a, b]$.

Definition 4.3. (Preliminaries for definition of definite integral)
Consider a function f which is continuous on some interval $[a, b]$.
Let R^+ be the region or regions which lie *below* the curve $y = f(x)$ and *above* the x-axis (i.e., regions in which the function is positive-valued) within the interval $[a, b]$.
Similarly, let R^- be the region or regions which lie *above* the curve $y = f(x)$ and *below* the x-axis (i.e., regions in which the function is negative-valued) within the interval $[a, b]$.
Finally, let $A(R^+)$ and $A(R^-)$ denote the *(total) areas* of these regions, respectively.

The definite integral of the function $f(x)$ from $x = a$ to $x = b$ is defined as the difference between these two areas, i.e., the *net area above the x-axis*, on the interval $[a, b]$. We have:

Definition 4.4. Let f be any function and let $[a, b]$ be any finite closed interval such that f is continuous on $[a, b]$. Let $A(R^+)$ and $A(R^-)$ be defined as above.
The symbol $\int_a^b f(x)dx$, called the definite integral of $f(x)$ from a to b, is defined as:

$$\int_a^b f(x)dx = A(R^+) - A(R^-)$$

Notice: Unlike an indefinite integral, which is a function, a definite integral is a *number*.

For some (only a few) functions, the regions R^+ and R^- have shapes which allow their areas to be calculated easily (e.g. regions whose shapes are rectangles, triangles, trapezoids, semi-circles, etc.). However, for the vast majority of functions, finding the areas of these regions is not so straightforward. (In a more rigorous study of calculus, such an area would be found by evaluating the limit at infinity of the sum of a number of approximating rectangles.)

Fortunately, there is a result which allows us to evaluate definite integrals using an entirely different approach, provided we know (or can find) an antiderivative of the integrand function.

Theorem 4.5. The Fundamental Theorem of Calculus
Let $f(x)$ be continuous on the interval [a,b], and let $F(x)$ be any antiderivative of $f(x)$ on [a,b]. Then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Definition 4.6. The numbers a and b, above, are called the *limits of integration.*

Thus we see that to evaluate $\int_a^b f(x)dx$ we simply need to:

1. find any antiderivative, F, of f,
2. evaluate $F(x)$ at $x = a$ and at $x = b$, and
3. calculate the difference $F(a) - F(b)$.

Notice: In doing this, we never actually calculate either $A(R^+)$ or $A(R^-)$.

Definition 4.7. We use $F(x)|_a^b$ or $[F(x)]_a^b$ to denote $F(b)-F(a)$, where $F(x)$ may be shown either by name or in functional form. (This would commonly be pronounced as "$F(x)$ evaluated from a to b".)

For instance,

$$\begin{aligned} \left[x^3 + x^2 - 3x\right]_1^2 &= [x^3 + x^2 - 3x]_{x=2} - [x^3 + x^2 - 3x]_{x=1} \\ &= (2^3 + 2^2 - 3(2)) - (1^3 + 1^2 - 3(1)) \end{aligned}$$

Example 1. Find $\int_1^2 x^3 dx$.

Solution: Notice that the integrand function is continuous on $[1, 2]$.

$\int_1^2 x^3 dx \quad = [\frac{x^4}{4}]_1^2$ (because $\frac{x^4}{4}$ is an antiderivative of x^3)

$= \frac{2^4}{4} - \frac{1^4}{4}$

$= 4 - \frac{1}{4} = \frac{15}{4}$

Corollary 4.8. *If the value of the function $f(x)$ is non-negative everywhere between $x = a$ and $x = b$, then the area between the graph of $f(x)$ and the x-axis is given by $\int_a^b f(x)dx$.*
(This is called the area under *$y = f(x)$ from $x = a$ to $x = b$.)*

This follows directly from our definition of $\int_a^b f(x)dx$ and the fact that since $f(x)$ is non-negative on $[a, b]$ then the graph of $y = f(x)$ lies (on or) above the x-axis throughout $[a, b]$, so that $A(R^-) = 0$.

