

Janusz Adamus

## EFFECTIVE REGULARITY CRITERIA FOR ANALYTIC MAPPINGS

**Abstract.** We survey recent developments in the study of singularities of complex-analytic mappings from a local algebraic viewpoint. We present several effective criteria for various modes of regularity of complex-analytic mappings in terms of vertical components in fibred powers of the mappings.

### Contents

1. Introduction	235
2. Analytic tensor product and fibred product	242
3. Openness vs. flatness	244
4. Vertical components and variation of fibre dimension	246
5. Vertical components and Gabrielov regularity	249
6. Homological algebra apparatus	252
7. Vertical components and flatness	254
8. Algebraic case	259
References	261

### 1. Introduction

The purpose of this paper is to provide a systematic exposition of a certain new method of analysis of the local geometry of complex-analytic mappings. We study the relationship between degeneracies in the family of fibres of an analytic mapping and the existence of the so-called *vertical components* in fibred powers of the mapping. There are, in fact, two natural notions of a vertical component (*algebraic* and *geometric*) and we are interested also in the relationship between them.

---

2000 *Mathematics Subject Classification*: Primary 13H05, 32B99; Secondary 13P99.

*Key words and phrases*: flat, open, Gabrielov regular, torsion-free, fibred power, vertical component, analytic tensor product, complex analytic geometry.

Research partially supported by the Natural Sciences and Engineering Research Council of Canada.

We gathered here a collection of criteria for various modes of regularity of analytic mappings, like Gabrielov regularity, openness, or flatness (see below). The main advantage of these results is that they reduce the continuous complexity of the family of fibres of a mapping to the discrete data that governs the presence (or lack thereof) of vertical components in a finite number of fibred powers of the mapping. In fact, in the algebraic setup, our criteria allow for a computer algorithm verification of openness and flatness, via Gröbner bases (see Section 8).

The study of vertical components began only in the late 90's with the works of M. Kwieciński [17] and P. Tworzewski [18], although the origins of this work are in M. Auslander's criterion for freeness (later generalized by Lichtenbaum [19] to the following statement).

**THEOREM 1.1.** (cf. [5, Thm. 3.2]) *Let  $R$  be a regular local ring of dimension  $n > 0$ , and let  $F$  be a finite  $R$ -module. Then  $F$  is  $R$ -free if and only if the  $n$ -fold tensor power  $F^{\otimes_R n}$  is a torsion-free  $R$ -module.*

They were followed by the first effective flatness criterion of A. Galligo and M. Kwieciński [9] (a special case of our Theorem 1.8 below), and numerous papers by the author (see, e.g., [1] and [3]). The next conceptual advance was only recently obtained by the author together with E. Bierstone and P. D. Milman [4], with the discovery of a certain phenomenon in local algebra, which we discuss in detail in Section 7. Most of the results presented in this survey are not original. The main regularity criteria (below) appeared in [1], [3] and [4].

### 1.1. Vertical components in fibred powers

**DEFINITION 1.2.** Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a morphism of germs of analytic spaces. An irreducible (isolated or embedded) component  $W_\xi$  of  $X_\xi$  is called *algebraic vertical* if there exists a nonzero element  $a \in \mathcal{O}_{Y,\eta}$  such that the pull-back  $\varphi_\xi^* a$  belongs to the associated prime  $\mathfrak{p}$  in  $\mathcal{O}_{X,\xi}$  corresponding to  $W_\xi$ . Equivalently,  $W_\xi$  is algebraic vertical if any sufficiently small representative of  $W_\xi$  is mapped into a proper analytic subset of a neighbourhood of  $\eta$  in  $Y$ . We say that  $W_\xi$  is *geometric vertical* if any sufficiently small representative of  $W_\xi$  is mapped into a nowhere dense subset of a neighbourhood of  $\eta$  in  $Y$ . (The notion of vertical component was introduced by Kwieciński [17], to mean what we now call a geometric vertical component.)

The concept of vertical component comes up naturally as an equivalent of torsion in algebraic geometry: Let  $\varphi : X \rightarrow Y$  be a polynomial map of algebraic varieties with  $Y$  irreducible. Then the coordinate ring of the source,  $A(X)$  has nonzero torsion as a module over the coordinate ring  $A(Y)$  of the target if and only if there exists a nonzero element  $a \in A(Y)$  such that its

pull-back  $\varphi^*a$  is a zerodivisor in  $A(X)$ . Since the set of zerodivisors equals the union of the associated primes, it follows from “prime avoidance” (see, e.g., [7, Lemma 3.3]) that  $A(X)$  has nonzero torsion over  $A(Y)$  if and only if there exists an irreducible (isolated or embedded) component of  $X$  whose image under  $\varphi$  is contained in a proper algebraic subset of  $Y$  (or, equivalently, is nowhere dense in  $Y$ ). There are two natural ways of generalizing this property of irreducible components to the analytic case. For a morphism  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  of germs of analytic spaces (with  $Y_\eta$  irreducible), one can either consider the components of the source that are mapped into nowhere dense subgerms of the target (the *geometric vertical* components), or the components that are mapped into proper analytic subgerms of the target (the *algebraic vertical* components).

The geometric approach has proved to be a very powerful tool (see Theorems 1.8 and 1.10 below). Note that in principle the existence of the algebraic vertical components is a stronger condition than the presence of the geometric vertical ones. Indeed, any algebraic vertical component (over an irreducible target) is geometric vertical, since a proper analytic subset of an irreducible analytic set has empty interior. The converse is not true though, as can be seen in the classical example of Osgood (cf. [10, Ch. II, § 5]):

$$\varphi : \mathbb{C}^2 \ni (x, y) \mapsto (x, xy, xye^y) \in \mathbb{C}^3.$$

Here the image of an arbitrarily small neighbourhood of the origin is nowhere dense in  $\mathbb{C}^3$ , but its Zariski closure has dimension 3 and therefore the image is not contained in a proper locally analytic subset of the target. (In general, the algebraic vertical components are precisely those geometric vertical ones along which the mapping is Gabrielov regular; see below.)

**REMARK 1.3.** On the other hand, the algebraic approach has an advantage that all the statements about algebraic vertical components (as opposed to geometric vertical) can be restated in terms of torsion freeness of the local rings. Namely,  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  has no (isolated or embedded) algebraic vertical components if and only if the local ring  $\mathcal{O}_{X,\xi}$  is a torsion-free  $\mathcal{O}_{Y,\eta}$ -module.

In view of Remark 1.3, it is interesting to know under what conditions are the two approaches equivalent. In other words, is there a class of analytic morphisms for which the existence of geometric vertical components is equivalent to the presence of the algebraic vertical ones. Theorem 1.10 below gives a partial answer to this question. On the other hand, as we show in Example 4.2, there are examples of bad behaviour of analytic mappings that can be detected by means of geometric vertical components but not by the algebraic vertical ones. The fundamental reason for this is that, in

general, the algebraic vertical components do not detect a hidden Gabrielov irregularity. Section 5 is devoted to the analysis of this problem.

Next, we want to extend the concept of vertical component to finitely generated  $\mathcal{O}_{X,\xi}$ -modules. Observe that, by the prime avoidance lemma,  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  has an algebraic (respectively, geometric) vertical component if and only if there exists a nonzero element  $m \in \mathcal{O}_{X,\xi}$  such that the zero-set germ  $\mathcal{V}(\text{Ann}_{\mathcal{O}_{X,\xi}}(m))$  of the annihilator of  $m$  in  $\mathcal{O}_{X,\xi}$  is mapped into a proper analytic (respectively, nowhere-dense) subgerm of  $Y_\eta$ . This can be generalized to  $\mathcal{O}_{X,\xi}$ -modules as follows.

**DEFINITION 1.4.** Let  $F$  be a finite  $\mathcal{O}_{X,\xi}$ -module, and let  $Z_\xi$  be the zero-set germ of the support of  $F$ ; i.e.,  $Z_\xi = \mathcal{V}(\bigcap\{\mathfrak{p} : \mathfrak{p} \in \text{Ass}_{\mathcal{O}_{X,\xi}}(F)\})$ . We say that  $F$  has an *algebraic* (respectively, *geometric*) *vertical component* over  $Y_\eta$  (or over  $\mathcal{O}_{Y,\eta}$ ) if  $Z_\xi$  has an algebraic (respectively, geometric) vertical component over  $Y_\eta$  in the sense of Definition 1.2; equivalently (by prime avoidance again), there exists a nonzero  $m \in F$  such that the  $\mathcal{V}(\text{Ann}_{\mathcal{O}_{X,\xi}}(m))$  is mapped into a proper analytic (respectively, nowhere-dense) subgerm of  $Y_\eta$ . In the geometric case, we will call such  $m$  a *geometric vertical element* (or simply a *vertical element*) of  $F$  over  $Y_\eta$  (or over  $\mathcal{O}_{Y,\eta}$ ).

Note that an analogous “algebraic vertical element” of  $F$  over  $Y_\eta$  is simply a (nonzero) zero-divisor of  $F$  over  $\mathcal{O}_{Y,\eta}$ , so there is no need to define algebraic vertical elements. A *vertical element* will always mean geometric vertical.

**REMARK 1.5.** In the special case that  $F = \mathcal{O}_{X,\xi}$ ,  $X_\xi$  has no geometric (respectively, algebraic) vertical components over  $Y_\eta$  if and only if  $\mathcal{O}_{X,\xi}$  (as an  $\mathcal{O}_{X,\xi}$ -module) has no vertical elements (respectively, no zero-divisors) over  $\mathcal{O}_{Y,\eta}$ .

Now let  $R$  denote a regular local analytic  $\mathbb{C}$ -algebra of dimension  $n$  (i.e.,  $R \cong \mathbb{C}\{y_1, \dots, y_n\}$  the ring of convergent power series in  $n$  variables). By a *local analytic  $R$ -algebra* we mean a ring of the form  $R\{x\}/I = \mathbb{C}\{y, x\}/I$ , where  $I$  is an ideal in  $\mathbb{C}\{y, x\} = \mathbb{C}\{y_1, \dots, y_n, x_1, \dots, x_m\}$ , with the canonical homomorphism  $R \rightarrow R\{x\}/I$ .

**DEFINITION 1.6.** Let  $F$  be an  $R$ -module. We say (following [9]) that  $F$  is an *almost finitely generated  $R$ -module*, when  $F$  is a finitely generated  $A$ -module, for some local analytic  $R$ -algebra  $A$ . In this case, there is a morphism of germs of analytic spaces  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  such that  $R \cong \mathcal{O}_{Y,\eta}$ ,  $A \cong \mathcal{O}_{X,\xi}$ ,  $R \rightarrow A$  is the induced homomorphism  $\varphi_\xi^* : \mathcal{O}_{Y,\eta} \rightarrow \mathcal{O}_{X,\xi}$ , and  $F$  is a finitely generated  $\mathcal{O}_{X,\xi}$ -module. We say that a nonzero element  $m \in F$  is *vertical* over  $R$  if  $m$  is vertical over  $\mathcal{O}_{Y,\eta}$  in the sense of Definition 1.4.

**REMARK 1.7.** It is easy to see that the notion of vertical element is well-defined; i.e., independent of a choice of local  $R$ -algebra  $A$  such that  $F$  is a finitely generated  $A$ -module (see [9] for details). In particular, given an almost finitely generated  $R$ -module  $F$ , we can assume without loss of generality that  $F$  is finitely generated over the regular ring  $A = R\{x\} \cong \mathbb{C}\{y, x\}$ , where  $x = (x_1, \dots, x_m)$ , for some  $m \geq 0$ .

**1.2. Main regularity criteria**

Recall that a morphism  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  of germs of analytic spaces is called *flat* when the pull-back homomorphism  $\varphi_\xi^* : \mathcal{O}_{Y,\eta} \rightarrow \mathcal{O}_{X,\xi}$  makes the local ring  $\mathcal{O}_{X,\xi}$  into a flat  $\mathcal{O}_{Y,\eta}$ -module. We say that a holomorphic mapping  $\varphi : X \rightarrow Y$  of analytic spaces is flat when  $\varphi_\xi$  is flat for every  $\xi \in X$ . A mapping  $\varphi : X \rightarrow Y$  is *open* when it sends open subsets of  $X$  onto open subsets of  $Y$ . A morphism  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is called open when  $\varphi_\xi$  is a germ at  $\xi$  of an open mapping. Finally, we say that  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is *Gabrielov regular* when, for every isolated irreducible component  $W_\xi$  of  $X_\xi$ ,  $\dim_\eta \varphi(W) = \dim_\eta \overline{\varphi(W)}$  for every sufficiently small representative  $W$  of  $W_\xi$ , where  $\overline{\varphi(W)}$  denotes the Zariski closure of  $\varphi(W)$  in a representative of  $Y_\eta$ .

