



Finite Determinacy and Stability of Flatness of Analytic Mappings

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Abstract. It is proved that flatness of an analytic mapping germ from a complete intersection is determined by its sufficiently high jet. As a consequence, one obtains finite determinacy of complete intersections. It is also shown that flatness and openness are stable under deformations.

1 Introduction

When dealing with singularities of analytic sets or mappings, particularly in explicit calculations, one is often tempted to forget the original infinite transcendental data and to work instead with its (sufficiently long) Taylor truncation. This approach is satisfactory in many circumstances. For example, the Milnor number of an isolated hypersurface singularity can be correctly calculated this way. In general, however, local analytic invariants of a given singularity may differ from those of its Taylor approximations of arbitrary length (*cf.* Example 3.5 and Remark 3.6).

In this paper we show that, roughly speaking, those algebro-geometric properties of an analytic mapping-germ $\phi = (\phi_1, \dots, \phi_n)$ that depend on the variation of its fibres are already determined by Taylor polynomials of ϕ_1, \dots, ϕ_n of sufficiently large degree.

1.1 Main Results

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $x = (x_1, \dots, x_m)$ and let \mathfrak{m}_x denote the maximal ideal in the ring of convergent power series $\mathbb{K}\{x\}$. For a natural number $k \in \mathbb{N}$ and a power series $f \in \mathbb{K}\{x\}$, the k -jet of f , denoted $j^k f$, is the image of f under the canonical epimorphism $\mathbb{K}\{x\} \rightarrow \mathbb{K}\{x\}/\mathfrak{m}_x^{k+1}$. For an s -tuple $\phi = (\phi_1, \dots, \phi_s) \in \mathbb{K}\{x\}^s$, we set $j^k \phi = (j^k \phi_1, \dots, j^k \phi_s)$.

An analytic mapping $\phi: X \rightarrow Y$ is called *flat* at a point $\xi \in X$ when the pullback homomorphism $\phi_\xi^*: \mathcal{O}_{Y, \phi(\xi)} \rightarrow \mathcal{O}_{X, \xi}$ makes $\mathcal{O}_{X, \xi}$ into a flat $\mathcal{O}_{Y, \phi(\xi)}$ -module. One of the major problems considered in this paper is whether flatness of a \mathbb{K} -analytic mapping germ is finitely determined (*i.e.*, determined by its k -jet for k large enough). We prove that this is indeed the case for mappings from complete intersections. More precisely, we have the following theorem.

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Theorem 1.1 *Let X be a \mathbb{K} -analytic subspace of \mathbb{K}^m . Suppose that $0 \in X$ and the local ring $\mathcal{O}_{X,0}$ is a complete intersection. If $\phi = (\phi_1, \dots, \phi_n): X \rightarrow \mathbb{K}^n$, $\phi(0) = 0$, is a \mathbb{K} -analytic mapping, then there exists $\mu_0 \geq 1$ such that the following conditions are equivalent:*

- (i) ϕ is flat at zero.
- (ii) For every $\mu \geq \mu_0$, $j^\mu \phi = (j^\mu \phi_1, \dots, j^\mu \phi_n): X \rightarrow \mathbb{K}^n$ is flat at zero.
- (iii) There exists $\mu \geq \mu_0$ such that $j^\mu \phi = (j^\mu \phi_1, \dots, j^\mu \phi_n): X \rightarrow \mathbb{K}^n$ is flat at zero.

The above theorem implies, in particular, *finite determinacy of complete intersections in $\mathbb{K}\{x\}$* : An s -tuple (f_1, \dots, f_s) forms a regular sequence in $\mathbb{K}\{x\}$ if and only if this is so for every $(g_1, \dots, g_s) \in \mathbb{K}\{x\}^s$ with $j^{\mu_0} g_i = j^{\mu_0} f_i$, $i = 1, \dots, s$ (Corollary 4.8). As a consequence, we obtain that $\phi: X \rightarrow \mathbb{K}^n$ is a flat mapping if and only if every $\psi: X \rightarrow \mathbb{K}^n$ satisfying $j^{\mu_0} \psi = j^{\mu_0} \phi$ is flat as well (Theorem 4.9).

Independently, one can prove an analogue of Theorem 1.1 for finite mappings. This is our Theorem 4.3, which requires no assumptions on the source X . Interestingly, Theorem 4.3 cannot be generalized to finitely generated modules. That is, in general, flatness of a finitely generated $\mathbb{K}\{x\}$ -module is not finitely determined (Example 4.5).

Another problem considered here is that of stability of flatness under deformations. Recall that given a morphism $\phi: X \rightarrow Y$ of \mathbb{K} -analytic spaces and a pointed space $(T, 0)$ (i.e., an analytic space with a distinguished point 0), a *deformation of ϕ over Y* is a Cartesian diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \phi \downarrow & & \downarrow \Phi \\ Y & \longrightarrow & Y \times T \\ \downarrow & & \downarrow \pi \\ \{0\} & \longrightarrow & T, \end{array}$$

such that \mathfrak{X} is flat over T (see, e.g., [6]). For $\theta \in T$, we denote by Φ^θ the specialization of Φ over θ , that is, the pullback of Φ by $Y \times \{\theta\} \hookrightarrow Y \times T$.

We prove that flatness of mappings into locally irreducible targets is stable under deformations.

Theorem 1.2 *Let $\phi: X \rightarrow Y$ be a flat morphism of \mathbb{K} -analytic spaces, with Y locally irreducible. If Φ is a deformation of ϕ parametrized by a locally irreducible T , then Φ^θ is a flat mapping for every $\theta \in T$ near zero.*

When $\mathbb{K} = \mathbb{C}$, we also have an analogous result about stability of openness (Proposition 6.3 and Corollary 6.4).

1.2 Plan of the Paper

Our main tool here is Hironaka's diagram of initial exponents. We recall this notion and its relevance to flatness in the next section. Section 3 is devoted to approximation

of the diagram of a given ideal in $\mathbb{K}\{x\}$ by the diagrams of its Taylor approximations. Theorems 1.1 and 1.2 are proved in Sections 4 and 5, respectively. The last section is devoted to the complex case. In the complex setting, we prove geometric analogues of our main results.

2 Diagram of Initial Exponents and Flatness

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $A = \mathbb{K}\{y\}/J$ be a local analytic \mathbb{K} -algebra, where $y = (y_1, \dots, y_n)$ and $J \subset \mathbb{K}\{y\}$ is a proper ideal. Let $x = (x_1, \dots, x_m)$ and define

$$A\{x\} := \mathbb{K}\{y, x\}/J \cdot \mathbb{K}\{y, x\}.$$

We will write x^β for $x_1^{\beta_1} \cdots x_m^{\beta_m}$, where $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$. By the canonical embedding $A\{x\} \hookrightarrow A[[x]]$, one can regard the elements of $A\{x\}$ as power series with coefficients in A .

Let \mathfrak{m}_A denote the maximal ideal of A . For a power series $F = \sum_{\beta \in \mathbb{N}^m} f_\beta x^\beta \in A\{x\}$, define its *evaluation at 0* as

$$F(0) = \sum_{\beta \in \mathbb{N}^m} f_\beta(0) x^\beta \in A/\mathfrak{m}_A\{x\} = \mathbb{K}\{x\},$$

and for an ideal I in $A\{x\}$ define $I(0) = \{F(0) : F \in I\}$, the *evaluated ideal*.

For $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{N}^m$, we write $|\beta| = \beta_1 + \cdots + \beta_m$ and call it the *length* of β . We define a total ordering of \mathbb{N}^m by lexicographic ordering of the $(m+1)$ -tuples $(|\beta|, \beta_1, \dots, \beta_m)$, where $\beta = (\beta_1, \dots, \beta_m)$. The *support* of $F = \sum_{\beta \in \mathbb{N}^m} f_\beta x^\beta$ is defined as $\text{supp}(F) = \{\beta \in \mathbb{N}^m : f_\beta \neq 0\}$, and $\exp(F) = \min\{\beta \in \text{supp}(F)\}$ denotes the *initial exponent* of F . Similarly,

$$\text{supp}(F(0)) = \{\beta \in \mathbb{N}^m : f_\beta(0) \neq 0\} \quad \text{and} \quad \exp(F(0)) = \min\{\beta \in \text{supp}(F(0))\},$$

for the evaluated series. Of course, $\text{supp}(F(0)) \subset \text{supp}(F)$.

If $\beta^* = \exp(F)$ is the initial exponent of $F = \sum f_\beta x^\beta$, then the “monomial” $f_{\beta^*} x^{\beta^*} \in A\{x\}$ is called the *initial term* of F and denoted in (F) .