For instance, for $f(x) = x^3$, we have $f(x) \geq 0$ on $[1, 2]$, so the area under the curve $y = x^3$ from $x = 1$ to $x = 2$ is given by $\int_1^2 x^3 dx = \frac{15}{4}$, from Example 1.

Let's look at more examples of evaluating definite integrals.

Example 2. Evaluate $\int_{-1}^2 (x^2 - x)dx$.

Solution: Since $\frac{x^3}{3} - \frac{x^2}{2}$ is an antiderivative of $x^2 - x$, we have:

$$\begin{aligned}
\int_{-1}^2 (x^2 - x)dx &= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^2 \\
&= \left(\frac{2^3}{3} - \frac{2^2}{2}\right) - \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2}\right) \\
&= \left(\frac{8}{3} - 2\right) - \left(-\frac{1}{3} - \frac{1}{2}\right) \\
&= \left(\frac{2}{3}\right) - \left(-\frac{2}{6} - \frac{3}{6}\right) \\
&= \frac{4}{6} - \left(-\frac{5}{6}\right) = \frac{9}{6} = \frac{3}{2}
\end{aligned}$$

Example 3. Evaluate $\int_1^2 \left(e^x - \frac{1}{x}\right) dx$.

Solution: (Notice that the integrand function, although not defined at $x = 0$, is continuous throughout $[1, 2]$.)
For $f(x) = e^x - \frac{1}{x}$, we see that $F(x) = e^x - \ln|x|$ is an antiderivative. So we get:

$$\begin{aligned}\int_1^2 \left(e^x - \frac{1}{x}\right) dx &= \left[e^x - \ln|x|\right]_1^2 \\ &= \left(e^2 - \ln 2\right) - \left(e^1 - \ln 1\right) \\ &= e^2 - e - \ln 2\end{aligned}$$

We now list some important properties of definite integrals. These should be studied carefully.

Theorem 4.9. *Let f be any function and $[a, b]$ be any interval such that $f(x)$ is continuous on $[a, b]$. Then:*

(1) $\boxed{\int_a^b cf(x)dx = c\int_a^b f(x)dx \text{ for any constant } c.}$

(2) $\boxed{\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx}$

(3) $\boxed{\int_a^b f(x)dx = \int_a^t f(x)dx + \int_t^b f(x)dx \text{ for any value of } t}$

Result (3) makes intuitive sense if t is some value between a and b. For instance, using $a = 1$, $b = 5$ and $t = 2$, we have

$$\int_1^5 f(x)dx = \int_1^2 f(x)dx + \int_2^5 f(x)dx$$

which says that the net area under $y = f(x)$ from $x = 1$ to $x = 5$ is equal to the net area under $y = f(x)$ from $x = 1$ to $x = 2$ plus the net area under $y = f(x)$ from $x = 2$ to $x = 5$. That is, if we divide our interval $[a, b]$ into sub-intervals, $[a, t]$ and $[t, b]$, with $a < t < b$, then the net area over the whole interval is equal to the sum of the net areas on the 2 sub-intervals.

We can extend this to include $t = a$ or $t = b$ using the following:

Definition 4.10. For any function $f(x)$ which is continuous at some value a we define:

$$\int_a^a f(x)dx = 0$$

This definition makes sense in terms of areas. $\int_a^a f(x)dx$ is the net area under $y = f(x)$ from $x = a$ to $x = a$. But there's no area there − the region is just a line segment, so the net area is 0.

We also define:

Definition 4.11. If $f(x)$ is continuous on $[a, b]$ then

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Notice: $\int_b^a f(x)dx$, where $b > a$, doesn't make much sense intuitively because it says "the net area under $y = f(x)$ from $x = b$ to $x = a$" where $b > a$, i.e., we're considering the interval *backwards*. However, using this definition, we see that property (3) from Theorem 4.9:

$$\int_a^b f(x)dx = \int_a^t f(x)dx + \int_t^b f(x)dx$$

makes sense even if t is outside of the interval $[a, b]$.