The failure of flatness, openness, or Gabrielov regularity of a morphism correspond to various levels of degeneracies in the family of fibres of the morphism. Douady [6] proved that every flat mapping is open. (This is rather nontrivial, but see Section 3 for a simple alternative proof). Section 3 contains also a simple example of an open and non-flat mapping. Open mappings are Gabrielov regular by definition. An example of a Gabrielov regular non-open map is given in Example 5.3.

The following criteria assert that flatness and openness of a complex-analytic mapping can be controlled by means of vertical components in *finitely many* fibred powers of the mapping (see Section 2 for definitions). In fact, it always suffices to check in the  $n$ -fold fibred power, where  $n$  is the dimension of the target.

**THEOREM 1.8.** ([4, Thm. 1.1]) *Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a morphism of germs of complex-analytic spaces, where  $Y_\eta$  is smooth, of dimension  $n$ . Then  $\varphi_\xi$  is flat if and only if the  $n$ -fold fibred power  $\varphi_{\xi^{\{n\}}}^{\{n\}} : X_{\xi^{\{n\}}}^{\{n\}} \rightarrow Y_\eta$  has no (isolated or embedded) geometric vertical components.*

A special case of Theorem 1.8 when  $X_\xi$  is pure-dimensional is a result of Galligo and Kwieciński, [9, Thm. 6.1]. To obtain Theorem 1.8 in full generality, one needs to actually prove even more, namely a variant with a finite  $\mathcal{O}_{X,\xi}$ -module. This is Theorem 7.1, which we state and prove in Section 7.

It remains an open problem, in general, whether the geometric vertical components can be replaced with the algebraic vertical ones in the statement

of Theorem 1.8 (compare with Theorem 1.10 below). The few cases known so far, when such a strengthening is possible are listed in the following theorem.

**THEOREM 1.9.** (cf. [3, Thm. 1.1]) *Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a morphism of germs of complex-analytic spaces, where  $Y_\eta$  is smooth, of dimension  $n$ . Suppose that one of the following conditions is satisfied:*

- (1)  $n < 3$ ;
- (2)  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is a Nash morphism of Nash germs;
- (3)  $X_\xi$  is pure-dimensional and its singular locus is mapped into a proper analytic subgerm of  $Y_\eta$ ;
- (4) the local ring of the source  $\mathcal{O}_{X,\xi}$  is Cohen-Macaulay.

*Then,  $\varphi_\xi$  is flat if and only if the  $n$ -fold fibred power  $X_{\xi\{n\}}^{\{n\}}$  contains no algebraic vertical components.*

In short, in all of the above cases, some of the geometric vertical components of Theorem 1.8 must, in fact, be algebraic vertical (see Section 7 for details).

**THEOREM 1.10.** ([1, Thm. 2.2]) *Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a morphism of germs of analytic spaces. Let  $X_\xi$  be of pure dimension, and let  $Y_\eta$  be irreducible, of dimension  $n$ . Then the following conditions are equivalent:*

- (1)  $\varphi_\xi$  is a germ of an open mapping;
- (2)  $X_{\xi\{n\}}^{\{n\}}$  has no isolated geometric vertical components;
- (3)  $X_{\xi\{n\}}^{\{n\}}$  has no isolated algebraic vertical components.

**REMARK 1.11.** In light of Remark 1.3, the equivalence (1)  $\Leftrightarrow$  (3) in the above theorem can be restated as follows:

*The morphism  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is open if and only if the reduced local ring  $(\mathcal{O}_{X\{n\},\xi\{n\}})_{\text{red}}$  is a torsion-free  $\mathcal{O}_{Y,\eta}$ -module.*

On the other hand, the equivalence (2)  $\Leftrightarrow$  (3) means that, under the pure-dimensionality assumption on the domain, at least one of the restrictions of  $\varphi_{\xi\{n\}}^{\{n\}}$  to an isolated geometric vertical component of  $X_{\xi\{n\}}^{\{n\}}$  must be Gabrielov regular.

Note that since openness is a local property, Theorem 1.10 can easily be “globalized” to the case of an analytic mapping  $\varphi : X \rightarrow Y$  of analytic spaces, where  $X$  is of pure dimension and  $Y$  is locally irreducible of dimension  $n$ . Thus, our result is a stronger version of a theorem by Kwieciński and Tworzewski [18, Thm. 3.2], asserting that openness is equivalent to the lack of geometric vertical components in the  $n$ ’th fibred power. We prove Theorem 1.10 in Section 4.

As we show in Example 4.2, the pure-dimensionality constraint on  $X$  in the above theorem is unavoidable. Roughly speaking, this is a consequence of the fact that fibred products do not behave well with respect to Gabrielov regularity. It is thus interesting to know when does the Gabrielov regularity of a morphism imply that of its fibred powers. This question is answered in Theorem 1.12 below.

For a morphism  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  of germs of analytic spaces, let  $k$  be its maximal fibre dimension and let  $(A_j)_\xi$  denote the germ of a locus of points of fibre dimension greater than or equal to  $j$ , for  $j = 0, 1, \dots, k$  (see Section 4 for details). Then we have:

**THEOREM 1.12.** ([3, Prop. 4.2]) *Suppose  $Y_\eta$  is irreducible. The following conditions are equivalent:*

- (1)  $\varphi_{\xi^{\{d\}}} : X_{\xi^{\{d\}}} \rightarrow Y_\eta$  is Gabrielov regular, for all  $d \geq 1$ ;
- (2) all the restrictions  $\varphi_\xi|_{(A_j)_\xi}$  are Gabrielov regular ( $j = 0, \dots, k$ ).

Notice the difference between the assumptions on the target space in Theorems 1.8 and 1.9 vs. Theorems 1.10 and 1.12. In fact, the assumptions of the latter theorems are the weakest possible that one needs to impose on the target to make sense of the notion of vertical component. (Otherwise, for instance, each of the irreducible components of the set  $X = \{xy = 0\}$  would be an algebraic vertical component for the identity mapping  $\text{id}_X : X \rightarrow X$ , which is absurd.) This is in contrast with Theorem 1.8, where any weakening of the regularity assumption on the target space leads to an open problem. As of today, it is not even known whether, for a singular irreducible  $Y_\eta$ , there exists a constant  $N$  (depending only on  $Y_\eta$ ) such that non-flatness of a morphism  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  could be detected by vertical components in the  $N$ -fold fibred power of  $\varphi_\xi$ . So far, the best result in this direction is the following theorem of Kwieciński.

**THEOREM 1.13.** ([17, Thm. 1.1]) *Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a morphism of germs of analytic spaces. Suppose that  $Y_\eta$  is irreducible. Then  $\varphi_\xi$  is non-flat if and only if there exists an (isolated or embedded) algebraic vertical component in the  $d$ -fold fibred power  $X_{\xi^{\{d\}}}$ , for some  $d \geq 1$ .*

It is also symptomatic that, contrary to openness, to characterize flatness, one needs to control both the isolated and embedded components in the fibred powers. (The embedded components are the zero-set germs of the embedded associated primes of the local ring). This reflects the fact that flatness often depends on infinitesimal data, which is invisible from the topological point of view. In Section 3 we give a simple example of an open mapping which is not flat (see also Example 2.1).

The structure of the paper is as follows. In the next section we recall the notions of fibred product and analytic tensor product. Section 3 is concerned with the comparison between openness and flatness: a simple proof of Douady's theorem on openness of a flat morphism is derived from Theorem 1.10. Section 4 contains the proof of Theorem 1.10 and some related results. In Section 5 we give a proof of Theorem 1.12 and an example of a Gabrielov regular mapping with irregular fibred powers. Sections 6 and 7 are devoted to the proof of Theorems 1.8 and 1.9. The last section contains computable criteria for openness and flatness of polynomial mappings, derived from Theorems 1.10 and 7.1.

## 2. Analytic tensor product and fibred product

Most of the arguments in this article are of local algebraic nature. To establish the notation and keep the article self-contained, we recall here the concepts of analytic tensor product and fibred product of analytic spaces, that are used throughout the paper.

Our main objects of study are morphisms of germs of complex-analytic spaces, that is, germs of holomorphic mappings between complex-analytic spaces. By a complex-analytic space we mean a  $\mathbb{C}$ -ringed space, whose underlying topological space is Hausdorff, and the local models are defined by coherent (not necessarily radical!) ideal sheaves on open sets in  $\mathbb{C}^n$  (readers not familiar with these notions may find a quick treatment in [8, Ch. 0]). In particular, unlike some classical expositions (e.g., [20]), we do not assume that our spaces are reduced. On the contrary, we are very much interested in their embedded nilpotent structure, especially in the study of flatness. Also, when studying fibred products of analytic spaces, the requirement that the spaces be reduced is not natural, because a fibred product of reduced spaces may as well be non-reduced, even when the product is over a smooth target (see Example 2.1 below).

The analytic tensor product is defined in the category of finitely generated modules over local analytic  $\mathbb{C}$ -algebras (i.e., rings of the form  $\mathbb{C}\{z_1, \dots, z_n\}/I$  for some ideal  $I$ ) by the usual universal mapping property for tensor product (cf. [10]): Let  $\varphi_i : R \rightarrow A_i$  ( $i = 1, 2$ ) be homomorphisms of local analytic  $\mathbb{C}$ -algebras. Then there is a unique (up to isomorphism) local analytic  $\mathbb{C}$ -algebra  $A_1 \tilde{\otimes}_R A_2$ , together with homomorphisms  $\theta_i : A_i \rightarrow A_1 \tilde{\otimes}_R A_2$  ( $i = 1, 2$ ), such that  $\theta_1 \circ \varphi_1 = \theta_2 \circ \varphi_2$ , and for every pair of homomorphisms of local analytic  $\mathbb{C}$ -algebras  $\psi_1 : A_1 \rightarrow B$ ,  $\psi_2 : A_2 \rightarrow B$  satisfying  $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$ , there is a unique homomorphism of local analytic  $\mathbb{C}$ -algebras  $\psi : A_1 \tilde{\otimes}_R A_2 \rightarrow B$  making the associated diagram commute. The algebra  $A_1 \tilde{\otimes}_R A_2$  is called the *analytic tensor product* of  $A_1$  and  $A_2$  over  $R$ .

For finite modules  $M_1$  and  $M_2$  over local analytic  $R$ -algebras  $A_1$  and  $A_2$ , respectively, there is a unique (up to isomorphism) finite  $A_1 \tilde{\otimes}_R A_2$ -module  $M_1 \tilde{\otimes}_R M_2$ , together with an  $R$ -bilinear mapping  $\rho : M_1 \times M_2 \rightarrow M_1 \tilde{\otimes}_R M_2$ , such that for every  $R$ -bilinear  $\kappa : M_1 \times M_2 \rightarrow N$ , where  $N$  is a finite  $A_1 \tilde{\otimes}_R A_2$ -module, there is a unique homomorphism of  $A_1 \tilde{\otimes}_R A_2$ -modules  $\lambda : M_1 \tilde{\otimes}_R M_2 \rightarrow N$  satisfying  $\kappa = \lambda \circ \rho$ . The module  $M_1 \tilde{\otimes}_R M_2$  is called the *analytic tensor product* of  $M_1$  and  $M_2$  over  $R$ .

