A MEYNIEL-TYPE CONDITION FOR BIPANCYCLICITY IN BALANCED BIPARTITE DIGRAPHS

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Abstract. We prove that a strongly connected balanced bipartite digraph D of order $2a$, $a \geq 3$, satisfying $d(u) + d(v) \geq 3a$ for every pair of vertices u, v with a common in-neighbour or a common out-neighbour, is either bipancyclic or a directed cycle of length $2a$.

1. Introduction

Recently, there has been a renewed interest in various Meyniel-type conditions for hamiltonicity in bipartite digraphs (see, e.g., [1, 2, 6, 8]). In particular, in [1], we proved the following bipartite variant of a conjecture of Bang-Jensen et al. [5]. (For details on terminology and notation, see Section 2.)

Theorem 1.1 (cf. [1, Thm. 1])**.** *Let D be a strongly connected balanced bipartite digraph with partite sets of cardinalities a, where $a \geq 3$. If*

$$d(u) + d(v) \geq 3a$$

for every pair of vertices $u, v \in V(D)$ with a common in-neighbour or a common out-neighbour, then D is hamiltonian.

In [6], the authors suggested that, modulo some exceptional digraphs, the hypotheses of Theorem 1.1 should, in fact, imply bipancyclicity of D. In the present note we prove that this is indeed the case.

First, it will be useful to introduce the following shorthand notation from [1].

Definition 1.2. Let D be a balanced bipartite digraph with partite sets of cardinalities a. We will say that D satisfies *condition* $(\mathcal{A})$ when

$$d(u) + d(v) \geq 3a$$

for every pair of vertices u, v with a common in-neighbour or a common out-neighbour.

Theorem 1.3. *Let D be a strongly connected balanced bipartite digraph with partite sets of cardinalities a, where $a \geq 3$. If D satisfies condition $(\mathcal{A})$, then D is either bipancyclic or a directed cycle of length $2a$.*

Remark 1.4. The bound in Theorem 1.3 is sharp, since there exist strongly connected balanced bipartite digraphs satisfying $d(u) + d(v) \geq 3a - 1$ for every pair of vertices u, v with a common in-neighbour or a common out-neighbour, that

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nonetheless do no contain a hamiltonian cycle (see, e.g., [2, Ex. 1.12]). On the other hand, it is natural to ask if for every $1 \leq l < a$ there is a $k \geq 1$ such that every strongly connected balanced bipartite digraph on $2a$ vertices contains cycles of all even lengths up to $2l$, provided $d(u) + d(v) \geq 3a - k$ for every pair of vertices u, v as above. We don't know the answer to this question.

2. Notation and terminology

We consider digraphs in the sense of [4]: A *digraph* D is a pair $(V(D), A(D))$, where $V(D)$ is a finite set (of *vertices*) and $A(D)$ is a set of ordered pairs of distinct elements of $V(D)$, called *arcs* (i.e., D has no loops or multiple arcs).

The number of vertices $|V(D)|$ is the *order* of D (also denoted by $|D|$). For vertices u and v from $V(D)$, we write $uv \in A(D)$ to say that $A(D)$ contains the ordered pair (u, v). If $uv \in A(D)$, then u is called an *in-neighbour* of v, and v is an *out-neighbour* of u.

For a vertex set $S \subset V(D)$, we denote by $N^+(S)$ the set of vertices in $V(D)$ *dominated* by the vertices of S; i.e.,

$$N^+(S) = \{u \in V(D) : vu \in A(D) \text{ for some } v \in S\}\,.$$

Similarly, $N^-(S)$ denotes the set of vertices of $V(D)$ *dominating* vertices of S; i.e,

$$N^-(S) = \{u \in V(D) : uv \in A(D) \text{ for some } v \in S\}\,.$$

If $S = \{v\}$ is a single vertex, the cardinality of $N^+(\{v\})$ (resp. $N^-(\{v\})$), denoted by $d^+(v)$ (resp. $d^-(v)$) is called the *outdegree* (resp. *indegree*) of v in D. The *degree* of v is $d(v) \coloneqq d^+(v) + d^-(v)$.