Given an ideal I in $A\{x\}$, we denote by $\mathfrak{N}(I)$ the *diagram of initial exponents* of I , that is,

$$\mathfrak{N}(I) = \{\exp(F) : F \in I \setminus \{0\}\}.$$

Similarly, for the evaluated ideal $I(0)$, we set

$$\mathfrak{N}(I(0)) = \{\exp(F(0)) : F \in I, F(0) \neq 0\}.$$

Note that every diagram $\mathfrak{N}(I)$ satisfies the equality $\mathfrak{N}(I) + \mathbb{N}^m = \mathfrak{N}(I)$. (Indeed, for $\beta \in \mathfrak{N}(I)$ and $\gamma \in \mathbb{N}^m$, one can choose $F \in I$ such that $\exp(F) = \beta$; then $x^\gamma F \in I$, and hence $\beta + \gamma = \exp(x^\gamma F)$ is in $\mathfrak{N}(I)$.)

Remark 2.1 It is not difficult to show that, for every ideal I , there exists a unique smallest (finite) set $V(I) \subset \mathfrak{N}(I)$ such that $V(I) + \mathbb{N}^m = \mathfrak{N}(I)$ (see, e.g., [3, Lem. 3.8]). The elements of $V(I)$ are called the *vertices* of the diagram $\mathfrak{N}(I)$.

We now recall Hironaka’s combinatorial criterion that expresses flatness in terms of the diagram of initial exponents. For an ideal I in $A\{x\}$, set $\Delta = \mathbb{N}^m \setminus \mathfrak{N}(I(0))$, and define $A\{x\}^\Delta = \{F \in A\{x\} : \text{supp}(F) \subset \Delta\}$. Consider the canonical projection

$A\{x\} \rightarrow A\{x\}/I$ and its restriction to $A\{x\}^\Delta$, called κ . The following two results will be used throughout the paper.

Proposition 2.2 ([7, §6, Prop. 9]) *The mapping $\kappa: A\{x\}^\Delta \rightarrow A\{x\}/I$ is surjective.*

Proposition 2.3 ([7, §6, Prop. 10]) *The ring $A\{x\}/I$ is flat as an A -module if and only if κ is bijective.*

3 Approximation of Diagrams

Let $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let $x = (x_1, \dots, x_m)$ and let \mathfrak{m}_x denote the maximal ideal of $\mathbb{K}\{x\}$. Recall that, for a natural number $k \in \mathbb{N}$ and a power series $f \in \mathbb{K}\{x\}$, the k -jet of f (denoted $j^k f$) is the image of f under the canonical epimorphism $\mathbb{K}\{x\} \rightarrow \mathbb{K}\{x\}/\mathfrak{m}_x^{k+1}$.

In this section we study the relations between the diagram of initial exponents of a given ideal in $\mathbb{K}\{x\}$ and those of its Taylor approximations. Throughout this section, we will use the following notation: Let f_1, \dots, f_s be a finite collection of power series in $\mathbb{K}\{x\}$ and let

$$I = (f_1, \dots, f_s) \cdot \mathbb{K}\{x\}.$$

For a natural number μ , let I_μ denote the ideal generated by the μ -jets $j^\mu f_i$, $i = 1, \dots, s$, that is,

$$I_\mu = (j^\mu f_1, \dots, j^\mu f_s) \cdot \mathbb{K}\{x\}.$$

The following simple observation will be used often in our considerations.

Remark 3.1 Given a power series $F \in \mathbb{K}\{x\}$, suppose that $\mu \geq |\exp(F)|$. Then

$$\exp(F) = \exp(G)$$

for every $G \in \mathbb{K}\{x\}$ with $j^\mu G = j^\mu F$.

We now show the connection between the diagram of initial exponents of I and those of its approximations I_μ .

Lemma 3.2 *Let I and $\{I_\mu\}_{\mu \in \mathbb{N}}$ be as above. Let l_0 be the maximum of lengths of vertices of the diagram $\mathfrak{N}(I)$. Then:*

- (i) $\mathfrak{N}(I_\mu) \supset \mathfrak{N}(I)$ for all $\mu \geq l_0$;
- (ii) for every $l \geq l_0$,

$$\mathfrak{N}(I_\mu) \cap \{\beta \in \mathbb{N}^m : |\beta| \leq l\} = \mathfrak{N}(I) \cap \{\beta \in \mathbb{N}^m : |\beta| \leq l\},$$

for all $\mu \geq l$.

Proof Fix $\mu \geq l_0$. By Remark 2.1, for the proof of (i) it suffices to show that the vertices of $\mathfrak{N}(I)$ are contained in $\mathfrak{N}(I_\mu)$. Let $F_1, \dots, F_q \in I$ be any set of representatives of the vertices of $\mathfrak{N}(I)$. We can write $F_k = \sum_{i=1}^s g_{ki} f_i$, $k = 1, \dots, q$, for some $g_{ki} \in \mathbb{K}\{x\}$. Then

$$j^\mu F_k = j^\mu \left(\sum_{i=1}^s g_{ki} f_i \right) = j^\mu \left(\sum_{i=1}^s g_{ki} \cdot j^\mu f_i \right),$$

since the power series of a product up to order μ depends only on the power series up to order μ of its factors. Hence, by Remark 3.1, we have equality of the initial exponents $\exp(F_k) = \exp(\sum_{i=1}^s g_{ki} \cdot j^\mu f_i)$. It follows that $\exp(F_k) \in \mathfrak{N}(I_\mu)$ for all k , which proves (i).

For the proof of part (ii), fix $l \geq l_0$. It now suffices to show that

$$\mathfrak{N}(I_\mu) \cap \{\beta \in \mathbb{N}^m : |\beta| \leq l\} \subset \mathfrak{N}(I) \cap \{\beta \in \mathbb{N}^m : |\beta| \leq l\},$$

for every $\mu \geq l$. Pick $\beta^* \in \mathbb{N}^m \setminus \mathfrak{N}(I)$ with $|\beta^*| \leq l$. Suppose that $\beta^* \in \mathfrak{N}(I_\mu)$ for some $\mu \geq l$. Then one can choose $G \in I_\mu$ with $\exp(G) = \beta^*$. Write $G = \sum_{i=1}^s g_i \cdot j^\mu f_i$ for some $g_i \in \mathbb{K}\{x\}$. We have $j^\mu G = j^\mu (\sum_{i=1}^s g_i \cdot j^\mu f_i) = j^\mu (\sum_{i=1}^s g_i f_i)$, and since $|\exp(G)| = |\beta^*| \leq \mu$, it follows that $\exp(G) = \exp(\sum_{i=1}^s g_i f_i)$, by Remark 3.1 again. Therefore, $\beta^* \in \mathfrak{N}(I)$, which is a contradiction. ■

Corollary 3.3 *Let I and $\{I_\mu\}_{\mu \in \mathbb{N}}$ be as above. Then*

$$\mathfrak{N}(I) = \bigcap_{\mu \geq l_0} \mathfrak{N}(I_\mu),$$

where l_0 is the maximum of lengths of vertices of the diagram $\mathfrak{N}(I)$.

Proof By Lemma 3.2(i), we have $\mathfrak{N}(I) \subset \bigcap_{\mu \geq l_0} \mathfrak{N}(I_\mu)$. On the other hand, by part (ii) of the lemma, if $\beta^* \in \mathbb{N}^m \setminus \mathfrak{N}(I)$, then $\beta^* \notin \mathfrak{N}(I_\mu)$ for all $\mu \geq \max\{|\beta^*|, l_0\}$. ■

Corollary 3.4 *Let I and $\{I_\mu\}_{\mu \in \mathbb{N}}$ be as above. Let l_0 be the maximum of lengths of vertices of the diagram $\mathfrak{N}(I)$. If I contains a k 'th power of the maximal ideal \mathfrak{m}_x , then $k \geq l_0$ and $\mathfrak{N}(I_\mu) = \mathfrak{N}(I)$ for all $\mu \geq k$.*

Proof If $I \supset \mathfrak{m}_x^k$ for some k , then $\mathfrak{N}(I) \supset \mathfrak{N}(\mathfrak{m}_x^k)$. Hence, $l_0 \leq k$ and

$$(\mathbb{N}^m \setminus \mathfrak{N}(I)) \cap \{\beta \in \mathbb{N}^m : |\beta| \leq l\} = \mathbb{N}^m \setminus \mathfrak{N}(I),$$

for all $l \geq k$. The statement thus follows from Lemma 3.2(ii). ■

It is important to observe that, in general, there need not be equality between the diagrams of I and I_μ , for μ arbitrarily large. This is shown in the following example.