For instance, if $a < b < t$, we have

$$\int_a^b f(x)dx = \int_a^t f(x)dx + \int_t^b f(x)dx = \int_a^t f(x)dx - \int_b^t f(x)dx$$

which says that the net area under $y = f(x)$ from $x = a$ to $x = b$ can be found as the net area under $y = f(x)$ from $x = a$ to some larger value $x = t$, minus the net area under $y = f(x)$ from $x = b$ to $x = t$, i.e., the part we didn't want.

Notice that

$$\int_a^b f(x)dx = \int_a^t f(x)dx - \int_b^t f(x)dx$$

can be rearranged to

$$\int_a^b f(x)dx + \int_b^t f(x)dx = \int_a^t f(x)dx$$

Also notice: Since for $b > a$ we have

$$\int_b^a f(x)dx = -\int_a^b f(x)dx = -[F(b)-F(a)] = -F(b)-(-F(a)) = -F(b)+F(a)$$

then we can simply use

$$\int_b^a f(x)dx = F(a) - F(b)$$

which feels like we're using Theorem 4.5, even though that theorem only directly applies when $a > b$.

Example 4. If the net area above the axis and below $y = f(x)$ on the interval $[1, 10]$ is 23 and $\int_2^{10} f(x)dx = 15$, find $\int_1^2 f(x)dx$.

Solution: We have $\int_1^{10} f(x)dx = 23$ and $\int_2^{10} f(x)dx = 15$, and we can express $\int_1^2 f(x)dx$ as

$$\int_1^2 f(x)dx = \int_1^{10} f(x)dx + \int_{10}^2 f(x)dx = \int_1^{10} f(x)dx - \int_2^{10} f(x)dx$$

so we see that $\int_1^2 f(x)dx = 23 - 15 = 8$.

Property (3) also allows us to extend the definition of $\int_a^b f(x)dx$ for some functions which are *not* continuous on $[a, b]$, provided that $f(x)$ is actually defined everywhere on $[a, b]$. Suppose $f(x)$ is defined by different (continuous) functions in different parts of its domain, and in particular that the function has different definitions on different parts of the interval $[a, b]$. Then as long as $f(x)$ is defined everywhere on $[a, b]$, we can evaluate $\int_a^b f(x)dx$ by breaking up the interval $[a, b]$ into subintervals such that the function has only one definition on each such subinterval. That is, we evaluate the definite integral from a to b by evaluating the sum of 2 or more definite integrals. This process will be most easily understood by looking at an example.

Example 5. Find $\int_{-2}^1 f(x)dx$, where $f(x) = \begin{cases} x+2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$

Solution: Notice that $f(x)$ is not continuous on $[-2, 1]$ because we have $\lim_{x\to 0^-} f(x) = 0 + 2 = 2$, but $\lim_{x\to 0^+} f(x) = 0^2 = 0$ so that $\lim_{x\to 0}$ does not exist and there is a discontinuity at $x = 0$. However, $f(x)$ *is* defined everywhere in $[-2, 1]$. Using property (3), we split $[-2, 1]$ up into $[-2, 0]$ and $[0, 1]$ to get:

$$\int_{-2}^1 f(x)dx = \int_{-2}^0 f(x)dx + \int_0^1 f(x)dx = \int_{-2}^0 (x+2)dx + \int_0^1 x^2 dx$$

Notice that, at this point, both integrals involve integrand functions which are continuous on the relevant intervals, so we have no difficulty evaluating these integrals. We get:

$$\begin{aligned}\int_{-2}^{1} f(x)dx &= \left[\frac{x^2}{2} + 2x\right]_{-2}^{0} + \left.\frac{x^3}{3}\right|_0^1 \\ &= 0 - (2-4) + \frac{1}{3} - 0 = \frac{7}{3}\end{aligned}$$

Another use of the definite integral is to give us the average value of a function (f_{ave}) over a specific interval.