More generally, for a vertex $v \in V(D)$ and a subdigraph E of D, we will denote the cardinality of $N^+(\{v\}) \cap V(E)$ by $d^+_E(v)$. Similarly, the cardinality of $N^-(\{v\}) \cap V(E)$ will be denoted by $d^-_E(v)$. We set $d_E(v) \coloneqq d^+_E(v) + d^-_E(v)$.

A directed cycle on vertices $v_1, \dots, v_m$ in D is denoted by $[v_1, \dots, v_m]$. We will refer to it as simply a *cycle* (skipping the term "directed"), since its non-directed counterpart is not considered in this article at all. A cycle passing through all the vertices of D is called *hamiltonian*. A digraph containing a hamiltonian cycle is called a *hamiltonian digraph*. A digraph containing cycles of all lengths is called *pancyclic*.

A digraph D is *strongly connected* when, for every pair of vertices $u, v \in V(D)$, D contains a path originating in u and terminating in v and a path originating in v and terminating in u. A digraph D in which, for every pair of vertices $u, v \in V(D)$ precisely one of the arcs uv, vu belongs to $A(D)$ is called a *tournament*.

A digraph D is *bipartite* when $V(D)$ is a disjoint union of independent sets V_1 and V_2 (the *partite sets*). It is called *balanced* if $|V_1| = |V_2|$. One says that a bipartite digraph D is *complete* when $d(x) = 2|V_2|$ for all $x \in V_1$. A complete bipartite digraph with partite sets of cardinalitites a and b will be denoted by $K^*_{a,b}$. A balanced bipartite digraph containing cycles of all even lengths is called *bipancyclic*.

3. Lemmas

The proof of Theorem 1.3 will be based on the four lemmas below and the following well-known theorem of Thomassen.

Theorem 3.1 ([7, Thm. 3.5])**.** *Let G be a strongly connected digraph of order n, $n \geq 3$, such that $d(u) + d(v) \geq 2n$ whenever u and v are non-adjacent. Then, G is either pancyclic, or a tournament, or n is even and G is isomorphic to $K^*_{\frac{n}{2},\frac{n}{2}}$.*

Throughout this section we assume that D is a strongly connected balanced bipartite digraph with partite sets of cardinalities $a \geq 3$, which satisfies condition $(\mathcal{A})$. Further, assume that C is a cycle of length $2a$ in D, and

$$d^+(u) \leq a - 1 \quad \text{and} \quad d^-(u) \leq a - 1 \quad \text{for every } u \in V(D)\,. \tag{3.1}$$

Lemma 3.2. *Suppose that D is not a cycle of length $2a$. Then, for every vertex $u \in V(D)$ there exists a vertex $v \in V(D) \setminus \{u\}$ such that u and v have a common in-neighbour or a common out-neighbour.*

Proof. For a proof by contradiction, suppose that D contains a vertex u_0 which has no common in-neighbour or out-neighbour with any other vertex in D. Let u_0^+ denote the successor of u_0 on C. Then, $d^-(u_0^+) = 1$, for else u_0^+ would be a common out-neighbour of u_0 and some other vertex. Similarly, $d^+(u_0^+) \leq a - 1$, for else u_0^+ would dominate both u_0^{++} and u_0 (where u_0^{++} denotes the successor of u_0^+ on C; note that $a \geq 3$ implies $u_0^{++} \neq u_0$). Consequently, $d(u_0^+) \leq a$, and hence any vertex v which would have a common in-neighbour or out-neighbour with u_0^+ would need to have $d(v) \geq 2a$, by condition $(\mathcal{A})$. Such a vertex v, however, would violate our assumption (3.1). It thus follows that u_0^+ has no common in-neighbour or out-neighbour with any other vertex in D.