In our considerations, it will be convenient to express the analytic tensor product of modules over a local analytic  $\mathbb{C}$ -algebra in terms of ordinary tensor product of certain naturally associated modules: Given homomorphisms of local analytic  $\mathbb{C}$ -algebras  $\varphi_i : R \rightarrow A_i$ , and finitely generated  $A_i$ -modules  $M_i$  ( $i = 1, 2$ ), the modules  $M_1 \tilde{\otimes}_R A_2$  and  $A_1 \tilde{\otimes}_R M_2$  are finitely generated over  $A_1 \tilde{\otimes}_R A_2$ , and there is a canonical isomorphism

$$M_1 \tilde{\otimes}_R M_2 \cong (M_1 \tilde{\otimes}_R A_2) \otimes_{A_1 \tilde{\otimes}_R A_2} (A_1 \tilde{\otimes}_R M_2).$$

In particular, if  $A_1 = R\{x\}/I_1$  and  $A_2 = R\{t\}/I_2$ , where  $x = (x_1, \dots, x_l)$ ,  $t = (t_1, \dots, t_m)$  are systems of variables and  $I_1 \subset R\{x\}$ ,  $I_2 \subset R\{t\}$  are ideals, then

$$\begin{aligned} A_1 \tilde{\otimes}_R A_2 &\cong (A_1 \tilde{\otimes}_R R\{t\}) \otimes_{R\{x\} \tilde{\otimes}_R R\{t\}} (R\{x\} \tilde{\otimes}_R A_2) \\ &\cong (R\{x, t\}/I_1 R\{x, t\}) \otimes_{R\{x, t\}} (R\{x, t\}/I_2 R\{x, t\}) \\ &\cong R\{x, t\}/(I_1 R\{x, t\} + I_2 R\{x, t\}). \end{aligned}$$

The fibred product of analytic spaces is defined by a dual universal mapping property (see [8]): Let  $\varphi_i : X_i \rightarrow Y$  ( $i = 1, 2$ ) denote holomorphic mappings of complex-analytic spaces. Then there exists a unique (up to isomorphism) complex-analytic space  $X_1 \times_Y X_2$ , together with holomorphic maps  $\pi_i : X_1 \times_Y X_2 \rightarrow X_i$  ( $i = 1, 2$ ), such that  $\varphi_1 \circ \pi_1 = \varphi_2 \circ \pi_2$ , and for every pair of holomorphic maps  $\psi_1 : X \rightarrow X_1$ ,  $\psi_2 : X \rightarrow X_2$  satisfying  $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$ , there is a unique holomorphic map  $\psi : X \rightarrow X_1 \times_Y X_2$  making the associated diagram commute. The space  $X_1 \times_Y X_2$  is called the *fibred product* of  $X_1$  and  $X_2$  over  $Y$  (more precisely, over  $\varphi_1$  and  $\varphi_2$ ). There is a canonical holomorphic mapping  $\varphi_1 \times_Y \varphi_2 : X_1 \times_Y X_2 \rightarrow Y$ , given by  $\varphi_1 \times_Y \varphi_2 = \varphi_i \circ \pi_i$  (where  $i = 1$  or  $2$ ).

Given a holomorphic map  $\varphi : X \rightarrow Y$  of complex-analytic spaces, the *d-fold fibred power* of  $X$  over  $Y$ , denoted  $X^{\{d\}}$ , is defined recursively as  $X^{\{2\}} = X \times_Y X$ ,  $X^{\{d\}} = X^{\{d-1\}} \times_Y X$ . Let further  $\xi \in X$ ,  $\eta = \varphi(\xi)$ , and let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  denote the germ of  $\varphi$  at  $\xi$ . We denote by  $\varphi^{\{d\}} : X^{\{d\}} \rightarrow Y$  the canonical map from the  $d$ -fold fibred power of  $X$  over  $Y$  to  $Y$ , and by  $\varphi_{\xi^{\{d\}}}^{\{d\}} : X_{\xi^{\{d\}}}^{\{d\}} \rightarrow Y_\eta$  its germ at the diagonal point  $\xi^{\{d\}} := (\xi, \dots, \xi) \in X^d$ .

**EXAMPLE 2.1.** ([13, Ch. III, Exercise 9.3]) The fibred product of reduced spaces need not be reduced itself: Let  $X \subset \mathbb{C}^4$  be a union of two copies of  $\mathbb{C}^2$  that intersect precisely at the origin, say

$$X_1 = \{(y_1, y_2, t_1, t_2) \in \mathbb{C}^4 : t_1 = t_2 = 0\} \quad \text{and}$$

$$X_2 = \{(y_1, y_2, t_1, t_2) \in \mathbb{C}^4 : t_1 - y_1 = t_2 - y_2 = 0\},$$

and let  $\varphi : X \rightarrow \mathbb{C}^2$  be the projection onto the  $y$  variables. Then the fibred power  $X^{\{2\}} = X \times_{\mathbb{C}^2} X$  has an embedded component, namely the origin in  $\mathbb{C}^4$ .

Suppose that  $\varphi_1 : X_1 \rightarrow Y$  and  $\varphi_2 : X_2 \rightarrow Y$  are holomorphic mappings of analytic spaces, with  $\varphi_1(\xi_1) = \varphi_2(\xi_2) = \eta$ . Then the local rings  $\mathcal{O}_{X_i, \xi_i}$  ( $i = 1, 2$ ) are  $\mathcal{O}_{Y, \eta}$ -modules and, by the uniqueness of fibred product and of analytic tensor product, the local ring  $\mathcal{O}_{Z, (\xi_1, \xi_2)}$  of the fibred product  $Z = X_1 \times_Y X_2$  at  $(\xi_1, \xi_2)$  is canonically isomorphic to  $\mathcal{O}_{X_1, \xi_1} \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \mathcal{O}_{X_2, \xi_2}$ . Therefore, given a holomorphic germ  $\varphi_\xi : X_\xi \rightarrow Y_\eta$ , we will identify the  $d$ -fold analytic tensor power  $\mathcal{O}_{X, \xi}^{\otimes_{\mathcal{O}_{Y, \eta}} d} = \mathcal{O}_{X, \xi} \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \dots \tilde{\otimes}_{\mathcal{O}_{Y, \eta}} \mathcal{O}_{X, \xi}$  with the local ring of the  $d$ -fold fibred power  $\mathcal{O}_{X^{\{d\}}, \xi^{\{d\}}}$ , for  $d \geq 1$ .

### 3. Openness vs. flatness

The relationship between Theorems 1.8 and 1.10 (the criteria for flatness and openness in terms of vertical components in fibred powers) shows that openness can be viewed as a geometric analogue of flatness. Flatness intuitively means that fibres of a morphism change in a continuous way. If one disregards the embedded structure of the fibres, that is, if one only considers their geometry, then continuity in the family of fibres means openness of the morphism. This roughly explains why for the study of openness it is enough to consider the isolated irreducible components in fibred powers, whilst both the isolated and embedded components must be considered in the study of flatness.

In fact, every flat morphism is open. This result was first obtained by A. Douady [6]; another proof was given by R. Kiehl [15]. Our openness criterion, Theorem 1.10, implies a quick alternative proof (see Proposition 3.2 below).

The proof of Proposition 3.2 below is based on the observation that if  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is flat then every restriction of  $\varphi_\xi$  to an irreducible component of  $X_\xi$  is flat as well (Lemma 3.1 below). The converse is not true though: flatness of all the restrictions to irreducible components does not imply flatness of the morphism, as can be observed in Example 2.1.

Observe that for openness the opposite is true: Clearly, openness of all the restrictions to isolated irreducible components implies openness of the morphism, but openness of a morphism does not imply openness of all its

restrictions to isolated irreducible components. Consider, for instance, the mapping given by

$$\varphi : X \ni (x, y, s, t) \mapsto (x + s, xy + t) \in Y = \mathbb{C}^2,$$

where  $X = X_1 \cup X_2$ ,  $X_1 = \{(x, y, s, t) \in \mathbb{C}^4 : s = t = 0\}$ , and  $X_2 = \{(x, y, s, t) \in \mathbb{C}^4 : x = 0\}$ . Then  $\varphi$  is open, but  $\varphi|_{X_1}$  is not. (In particular, the above is a simple example of an open and non-flat mapping, by Proposition 3.2 and Lemma 3.1 below.)

The following lemma is probably known, but we prove it here for completeness.

**LEMMA 3.1.** *If  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is a flat morphism of germs of analytic spaces with  $Y_\eta$  irreducible, then every restriction  $\varphi_\xi|_{W_\xi} : W_\xi \rightarrow Y_\eta$  to an (isolated or embedded) irreducible component  $W_\xi$  of  $X_\xi$  is also flat.*

**Proof.** Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a flat morphism of germs of analytic spaces with  $Y_\eta$  irreducible, and let  $W_\xi$  be an (isolated or embedded) irreducible component of  $X_\xi$ . Without loss of generality, we can assume that  $X_\xi$  is a subgerm of  $\mathbb{C}_0^m$  for some  $m \geq 1$ . We can then embed  $X_\xi$  into  $Y_\eta \times \mathbb{C}_0^m$  via the graph of  $\varphi_\xi$ . Therefore the local ring  $\mathcal{O}_{X,\xi}$  of the germ  $X_\xi$  can be thought of as a quotient of the local ring of  $Y_\eta \times \mathbb{C}_0^m$ ; i.e.,  $\mathcal{O}_{X,\xi} = \mathcal{O}_{Y,\eta}\{x\}/I$  for some ideal  $I$  in  $\mathcal{O}_{Y,\eta}\{x\}$ , where  $x = (x_1, \dots, x_m)$  is a system of  $m$  complex variables.

Let  $\mathfrak{p}$  be the associated prime of  $I$  in  $\mathcal{O}_{Y,\eta}\{x\}$  corresponding to  $W_\xi$ . Now,  $\varphi_\xi|_{W_\xi} : W_\xi \rightarrow Y_\eta$  is flat if and only if  $\mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}$  is a flat  $\mathcal{O}_{Y,\eta}$ -module. By a well known characterization of flatness [14, Prop. 6.2], the latter is equivalent to

$$\widetilde{\text{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}) = 0,$$

where  $\mathfrak{m}$  is the maximal ideal in  $\mathcal{O}_{Y,\eta}$  and  $\widetilde{\text{Tor}}$  denotes the derived functor for analytic tensor product (see Section 2).

The short exact sequence of  $\mathcal{O}_{Y,\eta}$ -modules

$$0 \rightarrow \mathfrak{p}/I \rightarrow \mathcal{O}_{Y,\eta}\{x\}/I \rightarrow \mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p} \rightarrow 0$$

induces a long exact sequence of the  $\widetilde{\text{Tor}}$  modules, of which a part is

$$\begin{aligned} \widetilde{\text{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{X,\xi}) &\rightarrow \widetilde{\text{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}) \xrightarrow{\psi} \\ &(\mathcal{O}_{Y,\eta}/\mathfrak{m}) \widetilde{\otimes}_{\mathcal{O}_{Y,\eta}} (\mathfrak{p}/I) \xrightarrow{\chi} (\mathcal{O}_{Y,\eta}/\mathfrak{m}) \widetilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi} \rightarrow \\ &(\mathcal{O}_{Y,\eta}/\mathfrak{m}) \widetilde{\otimes}_{\mathcal{O}_{Y,\eta}} (\mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}) \rightarrow 0. \end{aligned}$$

Since  $\mathcal{O}_{X,\xi}$  is  $\mathcal{O}_{Y,\eta}$ -flat, then  $\widetilde{\text{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{X,\xi}) = 0$ , and hence  $\psi$  is injective. Therefore,  $\widetilde{\text{Tor}}_1^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{Y,\eta}/\mathfrak{m}, \mathcal{O}_{Y,\eta}\{x\}/\mathfrak{p}) = 0$  if and only if  $\chi$  is injective.

Observe that  $\chi$  is *not* injective if and only if there exists a nonzero series  $f \in \mathfrak{p} \setminus I$  that factors (in  $\mathcal{O}_{Y,\eta}\{x\}$ , but not in  $\mathfrak{p}$ !) as  $f = gh$  for some  $g \in \mathfrak{m}$  and  $h \in \mathcal{O}_{Y,\eta}\{x\} \setminus \mathfrak{p}$ . Then  $g \in \mathfrak{p}$ , since  $\mathfrak{p}$  is prime and  $h \notin \mathfrak{p}$ . Thus, the maximal ideal  $\mathfrak{m}$  of  $\mathcal{O}_{Y,\eta}$  contains a nonzero element of an associated prime of  $I$ , and hence  $\mathcal{O}_{Y,\eta}$  contains a zero-divisor of  $\mathcal{O}_{X,\xi}$ . It follows from the characterization of flatness in terms of relations (see, e.g., [7, Cor. 6.5]) that  $\mathcal{O}_{Y,\eta}$  has a zero-divisor itself, which contradicts the assumption that  $Y_\eta$  be irreducible. We thus showed that  $\chi$  must be injective, which completes the proof. ■

**PROPOSITION 3.2.** *Let  $\varphi : X \rightarrow Y$  be a holomorphic map of complex-analytic spaces. If  $\varphi$  is flat, then it is open.*

**Proof.** We will proceed in three steps:

**Step 1.** Suppose first that  $Y$  is locally irreducible and  $X$  is of pure dimension. Let  $\xi \in X$  and  $\eta = \varphi(\xi)$ . Then flatness of  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  implies flatness of all its fibred powers (as flatness is preserved by any base change [14, Prop. 6.8], and compositions of flat mappings are flat). In particular,  $\mathcal{O}_{X\{n\},\xi\{n\}}$  is  $\mathcal{O}_{Y,\eta}$ -flat, hence torsion-free. By Remark 1.3,  $X_{\xi\{n\}}$  has no algebraic vertical components over  $Y_\eta$ , and so  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is open, by Theorem 1.10. Thus  $\varphi$  is open, because  $\xi$  was arbitrary.