We are familiar with the concept of the average value of two numbers. The average of a and b is given by $\frac{b+a}{2}$. It probably makes sense to you that the average value of x on the interval $[a, b]$ would also be given by $\frac{b+a}{2}$. For instance, the average value of x from $x = 0$ to $x = 2$ is $\frac{2+0}{2} = 1$.

We can also think of this as the average value of the function $f(x) = x$ on the interval $[0, 2]$, or in general on an interval $[a, b]$. We would like to extend this concept of average value to more complicated functions than $f(x) = x$.

Notice that $\frac{b+a}{2} = \frac{b+a}{2} \times \frac{b-a}{b-a} = \frac{(b+a)(b-a)}{2(b-a)} = \left(\frac{1}{b-a}\right)\left(\frac{b^2-a^2}{2}\right)$, and also that for $f(x) = x$ we have $\int_a^b f(x)dx = \int_a^b xdx = \left.\frac{x^2}{2}\right|_a^b = \left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b^2-a^2}{2}$.
We see that for the function $f(x) = x$, the average value of $f(x)$ on the interval $[a, b]$ can be thought of as being given by:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

In this configuration we can see how this concept could be extended to more complicated functions. This is precisely how we define the average value of any function f on an interval $[a, b]$.

Definition 4.12. For any function f which is continuous on $[a, b]$, we define the average value of $f(x)$ on the interval [a,b] to be given by:

$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx}$$

Example 6. Find the average value of $f(x) = e^x + 3x^2$ on the interval [0,2].

Solution: From the definition, with $a = 0$ and $b = 2$, we have:

$$\begin{aligned} f_{ave} &= \tfrac{1}{b-a}\int_a^b f(x)dx = \tfrac{1}{2-0}\int_0^2 (e^x + 3x^2)dx \\ &= \tfrac{1}{2}\left[e^x + x^3\right]_0^2 = \tfrac{1}{2}[(e^2 + 2^3) - (e^0 + 0^3)] \\ &= \tfrac{1}{2}(e^2 + 8 - 1) = \tfrac{e^2+7}{2} \end{aligned}$$

Suppose we have a function which is defined as a definite integral in which one of the limits of integration is a variable instead of a number. Consider the function $G(x)$ defined by $G(x) = \int_a^x f(t)dt$ for some value a. In order to find function values, we would need to actually perform the integration. For instance, the function value $G(b)$ is given by $G(b) = \int_a^b f(t)dt$, so we would evaluate $\int_a^b f(t)dt$. Alternatively, we could find a general formula (i.e. functional expression) for the function $G(x)$ by finding an antiderivative of $f(t)$, which we can call $F(t)$. Then $G(x) = \int_a^x f(t)dt = F(x) - F(a)$. We could then find $G(b)$ using this formula.

If, on the other hand, we are *only* interested in the *derivative* of $G(x)$, we don't have to do any work at all. Since we have $G(x) = F(x) - F(a)$ then $G'(x) = \frac{d}{dx}\left[F(x) - F(a)\right] = F'(x) - 0 = F'(x)$. But since F is an antiderivative of f, then $F'(x) = f(x)$ so we have $G'(x) = f(x)$ and we see that, since we already know the function f, we didn't need to integrate at all. Instead, we just write the integrand function using x in place of t. That is, we see that:

Theorem 4.13. *If $G(x) = \int_a^x f(t)dt$ then $G'(x) = f(x)$.*

Thus we can find values of the derivative function simply by evaluating the integrand function at the corresponding value.

Example 7. Find $G'(1)$, where $G(x) = \int_3^x \frac{t^2+4t-3}{(t+1)(t-2)(t+3)}dt$.