By repeating the above argument, one can now show that u_0^{++}, the successor of u_0^+ on C has no common in-neighbour or out-neighbour with any vertex in $V(D)$, and, inductively, that no vertex of D has a common in-neighbour or out-neighbour with any other vertex. The latter implies that $D = C$ is a cycle of length $2a$, contrary to the hypothesis of the lemma. □

Lemma 3.3. *Suppose that D is not a cycle of length $2a$. Then, for every two vertices $u, v \in V(D)$ from the same partite set of D, u and v have a common in-neighbour or a common out-neighbour.*

Proof. Observe first that, by (3.1), every vertex w of D satisfies $d(w) \leq 2a - 2$. Therefore, by Lemma 3.2 and condition $(\mathcal{A})$, every vertex $u \in V(D)$ satisfies

$$d(u) \geq 3a - (2a - 2) = a + 2\,. \tag{3.2}$$

It follows that, for any two vertices $u, v \in V(D)$, one has

$$2a + 4 \leq d(u) + d(v) = (d^-(u) + d^-(v)) + (d^+(u) + d^+(v))\,,$$

and hence $d^-(u) + d^-(v) > a$ or $d^+(u) + d^+(v) > a$. If now u and v belong to the same partite set of D, then the first of these inequalities implies that u and v have a common in-neighbour in D, while the second one implies that they have a common out-neighbour, as required. □

Lemma 3.4. *Suppose that D is not a cycle of length $2a$. Then, every vertex of D lies on a 2-cycle (i.e., for every $u \in V(D)$ there exists a vertex $v \in V(D) \setminus \{u\}$ such that $uv \in A(D)$ and $vu \in A(D)$).*

Proof. By (3.2), for every $u \in V(D)$, we have $d^+(u) + d^-(u) > a$, and hence $N^+(\{u\}) \cap N^-(\{u\}) \neq \varnothing$. □

From now on, we are going to denote the two partite sets of D by X and Y, with elements $\{x_1, \dots, x_a\}$ and $\{y_1, \dots, y_a\}$ respectively, ordered so that C is the cycle $[y_1, x_1, \dots, y_a, x_a]$.

We will associate with D two new digraphs, G_1 and G_2, constructed as follows. Set $V(G_1) \coloneqq \{v_1, \dots, v_a\}$, and $v_i v_j \in A(G_1)$ whenever $x_i y_j \in A(D)$, for $i, j \in \{1, \dots, a\}$, $i \neq j$. Similarly, set $V(G_2) \coloneqq \{w_1, \dots, w_a\}$, and $w_i w_j \in A(G_2)$ whenever $y_i x_j \in A(D)$, for $i, j \in \{1, \dots, a\}$, $i \neq j$. Note that $a \geq 3$, so G_1 and G_2 have at least three vertices each. Moreover, for every $1 \leq i \leq a$, we have

$$
\begin{aligned}
&d^+_{G_1}(v_i) \geq d^+_D(x_i) - 1, \quad d^-_{G_1}(v_i) \geq d^-_D(y_i) - 1, \quad \text{and} \\
&d^+_{G_2}(w_i) \geq d^+_D(y_i) - 1, \quad d^-_{G_2}(w_i) \geq d^-_D(x_i) - 1\,.
\end{aligned}
\tag{3.3}
$$

Lemma 3.5. *Suppose that D is not a cycle of length $2a$. Then, for any two vertices v_i, v_j in G_1 and for any two vertices w_i, w_j in G_2, we have $d_{G_1}(v_i) + d_{G_1}(v_j) \geq 2a$ and $d_{G_2}(w_i) + d_{G_2}(w_j) \geq 2a$.*

Proof. Pick any v_i and v_j from $V(G_1)$, and consider the corresponding vertices x_i, y_i and x_j, y_j of D. By Lemma 3.3 and condition $(\mathcal{A})$, we have $d_D(x_i) + d_D(x_j) \geq 3a$ and $d_D(y_i) + d_D(y_j) \geq 3a$. It follows that

$$6a \leq (d_D(x_i) + d_D(x_j)) + (d_D(y_i) + d_D(y_j))\,,$$

and hence

$$(d^+_D(x_i) + d^-_D(y_i)) + (d^+_D(x_j) + d^-_D(y_j)) \geq 6a - (d^-_D(x_i) + d^+_D(y_i) + d^-_D(x_j) + d^+_D(y_j))\,.$$

