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On solutions of linear equations with polynomial coefficients

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Abstract. We show that a linear functional equation with polynomial coefficients need not admit an arc-analytic solution even if it admits a continuous semialgebraic one. We also show that such an equation need not admit a Nash regulous solution even if it admits an arc-analytic one.

1. Introduction. The present note is concerned with existence of solutions to linear equations with polynomial coefficients in various classes of semialgebraic functions in $\mathbb{R}^n$. Recall that a set X in $\mathbb{R}^n$ is called *semialgebraic* if it can be written as a finite union of sets of the form $\{x \in \mathbb{R}^n : p(x) = 0, q_1(x) > 0, \dots, q_r(x) > 0\}$, where $r \in \mathbb{N}$ and $p, q_1, \dots, q_r$ are polynomial functions. Given $X \subset \mathbb{R}^n$, a *semialgebraic function* $f : X \to \mathbb{R}$ is one whose graph is a semialgebraic subset of $\mathbb{R}^{n+1}$.

A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *regulous* if there exist polynomial functions p and q such that the zero locus of q is nowhere dense in $\mathbb{R}^n$ and $f(x) = p(x)/q(x)$ whenever $q(x) \neq 0$. A real analytic semialgebraic function on $\mathbb{R}^n$ is called *Nash*. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *Nash regulous* if there exist Nash functions g and h such that the zero locus of h is nowhere dense in $\mathbb{R}^n$ and $f(x) = g(x)/h(x)$ whenever $h(x) \neq 0$. Finally, recall that a function $f : X \to \mathbb{R}$ is called *arc-analytic* if it is analytic along every arc, that is, $f \circ \gamma$ is analytic for every real analytic $\gamma : (-1, 1) \to X$. We shall denote the regulous, Nash regulous, and arc-analytic semialgebraic functions on $\mathbb{R}^n$ by $\mathcal{R}^0(\mathbb{R}^n)$, $\mathcal{N}^0(\mathbb{R}^n)$, and $\mathscr{A}_a(\mathbb{R}^n)$, respectively. We have

$$\mathcal{R}^0(\mathbb{R}^n) \subset \mathcal{N}^0(\mathbb{R}^n) \subset \mathscr{A}_a(\mathbb{R}^n). \tag{1.1}$$

The first inclusion is trivial and the second one follows from [8, Prop. 3.1]. Both inclusions are strict.

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The above classes of semialgebraic functions have been extensively studied recently (see, e.g., [1, 2, 6, 8] and the references therein), in particular, in the context of the following problem of Fefferman and Kollár [5].

Consider a linear equation

$$f_1\varphi_1 + \cdots + f_r\varphi_r = g, \tag{1.2}$$

where g and the f_j are continuous (real-valued) functions on $\mathbb{R}^n$. Fefferman–Kollár asked whether assuming that g and the f_j have some regularity properties, one could find a solution $(\varphi_1, \dots, \varphi_r)$ to (1.2) with similar regularity properties.

This is a difficult problem, even when the coefficients of (1.2) are polynomial. One line of attack is to instead consider a somewhat easier question:

Problem 1.1. *Suppose that* (1.2) *admits a solution* $(\varphi_1, \dots, \varphi_r)$ *within some class of functions. Does there exist then a solution to* (1.2) *within a strictly smaller class?*

In the semialgebraic setting, the most general positive answer to this problem is given by [5, Cor. 29(1)]: *If* $f_1, \dots, f_r$ *are polynomial,* g *is semialgebraic and* (1.2) *admits a continuous solution, then it admits a continuous semialgebraic solution.* In a similar vein, Kucharz and Kurdyka showed that, in case $n = 2$, if $f_1, \dots, f_r, g$ are regulous then (1.2) admits a continuous solution if and only if it admits a regulous solution (cf. [9, Cor. 1.7]).

On the other hand, the above is known to fail for $n \geq 3$. Namely, by [7, Ex. 6], there exist $f_1, f_2, g \in \mathbb{R}[x, y, z]$ such that $f_1\varphi_1 + f_2\varphi_2 = g$ admits a continuous solution, but no regulous one. Nonetheless, the solution from [7, Ex. 6] is Nash regulous, and in [8] Kucharz conjectured that existence of a continuous solution to (1.2) should imply the existence of a Nash regulous one, for any $n \geq 1$, provided $f_1, \dots, f_r, g$ are polynomial.

The main goal of this note is to prove that the latter is not the case. In Example 3.1, we show that there exists a linear equation with polynomial coefficients which admits a continuous solution, but no arc-analytic one. By (1.1), it follows that there is no Nash regulous solution either. Perhaps even more interestingly, in Example 3.2 we exhibit a linear equation with polynomial coefficients that *does* admit an arc-analytic solution and has no Nash regulous solution nonetheless. Both our examples are modifications of [7, Ex. 6].