Solution:

Approach 1 (not recommended)

1. Find the partial fraction decomposition of $\frac{t^2+4t-3}{(t+1)(t-2)(t+3)}$.

2. Use the partial fraction decomposition to find an antiderivative, $F(t)$, of $f(t) = \frac{t^2+4t-3}{(t+1)(t-2)(t+3)}$.

3. Use this antiderivative function $F(t)$ to evaluate the definite integral $\int_3^x \frac{t^2+4t-3}{(t+1)(t-2)(t+3)} = F(x) - F(3)$.

4. Differentiate $G(x) = F(x) - F(3)$ to get $G'(x)$.

5. Evaluate $G'(x)$ at $x = 1$.

Approach 2 (much preferred)
Since $G(x) = \int_3^x \frac{t^2+4t-3}{(t+1)(t-2)(t+3)} dt$ then $G'(x) = \frac{x^2+4x-3}{(x+1)(x-2)(x+3)}$ so that $G'(1) = \frac{1^2+4(1)-3}{(1+1)(1-2)(1+3)} = \frac{1+4-3}{2(-1)(4)} = \frac{2}{-8} = -\frac{1}{4}$.

Remember: The processes of integration and differentiation undo one another (except that any constant term is lost in differentiation, and an arbitrary constant is introduced in integration). That is:
$\frac{d}{dt}\left(\int f(t)dt\right) = f(t)$ and $\int \left[\frac{d}{dt}(F(t))\right] dt = F(t) + C$.

Substitution and Definite Integration

We have now learned the basics of definite integrals. Next, we look at how to apply the methods we know for solving more complex integrals (i.e., substitution and integration by parts) to definite integrals.

The formal statement of the substitution rule with definite integrals may look a bit cumbersome, but after applying it a few times it becomes quite easy. Suppose we have an integral which requires substitution. Then the integrand has the form $f(g(x))g'(x)dx$ where $g(x)$ is the candidate we select for u. In the original definite integral, we are integrating with respect to x, between say a and b. After substituting $u = g(x)$, our new integrand will be in terms of the new variable u and we will be integrating with respect to u. This means that we must change the limits of integration (a and b), which are the values that the variable x ranges through in the original integral, to new values which give the range of the new variable u. That is, instead of integrating $f'(g(x))g'(x)$ from $x = a$ to $x = b$, we need to integrate the simpler integrand $f(u)$ from $u = g(a)$ to $u = g(b)$. We get:

$$\boxed{\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \left(= \int_{u(a)}^{u(b)} f(u)du\right)}$$

Example 8. Evaluate $\int_1^2 \frac{3x^2+2x+1}{x^3+x^2+x-2}dx$.

Solution: We proceed as usual with the substitution, the only variation being that we must find the "new" limits of integration (which will depend on our choice of u). After a little thought, we see that we want to substitute:

$$\boxed{u = x^3 + x^2 + x - 2} \Rightarrow \boxed{du = (3x^2 + 2x + 1)dx}$$

Before we carry out the substitution, we must find the new limits of integration. With this choice of u we see that

$$\begin{aligned} & x = 1 \Rightarrow u = 1^3 + 1^2 + 1 - 2 = 1 \\ \text{and} \quad & x = 2 \Rightarrow u = 2^3 + 2^2 + 2 - 2 = 12 \end{aligned}$$

i.e. $u(1) = 1$ and $u(2) = 12$, so our new integral is:

$$\begin{aligned} \int_1^2 \frac{3x^2+2x+1}{x^3+x^2+x-2}dx &= \int_1^{12} \frac{1}{u}du \\ &= \ln|u|\,\Big|_1^{12} = \ln 12 - \ln 1 = \ln 12 \end{aligned}$$

Example 9. Evaluate $\int_0^1 (2x - x^2)e^{(x^3-3x^2)}dx$.