Example 3.5 Let I be an ideal in $\mathbb{K}\{x, y\}$ generated by f_1 and f_2 of the form

$$\begin{aligned} f_1 &= x^3 y + x y^4 + x y^5 + x y^6 + \cdots, \\ f_2 &= x^2 y^3 + y^6 + y^7 + y^8 + \cdots. \end{aligned}$$

Then, for every $\mu \geq 5$, we have $y^2 \cdot j^\mu f_1 - x \cdot j^\mu f_2 = x y^{\mu+1}$, hence $(1, \mu+1) \in \mathfrak{N}(I_\mu)$. However, $(1, k) \notin \mathfrak{N}(I)$ for any $k \geq 1$.

We prove the latter by contradiction. Suppose there exists $F \in I$ with $\exp(F) = (1, k_0)$ for some $k_0 \in \mathbb{N}$. Choose $h_1, h_2 \in \mathbb{K}\{x, y\}$ so that $F = h_1 f_1 + h_2 f_2$. Let $a x^{\alpha_1} y^{\alpha_2}$ and $b x^{\beta_1} y^{\beta_2}$ be the initial terms of h_1 and h_2 , respectively. Clearly, $\text{in}(h_1) \cdot \text{in}(f_1) + \text{in}(h_2) \cdot \text{in}(f_2) = 0$, because otherwise the x -component of $\exp(h_1 f_1 + h_2 f_2)$ would not be 1. Therefore, $a x^{\alpha_1+3} y^{\alpha_2+1} + b x^{\beta_1+2} y^{\beta_2+3} = 0$. It follows that $\alpha_1 + 1 = \beta_1$, $\alpha_2 = \beta_2 + 2$, and $a + b = 0$. Consequently,

$$(3.1) \quad \text{in}(h_1) \cdot f_1 + \text{in}(h_2) \cdot f_2 = 0.$$

Now, set $h_i^{(1)} := h_i - \text{in}(h_i)$, $i = 1, 2$. By (3.1), we get $h_1^{(1)} f_1 + h_2^{(1)} f_2 = F$. Hence, by repeating the above argument, $\text{in}(h_1^{(1)}) \cdot f_1 + \text{in}(h_2^{(1)}) \cdot f_2 = 0$. We can thus set $h_i^{(2)} := h_i^{(1)} - \text{in}(h_i^{(1)})$, $i = 1, 2$, and again obtain $h_1^{(2)} f_1 + h_2^{(2)} f_2 = F$. By induction, if $h_i^{(j)} = h_i^{(j-1)} - \text{in}(h_i^{(j-1)})$, $i = 1, 2$, then

$$(3.2) \quad h_1^{(j)} f_1 + h_2^{(j)} f_2 = F, \quad \text{for all } j.$$

Note that, for every $j \geq 1$, the initial exponent of $h_i^{(j+1)}$ is strictly greater than that of $h_i^{(j)}$, by construction. Therefore, by the Krull Intersection Theorem, the sequences $(h_1^{(j)})_{j \geq 1}$ and $(h_2^{(j)})_{j \geq 1}$ converge to zero in the Krull topology of $\mathbb{K}\{x, y\}$. It follows from (3.2) that $0 \cdot f_1 + 0 \cdot f_2 = F$, hence $F = 0$, which contradicts the choice of F . ■

Remark 3.6 Identifying those ideals I (or, more precisely, those systems of generators) in $\mathbb{K}\{x\}$ for which the sequence of diagrams $(\mathfrak{N}(I_\mu))_\mu$ stabilizes seems important from the point of view of singularity theory. In fact, the singularities defined by such ideals are precisely those whose Hilbert–Samuel function is finitely determined. Indeed, for an ideal J in $\mathbb{K}\{x\}$, let $H_J(k) = \dim_{\mathbb{K}} \mathbb{K}\{x\}/(J + \mathfrak{m}_x^{k+1})$ denote the Hilbert–Samuel function of $\mathbb{K}\{x\}/J$. It follows from Proposition 2.2 that

$$H_J(k) = \#(\mathbb{N}^m \setminus \mathfrak{N}(J)) \cap \{\beta \in \mathbb{N}^m : |\beta| \leq k\}.$$

Therefore, by Lemma 3.2(i), we have $\mathfrak{N}(I) = \mathfrak{N}(I_\mu)$ if and only if $H_I(k) = H_{I_\mu}(k)$ for all $k \in \mathbb{N}$.

We conjecture that the diagrams of I_μ stabilize in the case where I is generated by a regular sequence. Recall that an s -tuple (f_1, \dots, f_s) forms a $\mathbb{K}\{x\}$ -regular sequence when f_1, \dots, f_s generate a proper ideal, $f_1 \neq 0$, and f_{i+1} is not a zero divisor in $\mathbb{K}\{x\}/(f_1, \dots, f_i)$ for $i = 1, \dots, s-1$.

Conjecture 3.7 *Let $I = (f_1, \dots, f_s)$ be an ideal in $\mathbb{K}\{x\}$, and for $\mu \in \mathbb{N}$, let I_μ denote the ideal generated in $\mathbb{K}\{x\}$ by the μ -jets $j^\mu f_1, \dots, j^\mu f_s$. Suppose that f_1, \dots, f_s form a regular sequence in $\mathbb{K}\{x\}$. Then there exists $\mu_0 \geq 1$ such that $\mathfrak{N}(I) = \mathfrak{N}(I_\mu)$, for all $\mu \geq \mu_0$ (after a generic change of coordinates x , if needed).*

The above conjecture is partly justified by the following observation: Let I and $\{I_\mu\}_{\mu \in \mathbb{N}}$ be as above. If f_1, \dots, f_s form a regular sequence in $\mathbb{K}\{x\}$, then there exists $\mu_0 \geq 1$ such that

$$\dim(\mathbb{K}\{x\}/I) = \dim(\mathbb{K}\{x\}/I_\mu),$$

for all $\mu \geq \mu_0$.

Indeed, as an ideal generated by s elements in an m -dimensional ring, every I_μ satisfies the inequality $m-s \leq \dim(\mathbb{K}\{x\}/I_\mu)$. On the other hand, by [9, Ch. 2, Prop. 5.3], there exists $\mu_0 \geq 1$ such that

$$\dim(\mathbb{K}\{x\}/I_\mu) \leq \dim(\mathbb{K}\{x\}/I),$$

for all $\mu \geq \mu_0$. Finally, if f_1, \dots, f_s form a regular sequence, then $\dim(\mathbb{K}\{x\}/I) = m-s$.

4 Finite Determinacy of Flatness

We now turn to the problem of finite determinacy of flatness of analytic mappings. Throughout this section X will denote a \mathbb{K} -analytic space, and $\phi: X \rightarrow \mathbb{K}^n$ will be a \mathbb{K} -analytic mapping. Since our considerations are local, we can assume without loss of generality that X is a subspace of \mathbb{K}^m (for some $m \geq 1$), $0 \in X$, and $\phi(0) = 0$. Let us begin with finite mappings.

4.1 Finite Mappings

Proposition 4.1 *Let X be a \mathbb{K} -analytic subspace of \mathbb{K}^m , with $0 \in X$ and $\mathcal{O}_{X,0} = \mathbb{K}\{x\}/I$. Let $\phi = (\phi_1, \dots, \phi_n): X \rightarrow \mathbb{K}^n$, $\phi(0) = 0$, be a finite \mathbb{K} -analytic mapping. Then there exists $\mu_0 \geq 1$ such that the following conditions are equivalent:*

- (i) ϕ is flat at zero;
- (ii) for every $\mu \geq \mu_0$, $(j^\mu \phi_1, \dots, j^\mu \phi_n): X \rightarrow \mathbb{K}^n$ is flat at zero;
- (iii) there exists $\mu \geq \mu_0$ such that $(j^\mu \phi_1, \dots, j^\mu \phi_n): X \rightarrow \mathbb{K}^n$ is flat at zero.

If, moreover, $I = (h_1, \dots, h_s)$ and, for $\mu \in \mathbb{N}$, X_μ denotes a local model defined at $0 \in \mathbb{K}^m$ by $\mathcal{O}_{X_\mu,0} = \mathbb{K}\{x\}/(j^\mu h_1, \dots, j^\mu h_s)$, then the above conditions are equivalent to each of the following:

- (ii') for all $\mu \geq \mu_0$, $(j^\mu \phi_1, \dots, j^\mu \phi_n): X_\mu \rightarrow \mathbb{K}^n$ is flat at zero;
- (iii') there exists $\mu \geq \mu_0$ such that $(j^\mu \phi_1, \dots, j^\mu \phi_n): X_\mu \rightarrow \mathbb{K}^n$ is flat at zero.