2. Toolbox. The following facts will be needed in Examples 3.1 and 3.2.

Proposition 2.1. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a semialgebraic function. Then* f *is arc-analytic if and only if there exists a mapping* $\pi : \widetilde{R} \to \mathbb{R}^n$ *which is a finite sequence of blowings-up with smooth algebraic centers, such that the composite* $f \circ \pi$ *is a Nash function.*

Proof. This is a special case of [3, Thm. 1.4]. ∎

Functions satisfying the conclusion of Proposition 2.1 are called *blow-Nash*.

REMARK 2.2. A function $f : \mathbb{R} \to \mathbb{R}$ is arc-analytic if and only if it is real analytic. This follows directly from the definition of arc-analytic functions.

Recall that a Nash set (i.e., the zero set of a Nash function) in $\mathbb{R}^n$ is said to be *Nash irreducible* if it cannot be realized as a union of two proper Nash subsets. A set is called *Nash constructible* if it belongs to the Boolean algebra generated by the Nash subsets in $\mathbb{R}^n$.

REMARK 2.3 (cf. [10, Ex. 2.3]). The graph Γ_f of $f(x,y) = \sqrt{x^4+y^4}$ is not Nash constructible in $\mathbb{R}^3$.

Indeed, let $X := \{(x,y,z) \in \mathbb{R}^3 : z^2 = x^4+y^4\}$. We claim that X is Nash irreducible. First, note that $z^2 - x^4 - y^4$ is an irreducible element in the ring of convergent power series over $\mathbb{C}$. Hence, the set $\{z^2 - x^4 - y^4 = 0\} \subset \mathbb{C}^3$ has an irreducible (complex analytic) germ at the origin, of (complex) dimension 2. On the other hand, the (real analytic) germ of X at the origin is of (real) dimension 2. Hence, its complexification has to be given by precisely $\{z^2 - x^4 - y^4 = 0\}$. It follows that the germ X_0 is irreducible, and there is thus no way to decompose X into proper analytic subsets. (See [4] for details on real analytic germs and their complexifications.)

The irreducibility of X implies that X is the smallest Nash set in $\mathbb{R}^3$ containing Γ_f. Therefore, by [8, Prop. 2.1], if Γ_f were Nash constructible then it would need to contain the smooth locus of X. This is not the case, however, because X also contains the graph of $g(x,y) = -\sqrt{x^4+y^4}$.

The following result is new, though it follows easily from [8].

LEMMA 2.4. *Let $n \geq 1$ and let $f, g \in \mathscr{A}_a(\mathbb{R}^n)$. If the zero locus of g is nowhere dense in $\mathbb{R}^n$ and the function f/g extends continuously to $\mathbb{R}^n$, then this extension is in $\mathscr{A}_a(\mathbb{R}^n)$.*

Proof. By Proposition 2.1 above, there is a finite sequence $\pi : \widetilde{R} \to \mathbb{R}^n$ of blowings-up with smooth algebraic centers such that $f \circ \pi$ and $g \circ \pi$ are Nash functions on the Nash manifold $\widetilde{R}$. Continuity of f/g implies that $(f \circ \pi)/(g \circ \pi) : \widetilde{R} \to \mathbb{R}$ is a Nash regulous function. By [8, Prop. 3.1], Nash regulous functions are arc-analytic, and hence there is a finite sequence $\sigma : \widehat{R} \to \widetilde{R}$ of blowings-up with smooth algebraic centers such that

$$(f/g) \circ \pi \circ \sigma = \frac{f \circ \pi}{g \circ \pi} \circ \sigma : \widehat{R} \to \mathbb{R}$$

is Nash, by Proposition 2.1 again. Therefore, f/g is arc-analytic. ∎

3. Examples

EXAMPLE 3.1. Consider the equation

$$x^3y\varphi_1 + (x^3 - y^3z)\varphi_2 = x^4. \tag{3.1}$$

We claim that

$$\varphi_1(x,y,z) = z^{1/3}, \qquad \varphi_2(x,y,z) = \frac{x^3}{x^2 + xyz^{1/3} + y^2z^{2/3}}$$

is a continuous solution to (3.1), but no semialgebraic arc-analytic solution exists. The function φ_1 is clearly continuous. To see that φ_2 is continuous, first note that the set

$$\{(x,y,z) \in \mathbb{R}^3 : x^2 + xyz^{1/3} + y^2z^{2/3} = 0\}$$

is the union of the y-axis and the z-axis. Therefore, $x \to 0$ whenever (x,y,z) approaches the locus of indeterminacy of φ_2. On the other hand, we have

$$x^2 + xyz^{1/3} + y^2z^{2/3} \geq \tfrac{1}{2}(x^2 + y^2z^{2/3}),$$

which shows that

$$\frac{x^2}{x^2 + xyz^{1/3} + y^2z^{2/3}}$$

is bounded. Hence, φ_2 can be continuously extended by zero to $\mathbb{R}^3$.