Solution: After recognizing the form, we choose $u = x^3 - 3x^2$.
Then we have $\frac{du}{dx} = 3x^2 - 6x = 3(x^2 - 2x) = -3(2x - x^2)$ and so we have $-\frac{1}{3}du = (2x - x^2)dx$.
When $x = 0$ we have $u = 0^3 - 3(0^2) = 0$. Similarly, when $x = 1$ we have $u = 1^3 - 3(1^2) = -2$. Carrying out the substitution, we get:

$$\begin{aligned} \int_0^1 (2x - x^2)e^{(x^3-3x^2)}dx &= \int_0^{-2} \left(-\frac{1}{3}\right) e^u du = -\frac{1}{3}\int_0^{-2} e^u du \\ &= \frac{1}{3}\int_{-2}^{0} e^u du = \frac{1}{3}\left[e^u\right]_{-2}^{0} \\ &= \frac{1}{3}\left(e^0 - e^{-2}\right) = \frac{1}{3}\left(1 - \frac{1}{e^2}\right) = \frac{e^2-1}{3e^2} \end{aligned}$$

Notice: Be careful with the new limits. As we saw here, we can sometimes have $g(b) < g(a)$ even though $b > a$. Be sure you set up the new integral going **from** $g(a)$ **to** $g(b)$, even if $g(a) > g(b)$.

Integration By Parts and The Definite Integral

The integration by parts formula generalizes in the natural way to use with definite integrals. Formally stated, the formula is:

$$\boxed{\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du}$$

That is, we simply evaluate both parts of the right hand side of the integration by parts formula from a to b. Notice that since integration by parts does not actually change the variable of integration (that is, if the original integral was in terms of x, then after using the integration by parts formula the expression is still in terms of x) it is not necessary to change the limits of integration as we need to do when using substitution. A few of examples will illustrate the use of this formula.

Example 10. Evaluate $\int_1^2 x \ln x dx$

Solution: As always, we attempt to use substitution first, but we find that it will not work here (verify this for yourself). We turn to integration by parts. Letting $u = \ln x$ and $dv = xdx$, we get:

$$u = \ln x \Rightarrow du = \frac{1}{x}dx \qquad \text{and} \qquad dv = xdx \Rightarrow v = \frac{x^2}{2}$$

Substituting into the formula $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$, we get:

$$\begin{aligned} \int_1^2 x \ln x dx &= \left[\frac{x^2}{2}(\ln x)\right]_1^2 - \int_1^2 \frac{x^2}{2}\left(\frac{1}{x}\right)dx \\ &= \left[\frac{x^2 \ln x}{2}\right]_1^2 - \int_1^2 \frac{x}{2}dx \\ &= \left[\frac{x^2 \ln x}{2}\right]_1^2 - \left[\frac{x^2}{4}\right]_1^2 \\ &= \left(\frac{4\ln 2}{2} - \frac{1\ln 1}{2}\right) - \left(\frac{4}{4} - \frac{1}{4}\right) \\ &= 2\ln 2 - 0 - \frac{3}{4} \approx 0.6363 \end{aligned}$$

Example 11. Evaluate $\int_0^3 xe^{3x}dx$.

Solution: As always, we attempt to use substitution first but find that it will not work here (verify this for yourself). We turn to integration by parts.

Letting $u = x$ and $dv = e^{3x}dx$, we get:

$$u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx \qquad \text{and } dv = e^{3x}dx \Rightarrow v = \frac{e^{3x}}{3}$$

Substituting into the formula $\int_a^b u dv = [uv]_a^b - \int_a^b v du$ we get:

$$\begin{aligned}
\int_0^3 xe^{3x}dx &= \left[x\left(\frac{e^{3x}}{3}\right)\right]_0^3 - \int_0^3 \frac{e^{3x}}{3}dx \\
&= \left[\frac{xe^{3x}}{3}\right]_0^3 - \frac{1}{3}\int_0^3 e^{3x}dx \\
&= \left[\frac{xe^{3x}}{3}\right]_0^3 - \frac{1}{3}\left[\frac{e^{3x}}{3}\right]_0^3 \\
&= \left(\frac{3e^{3(3)}}{3} - \frac{0e^{3(0)}}{3}\right) - \frac{1}{3}\left(\frac{e^{3(3)}}{3} - \frac{e^{3(0)}}{3}\right) \\
&= (e^9 - 0) - \left(\frac{e^9}{9} - \frac{1}{9}\right) \\
&= \frac{9e^9}{9} - \frac{e^9 - 1}{9} \\
&= \frac{8e^9 + 1}{9}
\end{aligned}$$