Proof Identifying X with the graph of ϕ , we can write $\mathcal{O}_{X,0} = \mathbb{K}\{y, x\}/J$, where $y = (y_1, \dots, y_n)$, $x = (x_1, \dots, x_m)$, and

$$J = (h_1(x), \dots, h_s(x), y_1 - \phi_1(x), \dots, y_n - \phi_n(x)) \cdot \mathbb{K}\{y, x\}.$$

For $\mu \geq 1$, set

$$\begin{aligned} \tilde{J}_\mu &:= (h_1, \dots, h_s, y_1 - j^\mu \phi_1, \dots, y_n - j^\mu \phi_n) \cdot \mathbb{K}\{y, x\}, \\ J_\mu &:= (j^\mu h_1, \dots, j^\mu h_s, y_1 - j^\mu \phi_1, \dots, y_n - j^\mu \phi_n) \cdot \mathbb{K}\{y, x\}. \end{aligned}$$

The finiteness of ϕ implies that $J(0) \supset \mathfrak{m}_x^k$ for some integer k (where the evaluation is at $y = 0$). Let μ_0 denote the least such k . We shall prove that the theorem holds with this choice of μ_0 .

Let l_0 denote the maximum of lengths of the vertices of $\mathfrak{N}(J(0))$. By the proof of Lemma 3.2(i), we have

$$\mathfrak{N}(J(0)) \subset \mathfrak{N}(\tilde{J}_\mu(0)) \subset \mathfrak{N}(J_\mu(0)),$$

for all $\mu \geq l_0$. Hence, by Corollary 3.4,

$$\mathfrak{N}(J(0)) = \mathfrak{N}(\tilde{J}_\mu(0)) = \mathfrak{N}(J_\mu(0)),$$

for all $\mu \geq \mu_0$. Note also that $|\beta| \leq \mu_0$ for all $\beta \in \mathbb{N}^m \setminus \mathfrak{N}(J(0))$.

For the proof of (i) \Rightarrow (ii), suppose there exists $\mu \geq \mu_0$ such that

$$(j^\mu \phi_1, \dots, j^\mu \phi_n): X \rightarrow \mathbb{K}^n$$

is not flat at zero. Then, by Proposition 2.3, one can choose a nonzero $G \in \tilde{J}_\mu$ with $\exp(G) \in \mathbb{N}^m \setminus \mathfrak{N}(\tilde{J}_\mu(0))$. Write $G = \sum_{i=1}^s g_i \cdot h_i + \sum_{j=1}^n q_j \cdot (y_j - j^\mu \phi_j)$, for some

$g_i, q_j \in \mathbb{K}\{y, x\}$. Define $F := \sum_{i=1}^s g_i \cdot h_i + \sum_{j=1}^n q_j \cdot (y_j - \phi_j)$. Then $F \in J$ and $\exp(F) = \exp(G)$ (by Remark 3.1), hence $\exp(F) \in \mathbb{N}^m \setminus \mathfrak{N}(J(0))$. This proves that ϕ is not flat at zero, again by Proposition 2.3.

The implication (ii) \Rightarrow (iii) is trivial. To prove that (iii) \Rightarrow (i), suppose that ϕ is not flat at zero and choose $F = \sum_{i=1}^s g_i \cdot h_i + \sum_{j=1}^n q_j \cdot (y_j - \phi_j)$ with $\exp(F) \in \mathbb{N}^m \setminus \mathfrak{N}(J(0))$. Given an arbitrary $\mu \geq \mu_0$, set

$$G := \sum_{i=1}^s g_i \cdot h_i + \sum_{j=1}^n q_j \cdot (y_j - j^\mu \phi_j).$$

Then $G \in \tilde{J}_\mu$ and $\exp(G) = \exp(F) \in \mathbb{N}^m \setminus \mathfrak{N}(\tilde{J}_\mu(0))$, which proves that

$$(j^\mu \phi_1, \dots, j^\mu \phi_n): X \rightarrow \mathbb{K}^n$$

is not flat at zero.

The proof of implications (i) \Rightarrow (ii') and (iii') \Rightarrow (i) is analogous, with ideal \tilde{J}_μ replaced by J_μ . \blacksquare

Remark 4.2 Note that in Proposition 4.1 one can take μ_0 to be the Milnor number of the fibre $\phi^{-1}(0)$; i.e.,

$$\mu_0 = \dim_{\mathbb{K}}(\mathbb{K}\{x\}/J(0)).$$

Indeed, since $\dim_{\mathbb{K}}(\mathbb{K}\{x\}/J(0))$ is precisely the cardinality of the complement of $\mathfrak{N}(J(0))$, one readily sees that with this choice of μ_0 we have $J(0) \supset \mathfrak{m}_x^{\mu_0}$.

Proposition 4.1 implies that flatness of finite analytic mappings is finitely determined in the following sense.

Theorem 4.3 *Let X be a \mathbb{K} -analytic subspace of \mathbb{K}^m , with $0 \in X$ and $\mathcal{O}_{X,0} = \mathbb{K}\{x\}/I$. Let $\phi = (\phi_1, \dots, \phi_n): X \rightarrow \mathbb{K}^n$, $\phi(0) = 0$, be a finite \mathbb{K} -analytic mapping. Then there exists $\mu_0 \geq 1$ such that the following conditions are equivalent:*

- (i) ϕ is flat at zero;
- (ii) for every $\mu \geq \mu_0$ and for every analytic mapping $\psi = (\psi_1, \dots, \psi_n): X \rightarrow \mathbb{K}^n$ with $j^\mu \psi_j = j^\mu \phi_j$, $j = 1, \dots, n$, ψ is flat at zero;
- (iii) there exist $\mu \geq \mu_0$ and an analytic mapping $\psi = (\psi_1, \dots, \psi_n): X \rightarrow \mathbb{K}^n$ such that ψ is flat at zero and $j^\mu \psi_j = j^\mu \phi_j$, $j = 1, \dots, n$.

If, moreover, $I = (h_1, \dots, h_s)$ and, for $\mu \in \mathbb{N}$, X_μ denotes a local model defined at $0 \in \mathbb{K}^m$ by $\mathcal{O}_{X_\mu,0} = \mathbb{K}\{x\}/(j^\mu h_1, \dots, j^\mu h_s)$, then the above conditions are equivalent to each of the following:

- (ii') for every $\mu \geq \mu_0$ and for every analytic mapping $\psi = (\psi_1, \dots, \psi_n): X_\mu \rightarrow \mathbb{K}^n$ with $j^\mu \psi_j = j^\mu \phi_j$, $j = 1, \dots, n$, ψ is flat at zero;
- (iii') there exist $\mu \geq \mu_0$ and an analytic mapping $\psi = (\psi_1, \dots, \psi_n): X_\mu \rightarrow \mathbb{K}^n$ such that ψ is flat at zero and $j^\mu \psi_j = j^\mu \phi_j$, $j = 1, \dots, n$.

Proof Let $J = (h_1(x), \dots, h_s(x), y_1 - \phi_1(x), \dots, y_n - \phi_n(x)) \cdot \mathbb{K}\{y, x\}$, and let $\mu_0 = \dim_{\mathbb{K}}(\mathbb{K}\{x\}/J(0))$. Let $\psi = (\psi_1, \dots, \psi_n): X \rightarrow \mathbb{K}^n$ be an arbitrary analytic mapping with $j^{\mu_0} \psi_j = j^{\mu_0} \phi_j$, $j = 1, \dots, n$ and let

$$Q := (h_1(x), \dots, h_s(x), y_1 - \psi_1(x), \dots, y_n - \psi_n(x)) \cdot \mathbb{K}\{y, x\}.$$

We will show that then $\mathfrak{N}(Q(0)) = \mathfrak{N}(J(0))$. This proves that $\dim_{\mathbb{K}}(\mathbb{K}\{x\}/Q(0)) = \mu_0$ (hence ψ is a finite mapping), and, consequently, the theorem follows directly from Proposition 4.1 and Remark 4.2.

First, let us show that $\mathfrak{N}(Q(0))$ contains all the vertices of $\mathfrak{N}(J(0))$. Given β' a vertex of $\mathfrak{N}(J(0))$, let $F \in J$ be such that $\exp(F(0)) = \beta'$. Write $F = \sum_{i=1}^s g_i \cdot h_i + \sum_{j=1}^n q_j \cdot (y_j - \phi_j)$, for some $g_i, q_j \in \mathbb{K}\{y, x\}$. Define $G = \sum_{i=1}^s g_i \cdot h_i + \sum_{j=1}^n q_j \cdot (y_j - \psi_j)$. Then $G \in Q$, and, by assumption, $j^\mu G = j^\mu F$. In particular, $\exp(G(0)) = \exp(F(0)) = \beta'$, as $|\beta'| \leq \mu$. It follows that $\beta' \in \mathfrak{N}(Q(0))$.