By (3.3), the left hand side in the above inequality is less than or equal to $d_{G_1}(v_i) + d_{G_1}(v_j) + 4$, and thus, by (3.1), we get

$$d_{G_1}(v_i) + d_{G_1}(v_j) \geq 6a - 4(a-1) - 4 = 2a\,,$$

as required. The proof for G_2 is analogous. □

4. Proof of the main result

Proof of Theorem 1.3. Let D be a strongly connected balanced bipartite digraph with partite sets X and Y of cardinalities a, where $a \geq 3$. Suppose that D satisfies condition $(\mathcal{A})$. Then, by Theorem 1.1, D contains a cycle C of length $2a$. Suppose that D itself is not a cycle of length $2a$.

As in Section 3, we will denote the vertices of X and Y by $\{x_1, \dots, x_a\}$ and $\{y_1, \dots, y_a\}$ respectively, and assume that C is the cycle $[y_1, x_1, \dots, y_a, x_a]$.

Suppose first that condition (3.1) is not satisfied in D. This means that there exists a vertex on the hamiltonian cycle C which either dominates or is dominated by all the vertices of D from the opposite partite set. Clearly, in this case D contains cycles of all even lengths.

From now on we shall assume that D satisfies condition (3.1).

Let G_1 and G_2 be the digraphs associated with D, costructed in Section 3; i.e., $V(G_1) \coloneqq \{v_1, \dots, v_a\}$, with $v_i v_j \in A(G_1)$ whenever $x_i y_j \in A(D)$, and $V(G_2) \coloneqq \{w_1, \dots, w_a\}$, with $w_i w_j \in A(G_2)$ whenever $y_i x_j \in A(D)$, for $i, j \in \{1, \dots, a\}$, $i \neq j$. Then, G_1 is strongly connected because it contains a hamiltonian cycle $[v_1, \dots, v_a]$ (induced from C). By Lemma 3.5, it follows that G_1 satisfies the hypotheses of Theorem 3.1.

Notice that every cycle $[v_{i_1},\dots,v_{i_l}]$ of length l in G_1 corresponds to a cycle of length $2l$ in D, namely $[y_{i_1},x_{i_1},\dots,y_{i_l},x_{i_l}]$. Also, by Lemma 3.4, D contains a cycle of length 2. In light of Theorem 3.1, to complete the proof it thus suffices to consider the cases when G_1 is a tournament, or a is even and G_1 is isomorphic to $K^*_{\frac{a}{2},\frac{a}{2}}$.

First, suppose that G_1 is a tournament. Then, G_1 contains no cycle of length 2, and hence

$$d_{G_1}(v)=d^+_{G_1}(v)+d^-_{G_1}(v)\le a-1\,,\quad \text{for every } v\in V(G_1)\,.$$

It follows that, for any two vertices $v_i,v_j\in V(G_1)$, we have $d_{G_1}(v_i)+d_{G_2}(v_j)\le 2a-2$, which contradicts Lemma 3.5.

Suppose then that a is even and G_1 is isomorphic to $K^*_{\frac{a}{2},\frac{a}{2}}$. Since G_1 contains a hamiltonian cycle $[v_1,\dots,v_a]$, the two partite sets must be precisely $\{v_1,v_3,\dots,v_{a-1}\}$ and $\{v_2,v_4,\dots,v_a\}$. Moreover, we have $d^+_{G_1}(v_i)=\frac{a}{2}$ and $d^-_{G_1}(v_i)=\frac{a}{2}$, for every v_i in G_1. Hence, by (3.3),

$$d^+_D(x_i)\le\frac{a}{2}+1\quad\text{and}\quad d^-_D(y_i)\le\frac{a}{2}+1,\quad\text{for all } 1\le i\le a.$$