Example 12. Find the area under the curve $y = x\ln(x^2)$ between $x = 1$ and $x = 3$.

Solution: Since the curve $y = x\ln(x^2)$ lies (on or) above the x-axis throughout the interval $[1, 3]$, i.e. since $x\ln(x^2) \geq 0$ on $1 \leq x \leq 3$, then by Corollary 4.8, the area under the curve is given by $area = \int_1^3 x\ln(x^2)\,dx$.

Approach 1: It looks like a substitution may work here. If we let $z = x^2$, we get $xdx = \frac{1}{2}dz$. (*Note:* we can use any variable name other than x when we perform a susbtitution. The reason for not using u in this case will become evident momentarily.) Also, when $x = 1$, $z = 1^2 = 1$ and when $x = 3$, $z = 3^2 = 9$, so we get:

$$\int_1^3 x\ln\left(x^2\right)dx = \frac{1}{2}\int_1^9 \ln z dz$$

This, however, is still not an integral we recognize (except that we have done it before). We need to use integration by parts. Letting $u = \ln z$ and $dv = dz$, so that $du = \frac{1}{z}dz$ and $v = z$, we get:

$$\begin{aligned}
\frac{1}{2}\int_1^9 \ln z dz &= \frac{1}{2}\left\{[z\ln z]_1^9 - \int_1^9 z\frac{1}{z}dz\right\} \\
&= \left.\frac{z\ln z}{2}\right|_1^9 - \frac{1}{2}\int_1^9 dz \\
&= \left(\frac{9\ln 9}{2} - \frac{1\ln 1}{2}\right) - \left.\frac{z}{2}\right|_1^9 \\
&= \frac{9}{2}\ln 9 - 0 - \left(\frac{9}{2} - \frac{1}{2}\right) = \frac{9}{2}\ln 9 - 4
\end{aligned}$$

Approach 2: Since we ended up using integration by parts anyway, we could just use that approach right from the start, without the substitution. We have $\int_1^3 x\ln(x^2)\,dx$. We can work with this as is, or re-express $\ln(x^2)$ as $2\ln x$ (using $\ln a^r = r\ln a$) to get $\int_1^3 x\ln(x^2)\,dx = \int_1^3 2x\ln x dx$. (*Note:* it doesn't matter which or these we use – the second is slightly easier in the details, so let's use that one.)
Letting $u = \ln x$ and $dv = 2xdx$, we get $du = \frac{1}{x}dx$ and $v = x^2$, so we have:

$$\begin{aligned}
\int_1^3 2x\ln x dx &= \left. x^2\ln x\right|_1^3 - \int_1^3 x^2\left(\frac{1}{x}\right)dx = \left(3^2\ln 3 - 1^2\ln 1\right) - \int_1^3 xdx \\
&= 9\ln 3 - 0 - \left.\frac{x^2}{2}\right|_1^3 = 9\ln 3 - \left(\frac{9}{2} - \frac{1}{2}\right) = 9\ln 3 - 4
\end{aligned}$$

Notice: $\frac{9}{2}\ln 9 = \frac{9}{2}\ln 3^2 = \frac{9}{2}(2\ln 3) = 9\ln 3$, so the two answers are in fact the same. That is, either way, we see that the area under the curve $y = x\ln(x^2)$ from $x = 1$ to $x = 3$ is $9\ln 3 - 4$.