Now, choose $\beta'' \in \mathbb{N}^m \setminus \mathfrak{N}(J(0))$ and suppose that $\beta'' \in \mathfrak{N}(Q(0))$. Pick $G \in Q$ such that $\exp(G(0)) = \beta''$ and write $G = \sum_{i=1}^s g_i \cdot h_i + \sum_{j=1}^n q_j \cdot (y_j - \psi_j)$, for some $g_i, q_j \in \mathbb{K}\{y, x\}$. Define $F = \sum_{i=1}^s g_i \cdot h_i + \sum_{j=1}^n q_j \cdot (y_j - \phi_j)$. Then $F \in J$, and $j^\mu F = j^\mu G$, by assumption. Since $|\beta''| \leq \mu$, we get $\exp(F(0)) = \exp(G(0)) = \beta''$, which contradicts the choice of β'' . ■

Remark 4.4 Note that Theorem 4.3 cannot be generalized to finitely generated modules. More precisely, suppose that a finite $\mathbb{K}\{y\}$ -module M is given as the cokernel of a homomorphism $\Phi: \mathbb{K}\{y\}^q \rightarrow \mathbb{K}\{y\}^p$; i.e., $M = \mathbb{K}\{y\}^p / (\Phi(e_1), \dots, \Phi(e_q))$, where $\{e_1, \dots, e_q\}$ is the canonical basis of the free $\mathbb{K}\{y\}$ -module $\mathbb{K}\{y\}^q$. For $\mu \in \mathbb{N}$, let M_μ denote the module $\mathbb{K}\{y\}^p / (j^\mu \Phi(e_1), \dots, j^\mu \Phi(e_q))$. One might expect that flatness of M over $\mathbb{K}\{y\}$ is equivalent to flatness of M_μ for sufficiently large μ . This is not the case, however, as the following example shows.

Example 4.5 Consider f_1 and f_2 from Example 3.5, and let M be a $\mathbb{K}\{x, y\}$ -submodule of $\mathbb{K}\{x, y\}^3$ generated by

$$G_1 = (1, 0, f_1), \quad G_2 = (1, 1, f_2), \quad \text{and} \quad G_3 = (y^2 - x, -x, 0).$$

Let \mathcal{O} denote the structure sheaf of \mathbb{K}^2 and let \mathcal{M} be a coherent \mathcal{O} -module whose stalk at the origin is $\mathcal{M}_{(0,0)} = \mathbb{K}\{x, y\}^3 / M$. We have $G_1(0, 0) = (1, 0, 0)$, $G_2(0, 0) = (1, 1, 0)$, and $G_3(0, 0) = (0, 0, 0)$, hence the multiplicity of \mathcal{M} at the origin is $3 - 2 = 1$. On the other hand, G_3 being a combination of G_1 and G_2 (indeed, $G_3 = y^2 \cdot G_1 - x \cdot G_2$), the multiplicity of \mathcal{M} at any other point cannot be less than 1. Therefore, multiplicity of \mathcal{M} is constant, and so \mathcal{M} is flat at $(0, 0)$ (see, e.g., [6, Thm. 1.78]).

Let now \mathcal{M}^μ be a coherent \mathcal{O} -module with $\mathcal{M}_{(0,0)}^\mu = \mathbb{K}\{x, y\}^3 / M_\mu$, where M_μ is generated by $j^\mu G_1$, $j^\mu G_2$, and $j^\mu G_3$. We claim that \mathcal{M}^μ is not flat at the origin (equivalently, $\mathbb{K}\{x, y\}^3 / M_\mu$ is not $\mathbb{K}\{x, y\}$ -flat), for any $\mu \geq 5$.

Indeed, identifying M_μ with a matrix with rows $j^\mu G_i$, $i = 1, 2, 3$, we get

$$\det(M_\mu) = y^2 \cdot j^\mu f_1 - x \cdot j^\mu f_2,$$

which we know is equal to $x y^{\mu+1}$, provided $\mu \geq 5$ (cf. Example 3.5). Therefore, for any $a \neq 0$ and $b \neq 0$, the stalk $\mathcal{M}_{(a,b)}^\mu$ has multiplicity zero, while $\mathcal{M}_{(0,0)}^\mu$ has multiplicity one.

4.2 Mappings from Complete Intersections

Proof of Theorem 1.1 Let h_1, \dots, h_s be a regular sequence in $\mathbb{K}\{x\}$ for which $\mathcal{O}_{X,0} = \mathbb{K}\{x\} / (h_1, \dots, h_s)$. Identifying X with the graph of ϕ , we can write $\mathcal{O}_{X,0} = \mathbb{K}\{y, x\} / J$,

where

$$J = (h_1(x), \dots, h_s(x), y_1 - \phi_1(x), \dots, y_n - \phi_n(x)) \cdot \mathbb{K}\{y, x\}.$$

For $\mu \in \mathbb{N}$, set

$$J_\mu := (h_1(x), \dots, h_s(x), y_1 - j^\mu \phi_1(x), \dots, y_n - j^\mu \phi_n(x)) \cdot \mathbb{K}\{y, x\}.$$

By a well-known flatness criterion over regular local rings (see, e.g., [6, Thm. B.8.11]) and because the local ring $\mathcal{O}_{X,0}$ is Cohen–Macaulay, flatness of ϕ at zero is equivalent to

$$(4.1) \quad \dim(\mathbb{K}\{x\}/J(0)) = \dim \mathcal{O}_{X,0} - n = (m - s) - n.$$

Suppose then that ϕ is flat at zero. By [9, Ch. 2, Prop. 5.3], there exists $\mu' \geq 1$ such that $\dim(\mathbb{K}\{x\}/J_\mu(0)) \leq \dim(\mathbb{K}\{x\}/J(0))$, for all $\mu \geq \mu'$. On the other hand, as an ideal generated by $s+n$ elements, every $J_\mu(0)$ satisfies inequality $\dim(\mathbb{K}\{x\}/J_\mu(0)) \geq m - (s+n)$. Therefore, by (4.1), $\dim(\mathbb{K}\{x\}/J_\mu(0)) = (m - s) - n$, and so ϕ_μ is flat at zero. This proves (i) \Rightarrow (ii).

The implication (ii) \Rightarrow (iii) is trivial. For the proof of (iii) \Rightarrow (i), suppose that ϕ is not flat at the origin. Then, by Proposition 2.3, one can choose a nonzero $F \in J$ supported outside of $\mathfrak{N}(J(0))$. In particular, $\exp(F) \notin \mathfrak{N}(J(0))$. Write $F = \sum_{i=1}^s g_i h_i + \sum_{k=1}^n q_k \cdot (y_k - \phi_k)$, for some $g_i, q_k \in \mathbb{K}\{y, x\}$, and set $\mu'' := \max\{|\exp(F)|, l_0\}$, where l_0 is the maximum of lengths of vertices of the diagram $\mathfrak{N}(J(0))$. Fix $\mu \geq \mu''$, and define $G := \sum_{i=1}^s g_i h_i + \sum_{k=1}^n q_k \cdot (y_k - j^\mu \phi_k)$. Then $G \in J_\mu$ and $\exp(G) = \exp(F)$ (as $j^\mu G = j^\mu F$ and by Remark 3.1). By Lemma 3.2, $\exp(G) \notin \mathfrak{N}(J_\mu(0))$, which implies that ϕ_μ is not flat (by Propositions 2.3 and 2.2). The theorem thus holds for $\mu_0 = \max\{\mu', \mu''\}$. \blacksquare

Corollary 4.6 *Given a collection of power series $f_1, \dots, f_s \in \mathbb{K}\{x\}$, there exists $\mu_0 \geq 1$ such that the following conditions are equivalent:*

- (i) f_1, \dots, f_s form a regular sequence;
- (ii) for all $\mu \geq \mu_0$, $j^\mu f_1, \dots, j^\mu f_s$ form a regular sequence;
- (iii) there exists $\mu \geq \mu_0$ such that $j^\mu f_1, \dots, j^\mu f_s$ form a regular sequence.

Proof The equivalence follows immediately from Theorem 1.1 applied to the mapping $\phi := (f_1, \dots, f_s): \mathbb{K}^m \rightarrow \mathbb{K}^s$, and the fact that $h_1, \dots, h_k \in \mathbb{K}\{x\}$ form a regular sequence if and only if $(h_1, \dots, h_k): \mathbb{K}^m \rightarrow \mathbb{K}^k$ is flat (see, e.g., [6, Thm. B.8.11]). \blacksquare

For the next result, we will need the following useful observation.