**Step 2.** Next, assume only that  $Y$  is locally irreducible. Since openness of a germ  $\varphi_\xi$  follows from openness of every restriction  $\varphi_\xi|_{W_\xi}$  to an isolated irreducible component  $W_\xi$  of  $X_\xi$ , by Lemma 3.1 the problem reduces to Step 1.

**Step 3.** In the general case we may assume that  $Y$  is reduced. Let  $\nu : \tilde{Y} \rightarrow Y$  denote the normalization map. Then  $\tilde{Y}$  is locally irreducible and there is a commutative square

$$\begin{array}{ccc} X \times_Y \tilde{Y} & \longrightarrow & X \\ \tilde{\varphi} \downarrow & & \downarrow \varphi \\ \tilde{Y} & \xrightarrow{\nu} & Y. \end{array}$$

Now,  $\tilde{\varphi}$  is flat, because flatness is preserved by any base change (see [14, Prop. 6.8]), and hence open, by Step 2. Since  $Y$  has the quotient topology with respect to  $\nu$ , this implies that  $\varphi$  is open. ■

#### 4. Vertical components and variation of fibre dimension

In this section, we describe a relationship between the filtration of the target of an analytic mapping  $\varphi : X \rightarrow Y$  by fibre dimension and the isolated irreducible components of the  $n$ -fold fibred power  $X^{\{n\}}$ , where  $n = \dim Y$ .

We will also discuss the relationship between the occurrence of the two kinds of vertical components in fibred powers.

Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a morphism of germs of analytic spaces, where  $Y_\eta$  is irreducible and of dimension  $n$ . Let  $Y$  be an irreducible representative of  $Y_\eta$ , let  $X$  be a representative of  $X_\xi$ , such that the isolated irreducible components of  $X$  are precisely the representatives in  $X$  of the isolated irreducible components of  $X_\xi$ , and  $\varphi(X) \subset Y$ , where  $\varphi$  represents the germ  $\varphi_\xi$ . Let  $\text{fbd}_x\varphi$  denote the fibre dimension  $\dim_x \varphi^{-1}(\varphi(x))$  of  $\varphi$  at a point  $x \in X$ .

Put  $l := \min\{\text{fbd}_x\varphi : x \in X\}$ ,  $k := \max\{\text{fbd}_x\varphi : x \in X\}$ , and  $A_j := \{x \in X : \text{fbd}_x\varphi \geq j\}$ ,  $l \leq j \leq k$ . Then  $X = A_l \supset A_{l+1} \supset \dots \supset A_k$  and, by upper-semicontinuity of fibre dimension (see Cartan–Remmert Theorem [20, §V.3.3, Thm. 5]), the  $A_j$  are analytic in  $X$ . Define  $B_j := \varphi(A_j) = \{y \in Y : \dim \varphi^{-1}(y) \geq j\}$ ,  $l \leq j \leq k$ . Upper-semicontinuity of  $\text{fbd}_x\varphi$  (as a function of  $x$ ) implies that the germs  $(A_j)_\xi$  and  $(B_j)_\eta$  are independent of the choices of representatives made.

Note that, except for  $B_k$  (cf. proof of Proposition 4.1 below), the  $B_j$  may not even be semianalytic in general. This fact is responsible for a complicated relationship between the algebraic vertical and geometric vertical components in the fibred powers of  $X$  over  $Y$  (see Section 5), but will not affect our considerations here, which rely only on the properties of  $B_k$ .

**PROPOSITION 4.1.** ([1, Prop.2.1]) *Under the assumptions above, let  $\bigcup_{i \in I} W_i$  be the decomposition of  $(X^{\{n\}})_{\text{red}}$  into finitely many isolated irreducible components through  $\xi^{\{n\}}$ . Then:*

(1) *For each  $j = l, \dots, k$ , there is an index subset  $I_j \subset I$  such that*

$$B_j = \bigcup_{i \in I_j} \varphi^{\{n\}}(W_i).$$

(2) *Let  $y \in B_j$  and let  $s = \dim \varphi^{-1}(y)$  ( $s \geq j$ ). If  $Z$  is an isolated irreducible component of the fibre  $(\varphi^{\{n\}})^{-1}(y)$ , of dimension  $ns$ , and  $W$  is an irreducible component of  $X^{\{n\}}$  containing  $Z$ , then  $\varphi^{\{n\}}(W) \subset B_j$ .*

**Proof.** For (2), fix  $j \geq l + 1$ . (The statement is trivial for  $j = l$ , since  $B_l = \varphi(X)$ .) Suppose that there exists  $x = (x_1, \dots, x_n) \in W$  such that  $\varphi(x_1) \in Y \setminus B_j$  (and hence  $\varphi(x_i) \in Y \setminus B_j$ ,  $i \leq n$ ). Then  $\text{fbd}_{x_i}\varphi \leq j - 1$ ,  $i = 1, \dots, n$ ; hence  $\text{fbd}_x\varphi^{\{n\}} \leq n(j - 1) = nj - n$ . In particular, the generic fibre dimension of  $\varphi^{\{n\}}|_W$  is at most  $nj - n$ . Since  $\text{rank}(\varphi^{\{n\}}|_W) \leq \dim Y = n$ , then  $\dim W \leq (nj - n) + n = nj$  (see, e.g., [20, §V.3]).

Now we have  $W \supset Z$ ,  $\dim W \leq nj$ ,  $\dim Z = ns \geq nj$ , and both  $W$  and  $Z$  are irreducible analytic sets in  $X^{\{n\}}$ . This is possible only if  $W = Z$ ; hence  $\varphi^{\{n\}}(W) = \varphi^{\{n\}}(Z) = \{y\} \subset B_j$ ; a contradiction. Therefore  $\varphi^{\{n\}}(W) \subset B_j$ , completing the proof of (2).

Part (1) follows immediately, since if  $y \in B_j$  and  $Z$  is an irreducible component of  $(\varphi^{\{n\}})^{-1}(y)$  of the maximal dimension, then there exists an isolated irreducible component  $W$  of  $X^{\{n\}}$  that contains  $Z$ . ■

We are now ready to prove our openness criterion.

**Proof of Theorem 1.10.** First, notice that if  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is open, then  $\varphi_{\xi^{\{d\}}} : X_{\xi^{\{d\}}} \rightarrow Y_\eta$  is open for every  $d \geq 1$ , and hence every isolated irreducible component of  $X_{\xi^{\{d\}}}$  is dominating  $Y_\eta$ . This proves (1)  $\Rightarrow$  (2). The implication (2)  $\Rightarrow$  (3) is trivial, as every algebraic vertical component is geometric vertical over an irreducible target (see Section 1).

For the proof of (3)  $\Rightarrow$  (1), let us suppose that  $\varphi_\xi$  is not open. Let  $\varphi : X \rightarrow Y$  be its non-open representative, where  $Y$  is an irreducible representative of  $Y_\eta$ , of dimension  $n$ ,  $X$  is a representative of  $X_\xi$ , such that the isolated irreducible components  $X_1, \dots, X_s$  of  $X$  are precisely the representatives in  $X$  of the isolated irreducible components of  $X_\xi$ , and  $\varphi(X) \subset Y$ . Put  $m := \dim X$ . As above, let  $k = \max\{\text{fbd}_x \varphi : x \in X\}$ ,  $A_k = \{x \in X : \text{fbd}_x \varphi = k\}$ , and  $B_k = \varphi(A_k) = \{y \in Y : \dim \varphi^{-1}(y) = k\}$ . Then the fibre dimension of  $\varphi$  is constant on the analytic set  $A_k$ . By the Remmert Rank Theorem (see [20, §V.6, Thm. 1]),  $B_k$  is locally analytic in  $Y$ , of dimension  $\dim A - k$  (which is at most  $\dim X - k = m - k$ ). Shrinking  $Y$  if necessary, we can assume that  $B_k$  is an analytic subset of  $Y$ . Therefore, by Proposition 4.1 and irreducibility of the germ  $Y_\eta$ , it is enough to show that the analytic germ  $(B_k)_\eta$  is a proper subgerm of  $Y_\eta$ .

Suppose that  $k$  is the generic fibre dimension for all the components  $X_1, \dots, X_s$  simultaneously. Then, by the Remmert Rank Theorem again, either  $\dim X - k < n$  and then  $(B_k)_\eta = (\varphi(X))_\eta$  is a proper analytic subgerm of  $Y_\eta$ , or else  $\dim X - k = n$  and  $\varphi$  is open. Since an isolated algebraic vertical component in  $\varphi_\xi$  induces such a component in every fibred power  $\varphi_{\xi^{\{d\}}}$ , we may assume that, for some isolated irreducible component  $X_j$  of  $X$ , the generic fibre dimension of  $\varphi|_{X_j}$  is at most  $k - 1$ . Hence

$$\dim Y \geq \dim X_j - \text{generic fbd} \varphi|_{X_j} \geq m - k + 1 > \dim B_k.$$

Consequently,  $\dim(B_k)_\eta < \dim Y = \dim Y_\eta$ , so that  $(B_k)_\eta \subsetneq Y_\eta$ . ■

The pure-dimensionality constraint on  $X$  in our theorem is unavoidable. If  $X$  is not of pure dimension, it may happen that the exceptional fibres of one component are generic for a component of higher dimension, and therefore they do not give rise to an isolated algebraic vertical component in any fibred power. This phenomenon is illustrated in the following example, where not only is the morphism  $\varphi$  not open, but also it is not even regular in the sense of Gabrielov.

**EXAMPLE 4.2.** Let  $X = X_1 \cup X_2$ , where  $X_1 = \{(x, y, s, t, z) \in \mathbb{C}^5 : s = t = z = 0\}$  and  $X_2 = \{(x, y, s, t, z) \in \mathbb{C}^5 : x = 0\}$ . Define  $\varphi : X \rightarrow Y = \mathbb{C}^3$  as

$$\varphi(x, y, s, t, z) = (x + s, xy + t, xye^y + z).$$

Observe that (the germ at the origin of)  $\varphi|_{X_1}$  is an Osgood mapping (cf. Section 1) and hence it is not Gabrielov regular. Therefore  $\varphi$  is not open. But the exceptional fibre  $\{x = s = t = z = 0\}$  of  $\varphi|_{X_1}$  is in no sense exceptional for  $\varphi|_{X_2}$ . One can easily verify that in any fibred power of  $X$  over  $Y$ , any isolated (!) irreducible component is either *purely geometric* vertical (with the image equal to that of  $\varphi|_{X_1}$ ) or maps onto (the germ at the origin of)  $Y$  and so is not vertical in neither sense.

Fortunately, pathologies like the one above can be avoided in the mixed-dimensional case by assuming that the maximal fibre dimension is not generic along all components of  $X$  of maximal dimension.

**PROPOSITION 4.3.** *Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a morphism of germs of complex-analytic spaces. Suppose that  $Y_\eta$  is irreducible,  $\dim Y_\eta = n$ ,  $\dim X_\xi = m$ , and the maximal fibre dimension of  $\varphi_\xi$  is not generic on some  $m$ -dimensional irreducible component of  $X_\xi$ . Then the  $n$ -fold fibred power  $\varphi_{\xi^{\{n\}}}^{\{n\}} : X_{\xi^{\{n\}}}^{\{n\}} \rightarrow Y_\eta$  contains an isolated algebraic vertical component.*

**Proof.** As above, let  $\varphi : X \rightarrow Y$  be a representative of  $\varphi_\xi$ , where  $Y$  is irreducible and of dimension  $n$ . Let  $k = \max\{\text{fbd}_x \varphi : x \in X\}$ ,  $A_k = \{x \in X : \text{fbd}_x \varphi = k\}$ , and  $B_k = \varphi(A_k) = \{y \in Y : \dim \varphi^{-1}(y) = k\}$ . Then, as in the proof of Theorem 1.10 above, after shriking  $Y$  if necessary, we may assume that  $B_k$  is analytic in  $Y$ , of dimension at most  $\dim X - k$ . Let  $X_j$  be an isolated irreducible component of  $X$ , of dimension  $m = \dim X$ , for which the generic fibre dimension of  $\varphi|_{X_j}$  is at most  $k - 1$ . We have

$$\dim Y \geq \dim X_j - \text{generic fbd} \varphi|_{X_j} \geq m - k + 1.$$

Then  $\dim B_k \leq m - k < \dim Y$ ; hence  $\dim(B_k)_\eta < \dim Y = \dim Y_\eta$ , so that  $(B_k)_\eta \not\subseteq Y_\eta$ . ■

### 5. Vertical components and Gabrielov regularity

In this section we show that, in general, the fibred product does not behave well with respect to Gabrielov regularity. We then present a proof of Theorem 1.12, which gives a necessary and sufficient condition for Gabrielov regularity to be preserved by fibred powers.