Lemma 3.3 and condition $(\mathcal{A})$ then imply that, for any $i\ne j$,

$$\begin{aligned}&6a\le(d_D(x_i)+d_D(x_j))+(d_D(y_i)+d_D(y_j))=\\&(d^+_D(x_i)+d^-_D(y_i)+d^+_D(x_j)+d^-_D(y_j))+(d^-_D(x_i)+d^+_D(y_i)+d^-_D(x_j)+d^+_D(y_j))\le\\&\qquad 4(\frac{a}{2}+1)+(d^-_D(x_i)+d^+_D(y_i)+d^-_D(x_j)+d^+_D(y_j))\,,\end{aligned}$$

hence

$$d^-_D(x_i)+d^+_D(y_i)+d^-_D(x_j)+d^+_D(y_j)\ge 4(a-1)\,. \tag{4.1}$$

If the above inequality is strict for at least one pair of indices $\{i,j\}$, then at least one of the vertices x_i,y_i,x_j,y_j violates condition (3.1); a contradiction.

Suppose then that, for all $i\ne j$, we have equality in (4.1). Then we must also have equalities in all the inequalities that led to it. In particular, for every $i\in\{1,\dots,a\}$, we have

$$d^+_D(x_i)=\frac{a}{2}+1,\quad d^-_D(x_i)=a-1,\quad d^-_D(y_i)=\frac{a}{2}+1,\quad d^+_D(y_i)=a-1. \tag{4.2}$$

Now, if there exists i_0 such that $x^+_{i_0}x_{i_0}\notin A(D)$ (where $x^+_{i_0}$ denotes the successor of x_{i_0} on C), then x_{i_0} is dominated by all other vertices from Y, by (4.2). In this case, D clearly contains cycles of all even lengths greater than 3, and so D is bipancyclic, by Lemma 3.4.

We may thus suppose that $x^+_i x_i\in A(D)$ for all $1\le i\le a$. Since G_1 is bipartite and $d^+_D(x_i)=\frac{a}{2}+1$, it follows that $x_iy_i\in A(D)$ for all $1\le i\le a$, and so D contains a hamiltonian cycle $C'=[x_a,y_a,x_{a-1},y_{a-1},\dots,x_1,y_1]$. Consequently, G_2 is strongly connected as it contains the cycle $[w_a,w_{a-1},\dots,w_1]$ induced by C'. Repeating the preceding part of the proof for G_2 in place of G_1, we obtain that D is bipancyclic unless G_2 is bipartite. In the latter case, we have $d^+_{G_2}(w_i)\le\frac{a}{2}$ and $d^-_{G_2}(w_i)\le\frac{a}{2}$, for every w_i in G_2, hence, by (3.3),

$$d^+_D(y_i)\le\frac{a}{2}+1\quad\text{and}\quad d^-_D(x_i)\le\frac{a}{2}+1,\quad\text{for all } 1\le i\le a.$$

Lemma 3.3 and condition $(\mathcal{A})$ then imply that, for any $i \neq j$,

$$d^-_D(y_i) + d^+_D(x_i) + d^-_D(y_j) + d^+_D(x_j) \geq 4(a-1)\,. \tag{4.3}$$

If the above inequality is strict for at least one pair of indices $\{i, j\}$, then at least one of the vertices y_i, x_i, y_j, x_j violates condition (3.1); a contradiction. If, in turn, for all $i \neq j$, we have equality in (4.3), then we must also have, for every $i \in \{1, \dots, a\}$,

$$d^+_D(y_i) = \frac{a}{2} + 1, \quad d^-_D(y_i) = a - 1, \quad d^-_D(x_i) = \frac{a}{2} + 1, \quad d^+_D(x_i) = a - 1. \tag{4.4}$$

Combining (4.2) and (4.4), we get $\frac{a}{2} + 1 = a - 1$, hence $a = 4$. However, when $a = 4$ and G_1 is a bipartite digraph with partite sets $\{v_1, v_3\}$ and $\{v_2, v_4\}$, then (4.2) implies that $x_2y_1 \in A(D)$ and $x_4y_3 \in A(D)$. The existence of cycles C and C' then implies that D contains cycles $[x_1, y_2, x_2, y_1]$ and $[x_1, y_1, x_4, y_3, x_2, y_2]$. In light of Lemma 3.4, D is thus bipancyclic, which completes the proof. □

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