Remark 4.7 For an ideal I in $\mathbb{K}\{x\}$, the following conditions are equivalent, up to a generic linear change of coordinates x :

- (i) $\dim(\mathbb{K}\{x\}/I) \leq \dim \mathbb{K}\{x\} - k$;
- (ii) the diagram $\mathfrak{N}(I)$ has a vertex on each of the axes corresponding to x_1, \dots, x_k .

Indeed, condition (ii) clearly implies (i). On the other hand, (i) implies that (up to a generic linear change of coordinates) $\mathbb{K}\{x\}/I$ is a finite $\mathbb{K}\{\tilde{x}\}$ -module, where $\tilde{x} = (x_{k+1}, \dots, x_m)$. The latter is equivalent to saying that I contains a distinguished pseudo-polynomial $f_j \in \mathbb{K}\{\tilde{x}\}[x_j]$ for every $j = 1, \dots, k$ (see, e.g., [8, Ch. III, Sec. 2.2]), hence (ii).

Corollary 4.8 *Given a collection of power series $f_1, \dots, f_s \in \mathbb{K}\{x\}$, there exists $\mu_0 \geq 1$ such that the following conditions are equivalent:*

- (i) f_1, \dots, f_s form a regular sequence;
- (ii) for every $\mu \geq \mu_0$, every sequence g_1, \dots, g_s satisfying $j^\mu g_i = j^\mu f_i$ ($i = 1, \dots, s$) is regular in $\mathbb{K}\{x\}$;
- (iii) there exists $\mu \geq \mu_0$ such that every sequence g_1, \dots, g_s satisfying $j^\mu g_i = j^\mu f_i$ ($i = 1, \dots, s$) is regular in $\mathbb{K}\{x\}$.

Proof Suppose that f_1, \dots, f_s form a regular sequence in $\mathbb{K}\{x\}$. Then

$$\dim \mathbb{K}\{x\}/(f_1, \dots, f_s) = m - s.$$

By Remark 4.7, one can assume that the diagram $\mathfrak{N}(I)$ of the ideal $I = (f_1, \dots, f_s) \cdot \mathbb{K}\{x\}$ has a vertex on each of the axes corresponding to x_1, \dots, x_s ; say, β^1, \dots, β^s . Let μ' denote the maximum of lengths of β^1, \dots, β^s . Fix $\mu \geq \mu'$ and choose arbitrary $g_1, \dots, g_s \in \mathbb{K}\{x\}$ with $j^\mu g_i = j^\mu f_i$ for $i = 1, \dots, s$. Set $J := (g_1, \dots, g_s) \cdot \mathbb{K}\{x\}$. For $k = 1, \dots, s$, let $F_k \in I$ be such that $\exp(F_k) = \beta^k$. Write $F_k = \sum_{i=1}^s q_{ki} f_i$, for some $q_{ki} \in \mathbb{K}\{x\}$. Define $G_k := \sum_{i=1}^s q_{ki} g_i$, $k = 1, \dots, s$. Then $G_k \in J$ and $j^\mu G_k = j^\mu F_k$ for all k . As $|\beta^k| \leq \mu' \leq \mu$, it follows (by Remark 3.1) that $\exp(G_k) = \exp(F_k) = \beta^k$, and so β^1, \dots, β^s all belong to $\mathfrak{N}(J)$. Hence, by Remark 4.7 again, $\dim \mathbb{K}\{x\}/J \leq m - s$. Thus, g_1, \dots, g_s form a regular sequence, which proves (i) \Rightarrow (ii).

The implication (ii) \Rightarrow (iii) is trivial. For the proof of (iii) \Rightarrow (i), suppose that f_1, \dots, f_s do not form a regular sequence. Then, by Corollary 4.6, there exists μ'' such that, for all $\mu \geq \mu''$, $j^\mu f_1, \dots, j^\mu f_s$ do not form a regular sequence. The corollary thus holds for $\mu_0 := \max\{\mu', \mu''\}$. ■

It is now easy to see that flatness of mappings from complete intersections is finitely determined, in the following sense.

Theorem 4.9 *Let X be a \mathbb{K} -analytic subspace of \mathbb{K}^m . Suppose that $0 \in X$ and the local ring $\mathcal{O}_{X,0}$ is a complete intersection. Let $\phi = (\phi_1, \dots, \phi_n): X \rightarrow \mathbb{K}^n$, $\phi(0) = 0$, be a \mathbb{K} -analytic mapping. Then, there exists $\mu_0 \geq 1$ such that the following conditions are equivalent:*

- (i) ϕ is flat at zero;
- (ii) for every $\mu \geq \mu_0$, every analytic mapping $\psi = (\psi_1, \dots, \psi_n): X \rightarrow \mathbb{K}^n$ with $j^\mu \psi_i = j^\mu \phi_i$ ($i = 1, \dots, n$) is flat at zero;
- (iii) there exists $\mu \geq \mu_0$ such that every analytic mapping $\psi = (\psi_1, \dots, \psi_n): X \rightarrow \mathbb{K}^n$ with $j^\mu \psi_i = j^\mu \phi_i$ ($i = 1, \dots, n$) is flat at zero.

If, moreover, X is defined at $0 \in \mathbb{K}^m$ by a regular sequence h_1, \dots, h_s and, for $\mu \in \mathbb{N}$, X_μ denotes a local model defined at $0 \in \mathbb{K}^m$ by $\mathcal{O}_{X_\mu,0} = \mathbb{K}\{x\}/(j^\mu h_1, \dots, j^\mu h_s)$, then the above conditions are equivalent to each of the following:

- (ii') for every $\mu \geq \mu_0$, every analytic mapping $\psi = (\psi_1, \dots, \psi_n): X_\mu \rightarrow \mathbb{K}^n$ with $j^\mu \psi_i = j^\mu \phi_i$ ($i = 1, \dots, n$) is flat at zero;
- (iii') there exists $\mu \geq \mu_0$ such that every analytic mapping $\psi = (\psi_1, \dots, \psi_n): X_\mu \rightarrow \mathbb{K}^n$ with $j^\mu \psi_i = j^\mu \phi_i$ ($i = 1, \dots, n$) is flat at zero.

Proof Recall ([6, Thm. B.8.11]) that an analytic mapping $\phi = (\phi_1, \dots, \phi_n): X \rightarrow \mathbb{K}^n$ is flat at zero if and only if ϕ_1, \dots, ϕ_n form an $\mathcal{O}_{X,0}$ -regular sequence. Therefore, if X is defined at $0 \in \mathbb{K}^m$ by a regular sequence h_1, \dots, h_s , then the latter is equivalent to saying that $h_1, \dots, h_s, \phi_1, \dots, \phi_n$ form a $\mathbb{K}\{x\}$ -regular sequence. The theorem thus follows from Corollary 4.8 applied to the sequence $h_1, \dots, h_s, \phi_1, \dots, \phi_n$. ■

Question 4.10 *It would be interesting to know if one could relax the complete intersection assumption on the domain X in Theorem 4.9.*

5 Stability Under Deformations

Let $\phi: X \rightarrow Y$ be a morphism of \mathbb{K} -analytic spaces, and let $(T, 0)$ be a pointed space (i.e., a \mathbb{K} -analytic space with a distinguished point 0). By a *deformation of ϕ over Y* we shall mean a Cartesian diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \phi \downarrow & & \downarrow \Phi \\ Y & \longrightarrow & Y \times T \\ \downarrow & & \downarrow \pi \\ \{0\} & \longrightarrow & T \end{array}$$

such that \mathfrak{X} is flat over T (see, e.g., [6]). If \mathfrak{X} is of the form $X \times T$, then Φ is called an *unfolding* of ϕ .

In this section, we study the stability of flatness of ϕ under such deformations. Since our considerations are local, we can assume from the start that all spaces are local models; say, $\mathfrak{X} \subset \mathbb{K}^m$, $Y \subset \mathbb{K}^n$, $T \subset \mathbb{K}^k$, X and Y pass through the origins in \mathbb{K}^m and \mathbb{K}^n respectively, and $\phi(0) = 0$. Let $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_n)$, and $t = (t_1, \dots, t_k)$ denote the systems of variables in the respective ambient spaces of \mathfrak{X} , Y , and T .

For $\theta \in T$, we shall denote by Φ^θ the specialization of Φ over θ , that is, the unique mapping closing the following Cartesian diagram

$$\begin{array}{ccc} (\pi \circ \Phi)^{-1}(\theta) & \longrightarrow & \mathfrak{X} \\ \Phi^\theta \downarrow & & \downarrow \Phi \\ Y & \longrightarrow & Y \times T \\ \downarrow & & \downarrow \pi \\ \{\theta\} & \longrightarrow & T. \end{array}$$

For the proof of Theorem 1.2, we will first settle the case of smooth one-dimensional parameter space.