Recall that a morphism  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  of germs of analytic spaces is called Gabrielov regular when, for every isolated irreducible component  $W_\xi$  of  $X_\xi$ ,  $\dim_\eta \varphi(W) = \dim_\eta \overline{\varphi(W)}$  for every sufficiently small representative  $W$  of

$W_\xi$ , where  $\overline{\varphi(W)}$  denotes the Zariski closure of  $\varphi(W)$  in a representative of  $Y_\eta$ . Equivalently,  $\dim_\eta \overline{\varphi(W)}$  equals the generic rank of  $\varphi|_W$  (see, e.g., [21]).

Under the assumptions of Theorem 1.10, let as before  $Y$  be an irreducible representative of  $Y_\eta$ , let  $X$  be a pure-dimensional representative of  $X_\xi$ , such that the isolated irreducible components of  $X$  are precisely the representatives in  $X$  of the isolated irreducible components of  $X_\xi$ , and  $\varphi(X) \subset Y$ , where  $\varphi$  represents the germ  $\varphi_\xi$ . The following result is a simple corollary of Proposition 4.1. Define  $S = \{y \in Y : \dim \varphi^{-1}(y) > l\}$ , where as before  $l = \min\{\text{fbd}_x \varphi : x \in X\}$ . Then  $\dim S < \dim Y$ , since  $X$  is pure-dimensional.

**PROPOSITION 5.1.** ([1, Prop.3.1]) *Suppose that  $\dim_\eta \overline{S} = n$ , where  $\overline{S}$  denotes the Zariski closure of  $S$  in  $Y$ . Then  $X_{\xi^{\{n\}}}$  contains an isolated purely geometric vertical component; i.e., an isolated geometric vertical component which is not algebraic vertical.*

As a consequence, there exist Gabrielov regular morphisms  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  such that, for some  $d \geq 1$ ,  $\varphi_{\xi^{\{d\}}}$  is not Gabrielov regular (see Example 5.3 below).

**Proof of Proposition 5.1.** By Proposition 4.1(1), there exist irreducible components  $W_1, \dots, W_p$  of  $X^{\{n\}}$  such that  $S = \bigcup_{i=1}^p \varphi^{\{n\}}(W_i)$ . (Recall that  $S = B_{l+1}$  according to the notation from Section 4). We claim that  $\dim_\eta \overline{\varphi^{\{n\}}(W_j)} = n$  for some  $j \in \{1, \dots, p\}$ . Indeed, if  $\dim_\eta \overline{\varphi^{\{n\}}(W_i)} < n$  for all  $i$ , then we would have

$$n = \dim_\eta \overline{S} = \dim_\eta \overline{\bigcup_{i=1}^p \varphi^{\{n\}}(W_i)} = \max_{i=1, \dots, p} \dim_\eta \overline{\varphi^{\{n\}}(W_i)} < n,$$

a contradiction. So obtained  $W_j$  is not algebraic vertical, and it is geometric vertical, since  $\dim_\eta \varphi^{\{n\}}(W_j) \leq \dim_\eta S < n$ . ■

The proposition yields a necessary and sufficient condition for  $X_{\xi^{\{n\}}}$  to have purely geometric vertical components in the case of dominating mappings:

**COROLLARY 5.2.** ([1, Cor.3.2]) *Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a morphism of germs of complex-analytic spaces with  $X_\xi$  of pure dimension  $m$  and  $Y_\eta$  irreducible of dimension  $n$ . Assume that  $\varphi_\xi$  is dominating; i.e.,  $m - l = n$  (where  $l$  is the minimal fibre dimension). Then  $X_{\xi^{\{n\}}}$  has an isolated purely geometric vertical component if and only if (the germ at  $\eta$  of)  $Y$  equals (the germ at  $\eta$  of) the Zariski closure of  $S$ .*

**Proof.** Let  $Y$  and  $X$  be as before. If  $\dim_\eta \overline{S} < n$ , then there exists a proper locally analytic subset  $Z \subset Y$  containing  $S$  and such that  $\dim_\eta Z < n$ .

Then, for any isolated component  $W$  of  $X^{\{n\}}$ , either  $\varphi^{\{n\}}(W) \subset S$  or  $W \cap (\varphi^{\{n\}})^{-1}(Y \setminus S) \neq \emptyset$ . In the first case  $\varphi^{\{n\}}(W) \subset Z$ , so  $W$  is algebraic vertical. In the second case there exists a point  $z = (x_1, \dots, x_n) \in W$  such that  $\text{fbd}_{x_i}\varphi = l$ ,  $i = 1, \dots, n$ , hence the image under  $\varphi$  of an arbitrarily small neighbourhood of  $x_i$  has nonempty interior in  $Y$  (by Remmert's Rank Theorem and since  $m - l = n$ ). Therefore, the image under  $\varphi^{\{n\}}$  of any sufficiently small neighbourhood of  $z$  has nonempty interior in  $Y$ ; i.e.,  $W$  is not a vertical component in neither sense. ■

Next, we show an example of a Gabrielov regular mapping  $\varphi : X \rightarrow Y$ , with smooth  $X$  and  $Y$ , and such that the Zariski closure of  $S$  equals  $Y$ . By Proposition 5.1 above, the top fibred power of  $X$  over  $Y$  has an isolated irreducible purely geometric vertical component. This shows that a fibred power of a Gabrielov regular map may itself be irregular.

**EXAMPLE 5.3.** Let  $X = Y = \mathbb{C}^4$  and define  $\varphi$  as follows

$$(x, y, s, t) \mapsto (x, (x + s)y, x^2y^2e^y, x^2y^2e^{y(1+xe^y)} + st).$$

It is easy to check that the generic fibre dimension of  $\varphi$  equals 0, hence the generic rank of  $\varphi$  is 4, and so  $\varphi$  is Gabrielov regular.

On the other hand,  $S$  contains the image of the set  $\mathbb{C}^2 \times \{0\} \times \mathbb{C}$ , consisting of points  $(z_1, z_2, z_3, z_4)$  of the form  $(x, xy, x^2y^2e^y, x^2y^2e^{y(1+xe^y)})$ , whose Zariski closure equals  $\mathbb{C}^4$  (cf. [10, Ch. II, § 5]). Hence also  $\overline{S} = \mathbb{C}^4$ .

Finally, we shall prove the Gabrielov regularity criterion:

**Proof of Theorem 1.12.** Suppose first that  $\varphi|_{A_j}$  ( $j = l, \dots, k$ ) are regular. Fix a positive integer  $d$  and let  $W$  be an isolated irreducible component of  $X^{\{d\}}$ . Since the components of  $X^{\{d\}}$  are precisely the representatives of those of  $X_{\xi^{\{d\}}}^{\{d\}}$ , it suffices to show that  $\varphi^{\{d\}}|_W$  is regular.

Let  $q$  be the greatest integer for which a generic fibre  $F = F_1 \times \dots \times F_d$  of  $\varphi^{\{d\}}|_W$  contains a component  $F_s$  of dimension  $q$ . Then all the generic fibres of  $\varphi^{\{d\}}|_W$  have a  $q$ -dimensional component, and  $\varphi^{\{d\}}(W) \subset B_q = \varphi(A_q)$ . Indeed, by the upper semi-continuity, the fibre dimension can only drop in a small open neighbourhood of a given point. On the other hand, the sum  $\text{fbd}_{x_1}\varphi + \dots + \text{fbd}_{x_d}\varphi$  is already minimal along  $F$ , so all its summands must remain constant. The property of being a fibre of dimension  $q$  is an open condition on  $A_q$ . Hence  $W$  is induced by an irreducible component  $V$  of  $A_q$  with the generic fibre dimension of  $\varphi|_V$  equal to  $q$ , in the sense that there exists a component  $V$  of  $A_q$  such that  $\varphi^{\{d\}}(W) = \varphi(V)$ . By assumption,  $\dim_{\eta} \overline{\varphi(V)} = \dim_{\eta} \varphi(V)$  (where closure is in the Zariski topology in  $Y$ ), hence also  $\dim_{\eta} \overline{\varphi^{\{d\}}(W)} = \dim_{\eta} \varphi^{\{d\}}(W)$ ; i.e.,  $\varphi^{\{d\}}|_W$  is Gabrielov regular.

Suppose now that there exists  $j \in \{l, \dots, k\}$  for which  $\varphi|_{A_j}$  is not regular. We shall show that then regularity of  $\varphi_{\xi^{\{n\}}}^{\{n\}} : X_{\xi^{\{n\}}}^{\{n\}} \rightarrow Y_\eta$  fails, where  $n = \dim Y$ .

Fix  $j \in \{l, \dots, k\}$  such that  $\varphi|_{A_j}$  is not regular. Pick  $y \in B_j$  with  $\dim \varphi^{-1}(y) = j$ , and let  $Z$  be an isolated irreducible component of the fibre  $(\varphi^{\{n\}})^{-1}(y)$  of dimension  $nj$ . Let  $W$  be an isolated irreducible component of  $X^{\{n\}}$  containing  $Z$ . Then  $\varphi^{\{n\}}(W) \subset B_j$ , by Proposition 4.1(2). Moreover,  $\varphi^{\{n\}}|_W$  has no fibres of dimension less than or equal to  $n(j - 1)$ . Indeed, otherwise the generic fibre dimension of  $\varphi^{\{n\}}|_W$  would be at most  $n(j - 1)$ , so that  $\dim W \leq n(j - 1) + n = nj = \dim Z$ , and hence  $W = Z$ , a contradiction (see the proof of Proposition 4.1). Thus, a generic fibre  $F = F_1 \times \dots \times F_n$  of  $\varphi^{\{n\}}|_W$  contains a component  $F_s$  of dimension  $j$ .

Now, there is an isolated irreducible component  $V$  of  $A_j$  such that  $\dim_\eta \overline{\varphi(V)} > \dim_\eta \varphi(V)$  and the generic fibre dimension of  $\varphi|_V$  is  $j$ . Our  $y \in B_j$  can then be chosen from  $\varphi(V)$ , and  $Z$  a component of  $((\varphi|_V)^{-1}(y))^n$ . Since being a  $j$ -dimensional fibre is an open condition on  $A_j$ , then (as in the first part of the proof) we find that  $\varphi^{\{n\}}(W) = \varphi(V)$ , so that  $\dim_\eta \overline{\varphi^{\{n\}}(W)} > \dim_\eta \varphi^{\{n\}}(W)$ . Thus  $\varphi_{\xi^{\{n\}}}^{\{n\}} : X_{\xi^{\{n\}}}^{\{n\}} \rightarrow Y_\eta$  is not Gabrielov regular. ■

### 6. Homological algebra apparatus

This section is devoted to establishing the algebraic tools for the proof of our flatness criterion. First, we recall some homological properties of almost finitely generated modules (see Definition 1.6), established by Galligo and Kwieciński [9], that generalize the corresponding properties of finite modules used in Auslander’s [5]. We then generalize a lemma of Auslander [5, Lemma 3.1] to almost finitely generated modules (Lemma 6.3 below). Finally, we recall a lemma of [4] concerning zero-divisors in analytic tensor products.

Let  $R = \mathbb{C}\{y_1, \dots, y_n\}$  denote a regular local analytic  $\mathbb{C}$ -algebra of dimension  $n$ . Let  $\tilde{\otimes}_R$  denote the analytic tensor product over  $R$ , and let  $\widetilde{\text{Tor}}^R$  be the corresponding derived functor.

Let  $F$  denote an almost finitely generated  $R$ -module. We define the *flat dimension*  $\text{fd}_R(F)$  of  $F$  over  $R$  as the minimal length of an  $R$ -flat resolution of  $F$  (i.e., a resolution by  $R$ -flat modules). It is easy to see that

$$(6.1) \quad \text{fd}_R(F) = \max\{i \in \mathbb{N} : \widetilde{\text{Tor}}_i^R(F, N) \neq 0 \text{ for some } N\}.$$

Indeed, if  $M$  is an almost finitely generated  $R$ -module, then

$$(6.2) \quad M \text{ is } R\text{-flat} \Leftrightarrow \widetilde{\text{Tor}}_1^R(M, R/\mathfrak{m}_R) = 0,$$

where  $\mathfrak{m}_R$  is the maximal ideal of  $R$  (cf. [14, Prop. 6.2]). Let  $(A, \mathfrak{m}_A)$  be a

regular local  $R$ -algebra such that  $F$  is a finite  $A$ -module. Then (6.1) follows from (6.2) applied to the kernels of a minimal  $A$ -free (hence  $R$ -flat) resolution

$$\mathcal{F}_* : \dots \xrightarrow{\alpha_{i+1}} F_{i+1} \xrightarrow{\alpha_i} F_i \xrightarrow{\alpha_{i-1}} \dots \xrightarrow{\alpha_1} F_1 \xrightarrow{\alpha_0} F_0 \rightarrow F$$

of  $F$ . ( $\mathcal{F}_*$  minimal means that  $\alpha_i(F_{i+1}) \subset \mathfrak{m}_A F_i$ , for all  $i \in \mathbb{N}$ ).