Suppose now that (3.1) has an arc-analytic solution (ψ_1, ψ_2). Set

$$S \coloneqq \{(x,y,z) \in \mathbb{R}^3 : x^3 = y^3z\},$$

and note that y vanishes on S only when x does so. Therefore, x/y is a well defined function on $S \setminus \{x=0\}$, and thus, by (3.1), we obtain

$$\psi_1|_{S\setminus\{x=0\}} = \left.\frac{x}{y}\right|_{S\setminus\{x=0\}}.$$

Observe that every point $(0,0,c)$ of the z-axis can be approached within $S \setminus \{x = 0\}$, even by an analytic arc—indeed, for instance, by the arc $(\sqrt[3]{c}t, t, c)$ for $c \neq 0$ and the arc (t^2, t, t^3) for $c = 0$. This allows us to write

$$\lim_{(x,y,z)\to(0,0,c)} \psi_1(x,y,z) = \lim_{(x,y,z)\to(0,0,c)} \left.\frac{x}{y}\right|_{S\setminus\{x=0\}} = c^{1/3}.$$

Therefore, $\psi_1|_{z\text{-axis}} = z^{1/3}$, by continuity. This contradicts the arc-analyticity of ψ_1, by Remark 2.2. ∎

EXAMPLE 3.2. Consider now the equation

$$x^4y^2\varphi_1 + (x^4 - y^4(z^4 + w^4))\varphi_2 = x^6. \tag{3.2}$$

We claim that

$$\varphi_1 = \sqrt{z^4 + w^4}, \qquad \varphi_2 = \frac{x^4}{x^2 + y^2\sqrt{z^4 + w^4}}$$

is an arc-analytic solution to (3.2), but no Nash regulous solution exists. It is easy to see that the function $\sqrt{z^4+w^4}$ is blow-Nash, and hence arc-analytic, by Proposition 2.1. Thus, by Lemma 2.4, to see that φ_2 is arc-analytic, it suffices to show that it extends continuously to $\mathbb{R}^4$. First, note that the set

$$\{(x,y,z,w)\in\mathbb{R}^4 : x^2+y^2\sqrt{z^4+w^4}=0\}$$

is the union of the y-axis and the (z,w)-plane. Therefore, $x\to 0$ whenever (x,y,z,w) approaches the locus of indeterminacy of φ_2. On the other hand, the function

$$\frac{x^2}{x^2+y^2\sqrt{z^4+w^4}}$$

is clearly bounded. Hence, φ_2 can be continuously extended by zero to $\mathbb{R}^4$.

Suppose now that (3.2) has a Nash regulous solution (ψ_1,ψ_2). Set

$$S\coloneqq\{(x,y,z,w)\in\mathbb{R}^4 : x^4=y^4(z^4+w^4)\},$$

and note that y vanishes on S only when x does so. Therefore, $(x/y)^2$ is a well defined function on $S\setminus\{x=0\}$, and thus, by (3.2), we obtain

$$\psi_1|_{S\setminus\{x=0\}}=\left.\frac{x^2}{y^2}\right|_{S\setminus\{x=0\}}.$$

Note that the (z,w)-plane is contained in S, and every point $(0,0,c,d)$ of the (z,w)-plane can be approached within $S\setminus\{x=0\}$, even by an analytic arc. Indeed, for instance, by the arc $(\sqrt[4]{c^4+d^4}\,t,t,c,d)$ for $c^4+d^4\neq 0$ and the arc $(\sqrt[4]{2}\,t^2,t,t,t)$ for $c^4+d^4=0$. This allows us to write

$$\lim_{(x,y,z,w)\to(0,0,c,d)}\psi_1(x,y,z,w)=\lim_{(x,y,z,w)\to(0,0,c,d)}\left.\frac{x^2}{y^2}\right|_{S\setminus\{x=0\}}=\sqrt{c^4+d^4}.$$

Therefore, $\psi_1|_{(z,w)\text{-plane}}=\sqrt{z^4+w^4}$, by continuity. This is impossible for a Nash regulous function though, because by [8, Cor. 3.2] the graph of a Nash regulous function (and hence its intersection with any coordinate plane) is a closed Nash constructible set. However, the graph of $f(z,w)=\sqrt{z^4+w^4}$ is not Nash constructible, by Remark 2.3. □

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