Lemma 5.1 *Let $\phi: X \rightarrow Y$ be a flat morphism of \mathbb{K} -analytic spaces, with Y locally irreducible. If Φ is a deformation of ϕ parametrized by $T = \mathbb{K}$, then Φ^θ is a flat mapping for every $\theta \in T$ near zero.*

Proof Since flatness is an open property (see, e.g., [3, Thm. 7.15]) and because flatness is preserved by base change (see, e.g., [7, Prop. 6.8]), it suffices to show that Φ is flat at $0 \in \mathfrak{X}$. For a proof by contradiction, suppose that Φ is not flat at the origin. Let R denote the local ring $\mathcal{O}_{Y,0}$. After identifying \mathfrak{X} with the graph of Φ , we can regard $\mathcal{O}_{\mathfrak{X},0}$ as a quotient of $\mathcal{O}_{\mathbb{K}^m \times Y \times T, (0,0,0)}$; that is, $\mathcal{O}_{\mathfrak{X},0} = R\{t, x\}/I$ for some ideal $I = I(y, t, x)$ in $R\{t, x\}$.

Now, by Proposition 2.3, there exists a nonzero $F \in I$ such that

$$\text{supp}(F) \subset \mathbb{N}^m \setminus \mathfrak{N}(I(0, 0, x)),$$

where the evaluation is at $(y, t) = (0, 0)$. We have $F(y, 0, x) \in I(y, 0, x)$. Hence, flatness of ϕ at zero implies that $F(y, 0, x) = 0$ in $R\{x\}$, again by Proposition 2.3. The latter means that t divides F in $R\{t, x\}$. Let d be the maximum power of t that can be factored out from F in $R\{t, x\}$ and set $\tilde{F} := t^{-d} \cdot F$. Then $\tilde{F}(y, 0, x)$ is not zero anymore. But $\text{supp}(\tilde{F}) = \text{supp}(F)$, so applying Proposition 2.3 once more, we get that $\tilde{F} \notin I$. Consequently, t^d is a zero divisor in the local ring $\mathcal{O}_{\mathfrak{X},0}$, which contradicts the flatness of $\pi \circ \Phi$. \blacksquare

Proof of Theorem 1.2 As in the proof of the above lemma, it suffices to show that Φ is flat at $0 \in \mathfrak{X}$. Suppose first that T is smooth. The problem being local, we can thus assume that $T = \mathbb{K}^k$. We will prove by induction on k that flatness of $\pi \circ \Phi: \mathfrak{X} \rightarrow Y \times \mathbb{K}^k \rightarrow \mathbb{K}^k$ and $\Phi^0 = \phi: X \rightarrow Y$ at zero implies that $\Phi: \mathfrak{X} \rightarrow Y \times \mathbb{K}^k$ is flat at zero.

For $k = 0$ there is nothing to prove, so suppose that $k \geq 1$ and the statement holds for $k - 1$. Consider the flat mapping $\mathfrak{X}_1 \rightarrow Y \times \mathbb{K}^{k-1} \rightarrow \mathbb{K}^{k-1}$, defined as the pullback of the flat $\pi \circ \Phi$ by the inclusion $\mathbb{K}^{k-1} \hookrightarrow \mathbb{K}^k$. The inductive hypothesis implies that $\Phi_1: \mathfrak{X}_1 \rightarrow Y \times \mathbb{K}^{k-1}$ is flat. Next, consider the mapping $\mathfrak{X} \rightarrow (Y \times \mathbb{K}^{k-1}) \times \mathbb{K} \rightarrow \mathbb{K}$, which is flat as the composite of $\pi \circ \Phi$ with the projection $\mathbb{K}^k \rightarrow \mathbb{K}$. Applying Lemma 5.1 to the Cartesian diagram

$$\begin{array}{ccc} \mathfrak{X}_1 & \longrightarrow & \mathfrak{X} \\ \Phi_1 \downarrow & & \downarrow \Phi \\ Y \times \mathbb{K}^{k-1} & \longrightarrow & (Y \times \mathbb{K}^{k-1}) \times \mathbb{K} \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{K}, \end{array}$$

we conclude that Φ is flat at zero.

Finally, consider a general locally irreducible T . In this case, one can find a non-singular \mathbb{K} -analytic space Z and a dominant mapping $\sigma: Z \rightarrow T$, $\sigma(0) = 0$, with $\dim Z_0 = \dim T_0$ (for example, take a desingularization of T near the origin). Consider the pullback of $\pi \circ \Phi: \mathfrak{X} \rightarrow Y \times T \rightarrow T$ by $\sigma: Z \rightarrow T$ (which is flat by the flatness

of $\pi \circ \Phi$). One can easily check that this mapping factors as $\mathfrak{X}' \xrightarrow{\Phi'} Y \times Z \rightarrow Z$. By the first part of the proof, we thus get that $\Phi': \mathfrak{X}' \rightarrow Y \times Z$ is flat at zero. Moreover, the pullback of σ by π is clearly dominant, and so we get the following Cartesian square in which Φ' is flat and the bottom arrow is dominant:

$$\begin{array}{ccc} \mathfrak{X} & \longleftarrow & \mathfrak{X}' \\ \Phi \downarrow & & \downarrow \Phi' \\ Y \times T & \longleftarrow & Y \times Z. \end{array}$$

By assumption, $Y \times T$ is irreducible at $(0, 0)$. Hence, the analytic flatness descent (see [1, Prop. 2.1]) implies that Φ is flat at zero, as required. ■

6 Complex Case

In this section, we consider the case $\mathbb{K} = \mathbb{C}$. In the complex setting, flatness of a mapping $\phi: X \rightarrow Y$ has a natural geometric interpretation. Namely, it is equivalent to continuity in the family of fibres of ϕ . In fact, if Y is nonsingular and (the local ring of) X is Cohen–Macaulay (at every point), then flatness of ϕ is equivalent to openness (see, e.g., [5, §3.20]), and the latter simply means that all fibres of ϕ are of the same dimension. In particular, over $\mathbb{K} = \mathbb{C}$, in Theorems 1.1 and 4.9 “flatness” can be replaced with “openness”, since complete intersections are Cohen–Macaulay.

Over singular targets, the picture is (considerably) more complicated; nonetheless, it is still possible to interpret flatness in purely geometric terms. As we show in [2], a morphism $\phi: X \rightarrow Y$ of complex-analytic spaces (with Y locally irreducible) is flat at a point $\xi \in X$ if and only if every irreducible component of the fibred product $X \times_Y Z$ at (ξ, ζ) is dominant over Z_ζ , where $\sigma_\zeta: Z_\zeta \rightarrow Y_\eta$ is the local blowing up of Y at $\eta = \phi(\xi)$.

Below, we generalize this idea and construct test mappings to detect higher order discontinuities in the family of fibres of a given mapping. For an analytic mapping $\phi: X \rightarrow Y$ with locally irreducible Y and X of pure dimension, one can speak of the generic fibre dimension of ϕ , denoted λ_ϕ . Further, let κ_ϕ be the maximum fibre dimension of ϕ . We shall call the difference $\kappa_\phi - \lambda_\phi$ the *fibre defect* of ϕ . By the Remmert Open Mapping Theorem (see, e.g., [8, Ch. V, § 6, Thm. 2]), ϕ is open if and only if its fibre defect is zero.

6.1 Test Mappings

Consider a morphism $\phi: X \rightarrow Y$ of local models. Suppose that $X \subset \mathbb{C}^m$ is of pure dimension, $Y \subset \mathbb{C}^n$ is locally irreducible (of positive dimension), and $\phi(0) = 0$. Suppose further that $n = \text{edim } \mathcal{O}_{Y,0}$. After a linear change of coordinates in \mathbb{C}^n if needed, we can assume that y_n belongs to the tangent cone of Y at 0. The following proposition gives a method of testing for the degree of fibre defect of a given mapping.