The *depth*  $\text{depth}_R(F)$  of  $F$  as an  $R$ -module is defined as the length of a maximal  $F$ -sequence in  $R$  (i.e., a sequence  $a_1, \dots, a_s \in \mathfrak{m}_R$  such that  $a_j$  is not a zero-divisor in  $F/(a_1, \dots, a_{j-1})F$ , for  $j = 1, \dots, s$ ). Since all the maximal  $F$ -sequences in  $R$  have the same length, depth is well defined: As observed in [9, Lemma 2.4], the classical proof of Northcott-Rees for finitely generated modules (see, e.g., [16, §VI, Prop. 3.1]), carries over to the case of almost finitely generated modules.

**LEMMA 6.1.** *Let  $M$  and  $N$  be almost finitely generated  $R$ -modules. Then the following hold:*

- (1) Rigidity of  $\widetilde{\text{Tor}}^R$  [9, Prop. 2.2(4)]. If  $\widetilde{\text{Tor}}_{i_0}^R(M, N) = 0$  for some  $i_0 \in \mathbb{N}$ , then  $\widetilde{\text{Tor}}_i^R(M, N) = 0$  for all  $i \geq i_0$ .
- (2) Auslander–Buchsbaum-type formula [9, Thm. 2.7].

$$\text{fd}_R(M) + \text{depth}_R(M) = n.$$

- (3) Additivity of flat dimension [9, Prop. 2.10]. If  $\widetilde{\text{Tor}}_i^R(M, N) = 0$  for all  $i \geq 1$ , then

$$\text{fd}_R(M) + \text{fd}_R(N) = \text{fd}_R(M \widetilde{\otimes}_R N).$$

- (4) Verticality of  $\widetilde{\text{Tor}}^R$  (cf. [9, Prop. 4.5]). For all  $i \geq 1$ ,  $\widetilde{\text{Tor}}_i^R(M, N)$  is an almost finitely generated  $R$ -module, and every element of  $\widetilde{\text{Tor}}_i^R(M, N)$  is vertical over  $R$ .

**REMARK 6.2.** In regard to property (4) above, notice that, contrary to the finitely generated case, the analytic  $\widetilde{\text{Tor}}_i^R$  need not be torsion  $R$ -modules. It is an open problem whether they necessarily contain  $R$ -zero-divisors.

Finally, recall the following two lemmas of [4]. The proofs of these results are purely algebraic, and therefore we skip them here (detailed arguments may be found in [4]). Having said that, it is interesting to note the geometric meaning of Lemma 6.3(1) below: If  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  has no geometric vertical components in the  $n$ -fold fibred power, then there are no such components in any  $d$ -fold power, for  $d = 1, \dots, n$ .

**LEMMA 6.3.** ([4, Lemma 3.3], cf. [5, Lemma 3.1]) *Let  $A = R\{x\}$  denote a regular local analytic  $R$ -algebra,  $x = (x_1, \dots, x_m)$ . Let  $F$  be a finitely generated  $A$ -torsion-free module, and let  $N$  be a module which is finitely*

generated over  $B = A^{\tilde{\otimes}_R^j}$ , for some  $j \geq 1$ . Suppose that  $F \tilde{\otimes}_R N$  has no vertical elements over  $R$ . Then:

- (1)  $N$  has no vertical elements over  $R$ ;
- (2)  $\widetilde{\text{Tor}}_i^R(F, N) = 0$ , for all  $i \geq 1$ ;
- (3)  $\text{fd}_R(F) + \text{fd}_R(N) = \text{fd}_R(F \tilde{\otimes}_R N)$ .

**LEMMA 6.4.** ([4, Lemma 5.1]) *Let  $R = \mathbb{C}\{y_1, \dots, y_n\}$ , and let  $A$  and  $B$  be regular local analytic  $R$ -algebras. Suppose that  $M$  and  $N$  are finite  $A$ - and  $B$ -modules (respectively). Let  $g \in A$ ,  $h \in B$ , and  $m \in gM \tilde{\otimes}_R hN$  all be nonzero elements. If  $m = 0$  as an element of  $M \tilde{\otimes}_R N$ , then  $g \tilde{\otimes}_R h$  is a zero-divisor of  $gM \tilde{\otimes}_R hN$ .*

### 7. Vertical components and flatness

Our flatness criterion 1.8 follows immediately from a more general Theorem 7.1 below.

**THEOREM 7.1.** ([4, Thm. 1.10]) *Let  $R$  be a regular local analytic  $\mathbb{C}$ -algebra and let  $F$  denote an almost finitely generated  $R$ -module. Let  $n = \dim R$ . Then  $F$  is  $R$ -flat if and only if the  $n$ -fold analytic tensor power  $F^{\tilde{\otimes}_R^n}$  has no vertical elements over  $R$ .*

**Proof of Theorem 1.8.** Let  $R = \mathbb{C}\{y_1, \dots, y_n\}$ . Then  $\mathcal{O}_{Y,\eta} \cong R$  and  $\mathcal{O}_{X,\xi}$  can be viewed as an almost finitely generated  $R$ -module. (We can embed (a representative of)  $X_\xi$  in  $\mathbb{C}^m$  and identify  $X_\xi$  with the graph of  $\varphi_\xi$  in  $\mathbb{C}_\xi^m \times Y_\eta$  to realize  $\mathcal{O}_{X,\xi}$  as a quotient of  $R\{x_1, \dots, x_m\}$ .) Since  $\mathcal{O}_{X,\xi} \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \dots \tilde{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi} \cong \mathcal{O}_{X^{\{n\}}, \xi^{\{n\}}}$ , the left-hand side has no vertical elements over  $\mathcal{O}_{Y,\eta}$  if and only if the  $n$ -fold fibred power of  $X_\xi$  over  $Y_\eta$  has no geometric vertical components (by Remark 1.5). ■

We prove Theorem 7.1 by combining the analysis of vertical components with the homological algebraic tools of Auslander [5] (as extended in the previous section). The proof proceeds by induction on the fibre dimension  $m$  of  $A$  over  $R$ . The general phenomenon at the heart of the proof is that one can decompose  $F$  into two submodules, such that the homological structure of  $F$  as an  $R$ -module is inherited from that of one of the submodules.

We have a short exact sequence  $0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0$ , where  $K$  is the  $A$ -torsion submodule of  $F$ , and  $N$  is a suitably chosen  $A$ -torsion-free submodule. Then  $N$  can be treated by Auslander’s techniques of Section 6. The  $A$ -torsion module  $K$ , in turn, is supported over a subalgebra of  $A$  of strictly smaller fibre dimension over  $R$ , so it can be treated by induction. The last ingredient of the proof is to show that the vertical elements coming either from the powers of  $K$  or those of  $N$ , must embed into the corresponding powers of  $F$ .

**7.1. Proof of Theorem 7.1**

We can assume that  $F$  is an almost finitely generated module over  $R := \mathbb{C}\{y_1, \dots, y_n\}$ . By Remark 1.7, there exists  $m \geq 0$  such that  $F$  is finitely generated as a module over  $A = R\{x\}$ , where  $x = (x_1, \dots, x_m)$ . Let  $X$  and  $Y$  be connected open neighbourhoods of the origins in  $\mathbb{C}^{m+n}$  and  $\mathbb{C}^n$  (respectively), and let  $\varphi : X \rightarrow Y$  be the canonical coordinate projection. Let  $\mathcal{F}$  be a coherent sheaf of  $\mathcal{O}_X$ -modules whose stalk at the origin in  $X$  equals  $F$ , and let  $\mathcal{G}$  be a coherent  $\mathcal{O}_{X^{(n)}}$ -module whose stalk at the origin  $0^{\{n\}}$  in  $X^{\{n\}}$  equals  $F^{\otimes_R^n}$ . We can identify  $R$  with  $\mathcal{O}_{Y,0}$  and  $A$  with  $\mathcal{O}_{X,0}$ . Then  $F$  is  $R$ -flat if and only if  $\mathcal{F}_0$  is  $\mathcal{O}_{Y,0}$ -flat.

We first prove the “only if” direction of Theorem 7.1, by contradiction. Assume that  $F$  is  $R$ -flat. Since flatness is an open condition, by Douady’s theorem [6], we can assume that  $\mathcal{F}$  and  $\mathcal{G}$  are  $\mathcal{O}_Y$ -flat. Suppose that  $F^{\otimes_R^n}$  has a vertical element over  $\mathcal{O}_{Y,0}$ . In other words, (after shrinking  $X$  and  $Y$  if necessary) there exist a nonzero section  $\tilde{m} \in \mathcal{G}$  and an analytic subset  $Z \subset X^{\{n\}}$ , such that  $Z_0 = \mathcal{V}(\text{Ann}_{\mathcal{O}_{X^{\{n\}},0^{\{n\}}}}(\tilde{m}_0))$  and the image  $\varphi^{\{n\}}(Z)$  has empty interior in  $Y$ . Let  $\tilde{\varphi}$  denote the restriction  $\varphi^{\{n\}}|_Z : Z \rightarrow Y$ . Consider  $\xi \in Z$  such that the fibre dimension of  $\tilde{\varphi}$  at  $\xi$  is minimal. Then the fibre dimension  $\text{fbd}_x \tilde{\varphi}$  is constant on some open neighbourhood  $U$  of  $\xi$  in  $Z$ . By the Remmert Rank Theorem,  $\tilde{\varphi}(U)$  is locally analytic in  $Y$  near  $\eta = \tilde{\varphi}(\xi)$ . Since  $\tilde{\varphi}(Z)$  has empty interior in  $Y$ , it follows that there is a holomorphic function  $g$  in a neighbourhood of  $\eta$  in  $Y$ , such that  $(\tilde{\varphi}(U))_\eta \subset \mathcal{V}(g_\eta)$ . Therefore,  $\tilde{\varphi}_\xi^*(g_\eta) \cdot \tilde{m}_\xi = 0$  in  $\mathcal{G}_\xi$ ; i.e.,  $\mathcal{G}_\xi$  has a (nonzero) zero-divisor in  $\mathcal{O}_{Y,\eta}$ , contradicting flatness.

We will now prove the more difficult “if” direction of the theorem, by induction on  $m$ . If  $m = 0$ , then  $F$  is finitely generated over  $R$ , and the result follows from Auslander’s theorem 1.1 (because flatness of finitely generated modules over a local ring is equivalent to freeness, the analytic tensor product equals the ordinary tensor product for finite modules, and vertical elements in finite modules are just zero-divisors).

The inductive step will be divided into three cases:

- (1)  $F$  is torsion-free over  $A$ ;
- (2)  $F$  is a torsion  $A$ -module;
- (3)  $F$  is neither  $A$ -torsion-free nor a torsion  $A$ -module.

**Case (1).** We prove this case independently of the inductive hypothesis. We essentially repeat the argument of Galligo and Kwieciński [9], which itself is an adaptation of Auslander [5] to the almost finitely generated context.

Suppose that  $F^{\otimes_R^n}$  has no vertical elements over  $R$ . Then it follows from Lemma 6.3(1) that  $F^{\otimes_R^i}$  has no vertical elements, for  $i = 1, \dots, n$ . By

Lemma 6.3(3),

$$\text{fd}_R(F^{\tilde{\otimes}_R^n}) = \text{fd}_R(F) + \text{fd}_R(F^{\tilde{\otimes}_R^{n-1}}) = \dots = n \cdot \text{fd}_R(F).$$

On the other hand, since  $F^{\tilde{\otimes}_R^n}$  has no vertical elements over  $R$ , it has no zero-divisors over  $R$ , so that  $\text{depth}_R(F^{\tilde{\otimes}_R^n}) \geq 1$ . It follows from Lemma 6.1(2) that  $\text{fd}_R(F^{\tilde{\otimes}_R^n}) < n$ . Hence  $n \cdot \text{fd}_R(F) < n$ . This is possible only if  $\text{fd}_R(F) = 0$ ; i.e.,  $F$  is  $R$ -flat.