For $k \in \{0, \dots, n-1\}$, let $\sigma_k: \mathbb{C}^n \rightarrow \mathbb{C}^n$ denote the mapping

$$(y_1, \dots, y_n) \longmapsto (y_1, \dots, y_k, y_{k+1}y_n, \dots, y_{n-1}y_n, y_n).$$

In other words, σ_k is the restriction to the affine chart $\{y_n \neq 0\}$ of the blowing up of \mathbb{C}^n with centre $C_k = \{y_{k+1} = \dots = y_n = 0\}$. Denote by Y_k^{st} the strict transform of Y

under σ_k . We will consider a Cartesian square of the form

$$\begin{array}{ccc} X \times_Y Y_k^{st} & \longrightarrow & X \\ \phi'_k \downarrow & & \downarrow \phi \\ Y_k^{st} & \xrightarrow{\sigma_k|_{Y_k^{st}}} & Y. \end{array}$$

Proposition 6.1 *Suppose that ϕ has fibre defect greater than δ . Then:*

- (i) *at $(0, 0) \in X \times_Y Y_\delta^{st}$, the fibre $(\sigma_\delta|_{Y_\delta^{st}} \circ \phi'_\delta)^{-1}(0)$ has dimension greater than or equal to $\dim X$;*
- (ii) *the fibred product $X \times_Y Y_\delta^{st}$ has an isolated irreducible component at $(0, 0)$ that is mapped by $\sigma_\delta|_{Y_\delta^{st}} \circ \phi'_\delta$ into $C_\delta \cap Y$. Equivalently, y_n is a zero divisor in the reduced local ring $(\mathcal{O}_{X \times_Y Y_\delta^{st}, (0,0)})_{\text{red}}$.*

Proof Let $\delta \in \mathbb{N}$ and suppose that $\kappa_\phi - \lambda_\phi > \delta$. Since $\text{edim } \mathcal{O}_{Y,0} = n$, it follows that Y_0 is not contained in the germ of the center $(C_\delta)_0$. Consequently,

$$\dim(Y_\delta^{st} \cap \sigma_\delta^{-1}(C_\delta)) = \dim(\sigma_\delta|_{Y_\delta^{st}})^{-1}(C_\delta) = \dim Y_\delta^{st} - 1 = \dim Y - 1.$$

As the center itself is of dimension δ , we get that

$$\dim(\sigma_\delta|_{Y_\delta^{st}})^{-1}(0) \geq (\dim Y - 1) - \delta \geq \dim X - \lambda_\phi - 1 - \delta,$$

and hence

$$\text{fbd}_{(0,0)}(\sigma_\delta|_{Y_\delta^{st}} \circ \phi'_\delta) \geq (\dim X - \lambda_\phi - \delta - 1) + \kappa_\phi \geq \dim X,$$

which proves property (i).

On the other hand, since σ_δ is a biholomorphism outside $\sigma_\delta^{-1}(C_\delta)$, it follows that $\dim_{(\xi,\eta)} X \times_Y Y_\delta^{st} = \dim X$ for all (ξ, η) except at most those for which η is mapped by σ_δ into C_δ . Therefore, either the fibre $(\sigma_\delta|_{Y_\delta^{st}} \circ \phi'_\delta)^{-1}(0)$ itself contains an irreducible component of $X \times_Y Y_\delta^{st}$ at $(0, 0)$ or else it is contained in a component mapped into $C_\delta \cap Y$.

The last statement of the proposition follows from the fact that the $\mathcal{O}_{Y,0}$ -module structure of $\mathcal{O}_{X \times_Y Y_\delta^{st}, (0,0)}$ factors as $\mathcal{O}_{Y,0} \rightarrow \mathcal{O}_{Y_\delta^{st},0} \rightarrow \mathcal{O}_{X \times_Y Y_\delta^{st}, (0,0)}$ and the image in $\mathcal{O}_{Y_\delta^{st},0}$ of the ideal defining C_δ is the principal ideal generated by y_n . ■

Remark 6.2 It is evident from the proof above that if $Y = \mathbb{C}^n$ and the mapping $\phi: X \rightarrow Y$ is dominant (i.e., $\lambda_\phi = \dim X - n$), then, conversely, the equivalent conditions (i) and (ii) of the proposition imply that the fibre defect of ϕ is greater than δ .

6.2 Stability of Openness

In this section we prove that, like flatness, openness of complex-analytic mappings is stable under deformations. This follows from Theorem 1.2 for mappings from Cohen–Macaulay into smooth spaces (by [5, §3.20]), but in general an open mapping need not be flat. We have the following proposition.

Proposition 6.3 *Let $\phi: X \rightarrow Y$ be a morphism of local models, where $Y \subset \mathbb{C}^n$ is locally irreducible, $X \subset \mathbb{C}^m$ is of pure dimension, and $\phi(0) = 0$. Suppose that T is locally irreducible. Let $d \geq 1$ and suppose that ϕ is a dominant mapping with fibre defect less than d . If Φ is a deformation of ϕ over Y , parametrized by T and with \mathfrak{X} of pure dimension, then Φ^θ is dominant and has fibre defect less than d for every $\theta \in T$ near zero. (In particular, this is the case if Φ is an unfolding of ϕ .)*

Proof Set $l := \dim T$ and $r := \dim \mathfrak{X}$. As a flat mapping, $\pi \circ \Phi$ is open (by Douady [4]), and hence its fibre dimension is $r - l$ at every point $\xi \in \mathfrak{X}$. In particular, $X = (\pi \circ \Phi)^{-1}(0)$ is of pure dimension $r - l$. By dominance of ϕ , we have $\dim Y = \dim X - \lambda_\phi$, that is, $\dim Y = r - l - \lambda_\phi$. Since λ_ϕ is the generic fibre dimension of ϕ , it follows that Y contains an open subset Z adherent to $0 \in \mathbb{C}^n$ such that for all $\eta \in Z$, $\dim \phi^{-1}(\eta) = \lambda_\phi$. As $\phi^{-1}(\eta) = \Phi^{-1}(\eta, 0)$, it follows by upper semicontinuity of fibre dimension of Φ that $\lambda_{\Phi^\theta} \leq \lambda_\phi$ for $\theta \in T$ near zero.

On the other hand, one always has $\lambda_{\Phi^\theta} \geq \dim(\pi \circ \Phi)^{-1}(\theta) - \dim Y = r - l - \dim Y$. Therefore, $\lambda_{\Phi^\theta} \geq (r - l) - (r - l - \lambda_\phi) = \lambda_\phi$, and so $\lambda_{\Phi^\theta} = \lambda_\phi$ for all $\theta \in T$ near zero. Since all $(\pi \circ \Phi)^{-1}(\theta)$ are of the same dimension as X , the dominance of Φ^θ follows.

Finally, κ_ϕ is equal to the dimension of $\phi^{-1}(0)$, and hence $\kappa_{\Phi^\theta} \leq \kappa_\phi$, by upper semicontinuity of fibre dimension of Φ again. Thus, for all $\theta \in T$ near zero, $\kappa_{\Phi^\theta} - \lambda_{\Phi^\theta} \leq \kappa_\phi - \lambda_\phi < d$, as required. ■

Corollary 6.4 *Openness is stable under deformations: if ϕ is an open mapping and Φ is its deformation as in Proposition 6.3, then Φ^θ is open for every $\theta \in T$ near zero.*

6.3 Finite Determinacy of Flatness of Complex-analytic Mappings

We conclude the paper with a comment on Theorem 4.9 in the complex case.

Recall that, for a d -dimensional complex analytic set X in \mathbb{C}^m and a point $\xi \in X$, one defines the *multiplicity* $\mu_\xi(X)$ of X at ξ as follows: In a generic system of coordinates x at ξ in \mathbb{C}^m , the local ring $\mathcal{O}_{X,\xi}$ is a finite $\mathbb{C}\{\tilde{x}\}$ -module, where $\tilde{x} = (x_1, \dots, x_d)$. We set $\mu_\xi(X)$ to be the rank of this module. Equivalently, $\mu_\xi(X)$ is the generic cardinality of a fibre of a projection of X to a generic d -dimensional linear subspace of \mathbb{C}^m (in a neighbourhood of ξ).

Proposition 6.5 *Under the notations of Theorem 4.9, let Z denote the fibre $\phi^{-1}(0)$ (that is, $Z_0 = \mathcal{V}(J)$, where $J = (h_1, \dots, h_s, \phi_1, \dots, \phi_n)$). Then, the implications (i) \Rightarrow (ii) and (i) \Rightarrow (iii) in Theorem 4.9 hold with $\mu_0 := \mu_0(Z)$.*

Proof Indeed, directly from the definition of $\mu_0(Z)$, it follows that (after a linear change of variables x , if needed) the classes in $\mathbb{C}\{x\}/J$ of x_1, \dots, x_{n+s} are integral over $\mathbb{C}\{\tilde{x}\}$, where $\tilde{x} = (x_{n+s+1}, \dots, x_m)$. Hence, J contains a distinguished pseudo-polynomial $P_j \in \mathbb{C}\{\tilde{x}\}[x_j]$ for every $j = 1, \dots, n + s$. Therefore, $\mathfrak{N}(J)$ has a vertex on each of the axes corresponding to x_1, \dots, x_{n+s} . Since $\deg(P_j) \leq \mu_0(Z)$ for all j , we get that each of these vertices is of length at most $\mu_0(Z)$, and so the claim follows from the proof of Corollary 4.8. ■

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