**Case (2).** Suppose that  $F$  is not  $R$ -flat. We will show that then  $F^{\tilde{\otimes}_R^n}$  contains vertical elements over  $R$ . Let  $I = \text{Ann}_A(F)$ . Since every element of  $F$  is annihilated by some nonzero element of  $A$ , and  $F$  is finitely generated over  $A$ , then  $I$  is a nonzero ideal in  $A$ . Put  $B = A/I$ ; then  $F$  is finitely generated over  $B$ . Let  $I(0)$  denote the *evaluation* of  $I$  at  $y = 0$  (i.e.,  $I(0)$  is the ideal generated by  $I$  in  $A(0) := A \tilde{\otimes}_R R/\mathfrak{m}_R \cong \mathbb{C}\{x_1, \dots, x_m\}$ ).

First suppose that  $I(0) \neq (0)$ . Then there exists  $g \in I$  such that  $g(0) := g(0, x) \neq 0$ , and  $F$  is a finite  $A/(g)A$ -module. It follows that (after an appropriate linear change in the  $x$ -coordinates)  $g$  is regular in  $x_m$  and hence, by the Weierstrass Preparation Theorem, that  $F$  is finite over  $R\{x_1, \dots, x_{m-1}\}$ . Therefore,  $F^{\tilde{\otimes}_R^n}$  has a vertical element over  $R$ , by the inductive hypothesis.

On the other hand, suppose that  $I(0) = (0)$ . Then  $B \tilde{\otimes}_R R/\mathfrak{m}_R = A(0)/I(0)$  equals  $\mathbb{C}\{x_1, \dots, x_m\}$ . Let  $Z$  be a closed analytic subspace of  $X$  such that  $\mathcal{O}_{Z,0} \cong B$ , and let  $\tilde{\varphi} := \varphi|_Z$ . It follows that the fibre  $\tilde{\varphi}^{-1}(0)$  equals  $\mathbb{C}^m$ . Of course,  $m$  is not the generic fibre dimension of  $\tilde{\varphi}$  restricted to any component, because otherwise all its fibres would equal  $\mathbb{C}^m$ , so we would have  $B = A$  and  $I = (0)$ , contrary to the choice of  $I$ . Therefore, by Proposition 4.3, there is an isolated algebraic vertical component in the  $n$ -fold fibred power of  $\tilde{\varphi}_0$ ; i.e.,  $B^{\tilde{\otimes}_R^n}$  has a zero-divisor in  $R$ . But  $F^{\tilde{\otimes}_R^n}$  is a finitely generated  $B^{\tilde{\otimes}_R^n}$ -module, so itself it has a zero-divisor (hence a vertical element) over  $R$ .

**Case (3).** Suppose that  $F$  is not  $R$ -flat,  $F$  has zero-divisors in  $A$ , but  $\text{Ann}_A(F) = (0)$ . Let

$$K := \{f \in F : af = 0 \text{ for some nonzero } a \in A\};$$

i.e.,  $K$  is the  $A$ -torsion submodule of  $F$ . Since  $K$  is a submodule of a finitely generated module over a Noetherian ring,  $K$  is finitely generated; say  $K = \sum_{i=1}^s A \cdot f_i$ . Take  $a_i \in A \setminus \{0\}$  such that  $a_i f_i = 0$ , and put  $g = a_1 \dots a_s$ . Then the sequence of  $A$ -modules

$$(7.1) \quad 0 \rightarrow K \rightarrow F \xrightarrow{g} gF \rightarrow 0$$

is exact, and  $gF$  is a torsion-free  $A$ -module.

First suppose that  $gF$  is  $R$ -flat. By applying  $\tilde{\otimes}_R K$  and  $F\tilde{\otimes}_R$  to (7.1), we get short exact sequences

$$\begin{aligned} 0 &\rightarrow K\tilde{\otimes}_R K \rightarrow F\tilde{\otimes}_R K \rightarrow gF\tilde{\otimes}_R K \rightarrow 0, \\ 0 &\rightarrow F\tilde{\otimes}_R K \rightarrow F\tilde{\otimes}_R F \rightarrow F\tilde{\otimes}_R gF \rightarrow 0. \end{aligned}$$

So we have injections

$$K\tilde{\otimes}_R K \hookrightarrow F\tilde{\otimes}_R K \hookrightarrow F\tilde{\otimes}_R F,$$

and by induction, an injection  $K\tilde{\otimes}_R^i \hookrightarrow F\tilde{\otimes}_R^i$ , for all  $i \geq 1$ . In particular,  $K\tilde{\otimes}_R^n$  is a submodule of  $F\tilde{\otimes}_R^n$ . Since  $gF$  is  $R$ -flat and  $F$  is not  $R$ -flat, it follows that  $K$  is not  $R$ -flat. Therefore, by Case (2),  $K\tilde{\otimes}_R^n$  (and hence  $F\tilde{\otimes}_R^n$ ) has a vertical element over  $R$ .

Now suppose that  $gF$  is not  $R$ -flat. Then  $(gF)\tilde{\otimes}_R^n$  has a vertical element over  $R$ , by Case (1). We will show that  $(gF)\tilde{\otimes}_R^n$  embeds into  $F\tilde{\otimes}_R^n$ , and hence so do its vertical elements. By Lemma 6.4, in order for  $(gF)\tilde{\otimes}_R^n$  to embed into  $F\tilde{\otimes}_R^n$ , it suffices to prove that  $g\tilde{\otimes}_R^n$  is not a zero-divisor of  $(gF)\tilde{\otimes}_R^n$ .

To simplify the notation, let  $B$  denote the ring  $A\tilde{\otimes}_R^n$ , and let  $h := g\tilde{\otimes}_R^n \in B$ . Since  $(gF)\tilde{\otimes}_R^n$  is a finite  $B$ -module, we can write  $(gF)\tilde{\otimes}_R^n = B^q/M$ , where  $q \geq 1$  and  $M$  is a  $B$ -submodule of  $B^q$ . Given  $b \in B$ , let  $M : b$  denote the  $B$ -submodule of  $B^q$  consisting of those elements  $m \in B^q$  for which  $b \cdot m \in M$ . Since

$$M : h \subset M : h^2 \subset \dots \subset M : h^l \subset \dots$$

is an increasing sequence of submodules of a Noetherian module  $B^q$ , it stabilizes; i.e., there exists  $k \geq 1$  such that  $M : h^{k+1} = M : h^k$ . In other words, there exists  $k \geq 1$  such that  $h$  is not a zero-divisor in  $h^k \cdot B^q/M$ ; i.e.,  $g\tilde{\otimes}_R^n$  is not a zero-divisor in  $(g^{k+1}F)\tilde{\otimes}_R^n$ .

Observe though that (because  $gF$  is  $A$ -torsion-free), multiplication by  $g$  induces an isomorphism  $gF \rightarrow g^2F$  of  $A$ -modules, and in general,  $gF \cong g^lF$ , for  $l \geq 1$ . We thus have isomorphisms  $(gF)\tilde{\otimes}_R^n \cong (g^lF)\tilde{\otimes}_R^n$  of  $B$ -modules, for  $l \geq 1$ . In particular, for every  $l \geq 1$ ,  $g\tilde{\otimes}_R^n$  is a zero-divisor of  $(gF)\tilde{\otimes}_R^n$  if and only if it is a zero-divisor of  $(g^lF)\tilde{\otimes}_R^n$ . Therefore, by Lemma 6.4, we have an embedding  $(gF)\tilde{\otimes}_R^n \hookrightarrow F\tilde{\otimes}_R^n$ . This completes the proof of Theorem 7.1. ■

### 7.2. Proof of Theorem 1.9

In light of Remark 1.3, the “only if” implication of the theorem is automatic (without any assumptions on  $\varphi_\xi$ ). Indeed, if  $\mathcal{O}_{X,\xi}$  is  $\mathcal{O}_{Y,\eta}$ -flat, then so are all its analytic tensor powers (as flatness is preserved by any base change and compositions of flat mappings are flat). In particular, the  $n$ -fold power  $\mathcal{O}_{X,\xi}^{\tilde{\otimes}_n \mathcal{O}_{Y,\eta}} \cong \mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$  is flat, hence torsion-free, over  $\mathcal{O}_{Y,\eta}$ .

For the proof of the “if” direction we will proceed in four cases, according to the conditions in Theorem 1.9.

**Case (1).** Suppose that  $\dim Y_\eta = n < 3$  and  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is not flat. Then, by Theorem 1.8, there exists a geometric vertical component in  $X_{\xi\{n\}}^{\{n\}}$ . We will show that the restriction of  $\varphi_{\xi\{n\}}^{\{n\}}$  to this component is Gabrielov regular, and hence the component is algebraic vertical.

In fact, every morphism  $\psi_\zeta : Z_\zeta \rightarrow Y_\eta$  of germs of analytic spaces with target of dimension less than 3 is regular: By the Remmert Rank Theorem the Gabrielov-non-regular locus  $\text{NR}(\psi)$  is contained in the locus of non-generic fibre dimension of  $\psi$ , and thus the image of  $\text{NR}(\psi)$  is of codimension (at least) two in the image  $\psi(Z)$ . If  $Z_\zeta$  is geometric vertical, then the image  $\psi(Z)$  is already of codimension (at least) one in  $Y$ , and hence

$$\dim \psi(\text{NR}(\psi)) \leq \dim Y - 1 - 2 \leq -1.$$

Thus,  $\psi(\text{NR}(\psi)) = \emptyset$  and  $\psi$  is Gabrielov regular.

**Case (2).** Suppose that  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  is a non-flat Nash morphism of Nash germs. By Theorem 1.8 again, there exists a geometric vertical component in  $X_{\xi\{n\}}^{\{n\}}$ . Such a component is necessarily a Nash germ itself, by [2, Prop. 5.1], and hence the restriction of  $\varphi_{\xi\{n\}}^{\{n\}}$  to this component is Gabrielov regular, by [23, Thm. 2.10] and Chevalley Theorem [20, Ch. VII, § 8.3].

**Case (3).** Suppose now that  $X_\xi$  is pure-dimensional,  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  maps the singular locus of  $X_\xi$  into a proper analytic subgerm  $Z_\eta$  of  $Y_\eta$ , and  $X_{\xi\{n\}}^{\{n\}}$  has no algebraic vertical components. In particular, there are no *isolated* algebraic vertical components in  $X_{\xi\{n\}}^{\{n\}}$ , and hence  $\varphi_\xi$  is open, by Theorem 1.10.

Openness being an open condition, we can extend  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  to an open holomorphic mapping  $\varphi : X \rightarrow Y$  of analytic spaces, where  $Y$  is smooth of dimension  $n$ . Moreover, this can be done so that  $Z_\eta$  extends to a proper analytic subset  $Z$  of  $Y$  with  $\varphi(X^{\text{sing}}) \subset Z$ . We may now conclude that  $\varphi$  is flat over  $Y \setminus Z$ , as for a mapping of smooth spaces openness is equivalent to flatness (see, e.g., [8, Prop. 3.20]). Hence also  $\varphi^{\{d\}} : X^{\{d\}} \rightarrow Y$  is flat over  $Y \setminus Z$  for every  $d \geq 1$ , because this is so locally.

For simplicity of notation, put  $R := \mathcal{O}_{Y,\eta}$ ,  $M := \mathcal{O}_{X,\xi}$ , and  $N := \mathcal{O}_{X^{\{n-1\}},\xi^{\{n-1\}}}$ . We shall show that  $\widetilde{\text{Tor}}_i^R(M, N) = 0$  for all  $i \geq 1$ . By torsion-freeness of  $M \widetilde{\otimes}_R N = \mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$  and injectivity of the morphism

$$M \ni m \mapsto m \widetilde{\otimes}_R 1 \widetilde{\otimes}_R \dots \widetilde{\otimes}_R 1 \in M \widetilde{\otimes}_R N,$$

we get that  $M$  is torsion-free over  $R$ . Then, by [9, Lem. 5.2], we have an

exact sequence of almost finitely generated  $R$ -modules

$$0 \rightarrow M \rightarrow F \rightarrow F/M \rightarrow 0,$$

where  $F$  is  $R$ -flat. After tensoring with  $N$ , we get an exact sequence

$$0 \rightarrow \widetilde{\text{Tor}}_1^R(F/M, N) \xrightarrow{\lambda} M \tilde{\otimes}_R N \rightarrow F \tilde{\otimes}_R N \rightarrow F/M \tilde{\otimes}_R N \rightarrow 0$$

and isomorphisms

$$(7.2) \quad \widetilde{\text{Tor}}_{i+1}^R(F/M, N) \cong \widetilde{\text{Tor}}_i^R(M, N) \quad \text{for all } i \geq 1.$$

Flatness of  $\varphi^{\{n-1\}} : X^{\{n-1\}} \rightarrow Y$  over  $Y \setminus Z$  implies that  $\widetilde{\text{Tor}}_1^R(F/M, N)$  is supported over  $Z_\eta$ ; i.e.,  $m \cdot r = 0$  for every  $m \in \widetilde{\text{Tor}}_1^R(F/M, N)$  and  $r \in \mathfrak{J}$ , where  $\mathfrak{J} \subset R$  is the ideal of  $Z_\eta$ . Since  $Z_\eta \subsetneq Y_\eta$ , it follows that there is a nonzero  $r \in R$  such that  $r \cdot \lambda(m) = 0$  for every  $m \in \widetilde{\text{Tor}}_1^R(F/M, N)$ . Hence  $\widetilde{\text{Tor}}_1^R(F/M, N) = 0$ , by injectivity of  $\lambda$  and torsion-freeness of  $M \tilde{\otimes}_R N$ . Consequently,  $\widetilde{\text{Tor}}_{i+1}^R(F/M, N) = 0$ , by rigidity of  $\widetilde{\text{Tor}}^R$  (Lemma 6.1(1)), and so  $\widetilde{\text{Tor}}_i^R(M, N) = 0$  for all  $i \geq 1$ , by (7.2).

Therefore, as in the proof of Case (1) of Theorem 7.1 above,

$$n - 1 \geq \text{fd}_R(M \tilde{\otimes}_R N) = \text{fd}_R(M) + \text{fd}_R(N) = \dots = n \cdot \text{fd}_R(M),$$

by Lemma 6.3(3), so that  $\text{fd}_R(M) = 0$ ; i.e.,  $\mathcal{O}_{X,\xi}$  is  $\mathcal{O}_{Y,\eta}$ -flat.

**Case (4).** Let  $\varphi_\xi : X_\xi \rightarrow Y_\eta$  be a morphism of germs of analytic spaces, where  $\mathcal{O}_{X,\xi}$  is Cohen–Macaulay and  $\mathcal{O}_{Y,\eta}$  is regular. Then, by [8, Prop. 3.20],  $\varphi_\xi$  is flat if and only if it is open. Hence, if there are no algebraic vertical components in  $\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$ , the flatness of  $\varphi_\xi$  follows from Theorem 1.10. ■

### 8. Algebraic case

In this final section, we present the algebraic analogues of our openness and flatness criteria (Theorems 1.10 and 7.1). Although these are not exactly within the main scope of this survey, Theorems 8.1 and 8.2 below provide important motivation for our main work in that they allow for actual *computation* of openness and flatness, as explained in Remark 8.4.

**THEOREM 8.1.** *Let  $\varphi : X \rightarrow Y$  be a polynomial mapping of affine algebraic varieties (over  $\mathbb{C}$ ). Assume that  $Y$  is (analytically) locally irreducible, of dimension  $n$ . Then  $\varphi$  is open if and only if the coordinate ring  $A(X^{\{n\}})$  of the  $n$ -fold fibred power  $X^{\{n\}}$  is a torsion-free  $A(Y)$ -module.*

The following flatness criterion generalizes Vasconcelos’s conjecture [24, Conj. 6.2] in the case of  $\mathbb{C}$ -algebras.

**THEOREM 8.2.** ([4, Thm. 1.3]) *Let  $R$  be a regular  $\mathbb{C}$ -algebra of finite type. Let  $A$  denote an  $R$ -algebra of finite type and let  $F$  denote a finitely generated*

*A*-module. If  $\mathfrak{p} \subset A$  is a prime ideal, then the localization  $F_{\mathfrak{p}}$  is  $R$ -flat if and only if the  $n$ -fold tensor power  $F_{\mathfrak{p}}^{\otimes_R^n}$  is a torsion-free  $R$ -module, where  $n = \dim R$ .

Equivalently, if  $\mathfrak{p} \subset A$  is a prime ideal and  $\mathfrak{q} = \mathfrak{p} \cap R$ , then  $F_{\mathfrak{p}}$  is  $R_{\mathfrak{q}}$ -flat if and only if  $F_{\mathfrak{p}}^{\otimes_R^n}$  is  $R_{\mathfrak{q}}$ -torsion-free.

**COROLLARY 8.3.** *With the assumptions of Theorem 8.2,  $F$  is  $R$ -flat if and only if  $F^{\otimes_R^n}$  is  $R$ -torsion-free.*

**REMARK 8.4.** By Corollary 8.3, and the prime-avoidance lemma [7, Lemma 3.3], in order to verify that  $F$  is not  $R$ -flat, it is enough to find an associated prime of  $F^{\otimes_R^n}$  in  $A^{\otimes_R^n}$  which contains a nonzero element  $r \in R$ . Similarly, by Theorem 8.1, to verify non-openness of a polynomial map  $\varphi : X \rightarrow Y$ , it is enough to find a *minimal* associated prime of  $A(X)^{\otimes_{A(Y)}^n}$  which contains a nonzero element  $r \in A(Y)$ .

In other words, the study of openess and flatness in the algebraic case, boils down to showing that the contraction of a certain prime ideal to the base ring is non-trivial. Thus Theorems 8.2 and 8.1 together with Gröbner-basis algorithms for primary decomposition (see [25] or [11]) provide tools for checking flatness and openess by effective computation.

**Proof of Theorem 8.1.** For simplicity of notation, write  $R := A(Y)$  and  $S := A(X)$ . First suppose that  $\varphi$  is open. Then it is universally open (by [12, Cor. 14.4.3]), and hence all its fibred powers are open, too. In particular, the canonical mapping  $\varphi^{\{n\}} : X^{\{n\}} \rightarrow Y$  is dominating, when restricted to every *isolated* irreducible component of  $X^{\{n\}}$ . Consequently, the minimal associated primes of  $S^{\otimes_R^n}$  contract to zero in  $R$ ; i.e., the coordinate ring  $A(X^{\{n\}}) = (S^{\otimes_R^n})_{\text{red}}$  is a torsion-free  $R$ -module.

Now suppose that  $\varphi$  is not open. Then it is not open at some point  $\xi \in X$ , and so  $\varphi_{\xi} : X_{\xi} \rightarrow Y_{\eta}$  is a non-open morphism of germs of analytic spaces. By Theorem 1.10 and Remark 1.11, the reduced local ring  $(\mathcal{O}_{X^{\{n\}}, \xi^{\{n\}}})_{\text{red}}$  has a zero-divisor in  $\mathcal{O}_{Y, \eta}$ , where  $\eta = \varphi(\xi)$ . Hence  $(S^{\otimes_R^n})_{\text{red}}$  is not torsion-free over  $R$ . ■

**Proof of Theorem 8.2.** If  $F_{\mathfrak{p}}$  is  $R$ -flat, then  $F_{\mathfrak{p}}^{\otimes_R^k}$  is  $R$ -flat and therefore  $R$ -torsion-free, for all  $k$ .

On the other hand, flatness and torsion-freeness are both local properties. I.e.,  $F_{\mathfrak{p}}$  is  $R$ -flat (respectively,  $R$ -torsion-free) if and only if  $F_{\mathfrak{m}}$  is  $R$ -flat (respectively,  $R$ -torsion-free), for every maximal ideal  $\mathfrak{m}$  of  $A$  containing  $\mathfrak{p}$ . Suppose that  $F_{\mathfrak{p}}$  is not  $R$ -flat. Then there is a maximal ideal  $\mathfrak{m}$  containing  $\mathfrak{p}$  such that  $F_{\mathfrak{m}}$  is not  $R$ -flat, and it suffices to prove that  $F_{\mathfrak{m}}^{\otimes_R^n}$  is not  $R$ -torsion-free. Let  $\varphi : X \rightarrow Y$  be the morphism of complex-analytic spaces

associated to the morphism  $\text{Spec } A \rightarrow \text{Spec } R$  and let  $\mathcal{F}$  be the coherent sheaf of  $\mathcal{O}_X$ -modules associated to  $F$ . Let  $\xi \in X$  be the point corresponding to the maximal ideal  $\mathfrak{m}$  of  $\text{Spec } A$ . It follows from faithful flatness of completion that  $\mathcal{F}_\xi$  is not  $\mathcal{O}_{Y,\eta}$ -flat, where  $\eta = \varphi(\xi)$ . By Theorem 7.1,  $\mathcal{F}_\xi^{\otimes^n_{\mathcal{O}_{Y,\eta}}}$  has a vertical element over  $\mathcal{O}_{Y,\eta}$ . Since  $\varphi^{\{n\}}$  is the holomorphic map induced by the ring homomorphism  $R \rightarrow A^{\otimes^n_R}$ , it follows from Chevalley's Theorem that  $\mathcal{F}_\xi^{\otimes^n_{\mathcal{O}_{Y,\eta}}}$  has a zero-divisor in  $\mathcal{O}_{Y,\eta}$ . Then  $F_{\mathfrak{m}}^{\otimes^n_R}$  has a zero-divisor in  $R$ ; i.e.,  $F_{\mathfrak{m}}^{\otimes^n_R}$  is not  $R$ -torsion-free. ■

### References

- [1] J. Adamus, *Vertical components in fibre powers of analytic spaces*, J. Algebra 272 (2004), 394–403.
- [2] J. Adamus, *Vertical components and flatness of Nash mappings*, J. Pure Appl. Algebra 193 (2004), 1–9.
- [3] J. Adamus, *Flatness testing and torsion freeness of analytic tensor powers*, J. Algebra 289 (2005), 148–160.
- [4] J. Adamus, E. Bierstone, P. D. Milman, *Geometric Auslander criterion for flatness of an analytic mapping*, preprint arXiv:0901.2744v2 (2009), 21 pages.
- [5] M. Auslander, *Modules over unramified regular local rings*, Illinois J. Math. 5 (1961), 631–647.
- [6] A. Douady, *Le problème des modules pour les sous-espaces analytiques compacts d'un espace donné*, Ann. Inst. Fourier 16:1 (1966), 1–98.
- [7] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Grad. Texts in Math., Vol. 150, Springer, New York, 1995.
- [8] G. Fischer, *Complex analytic geometry, Lecture Notes in Math.* 538, Springer, Berlin, Heidelberg, New York, 1976.
- [9] A. Galligo, M. Kwieciński, *Flatness and fibred powers over smooth varieties*, J. Algebra 232 (2000), 48–63.
- [10] H. Grauert, R. Remmert, *Analytische Stellenalgebren*, Springer, Berlin, Heidelberg, New York, 1971.
- [11] G.-M. Greuel, G. Pfister, *A Singular Introduction to Commutative Algebra*, Second edition, Springer, Berlin, 2008.
- [12] A. Grothendieck, *Éléments de géométrie algébrique IV. Etudes locale des schémas et des morphismes de schémas*, Publ. Math. I.H.E.S. 28, 1966.
- [13] R. Hartshorne, *Algebraic Geometry*, Springer, New York, 1977.
- [14] H. Hironaka, *Stratification and flatness, Real and Complex Singularities*, Proc. Oslo 1976, ed. Per Holm, Sijthoff and Noordhoff (1977), 199–265.
- [15] R. Kiehl, *Äquivalenzrelationen in analytischen Räumen*, Math. Z. 105 (1968), 1–20.
- [16] E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, Basel, Boston, Berlin, 1985.
- [17] M. Kwieciński, *Flatness and fibred powers*, Manuscripta Math. 97 (1998), 163–173.
- [18] M. Kwieciński, P. Tworzewski, *Fibres of analytic maps*, Bull. Polish Acad. Sci. Math. 47 (1999), 45–55.

- [19] S. Lichtenbaum, *On the vanishing of Tor in regular local rings*, Illinois J. Math. 10 (1966), 220–226.
- [20] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser, Basel, Boston, Berlin, 1991.
- [21] W. Pawłucki, *On Gabrielov's regularity condition for analytic mappings*, Duke Math. J. 65 (1992), 299–311.
- [22] J.-P. Serre, *Local Algebra*, Springer Monographs in Math., Springer, Berlin, 2000.
- [23] P. Tworzewski, *Intersections of analytic sets with linear subspaces*, Ann. Soc. Norm. Super. Pisa 17 (1990), 227–271.
- [24] W. V. Vasconcelos, *Flatness testing and torsionfree morphisms*, J. Pure Appl. Algebra 122 (1997), 313–321.
- [25] W. V. Vasconcelos, *Computational methods in commutative algebra and algebraic geometry, Algorithms and Computation in Mathematics*, Vol. 2, Springer, Berlin, Heidelberg, 1998.

DEPARTMENT OF MATHEMATICS  
THE UNIVERSITY OF WESTERN ONTARIO  
LONDON, ONTARIO N6A 5B7 CANADA

INSTITUTE OF MATHEMATICS  
JAGIELLONIAN UNIVERSITY  
ul. Łojasiewicza 6  
30-348 KRAKÓW, POLAND  
E-mail: jadamus@uwo.ca

*Received January 27, 2010.*