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GLOBALLY SUBANALYTIC ARC-SYMMETRIC SETS

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Abstract. We show that a globally subanalytic set can be realized as the image of a semianalytic set by a finite composite of global blowings-up. As an application, we prove that a globally subanalytic arc-symmetric set of pure dimension is the image under such a composite of a real analytic manifold of the same dimension, and derive basic geometric properties of the class of globally subanalytic arc-symmetric sets. As another application, we show that globally subanalytic arc-analytic functions are blow-analytic in the sense of Kuo.

1. Introduction

The purpose of this article is to initiate a systematic study of a certain important class of subanalytic sets, which are closed under analytic continuation. Let us begin by recalling some basic notions. A set $X \subset \mathbb{R}^n$ is called *semianalytic*, when every point $x \in \mathbb{R}^n$ has an open neighbourhood U such that $X \cap U$ is a finite union of sets of the form

$$\{y \in U \ : \ f(y) = 0, g_1(y) > 0, \dots, g_k(y) > 0\} ,$$

where $f, g_1, \dots, g_k$ are real analytic functions on U. A set $Y \subset \mathbb{R}^n$ is called *subanalytic*, when for every point $x \in \mathbb{R}^n$ there are an open neighbourhood U and a bounded semianalytic set $X \subset \mathbb{R}^{n+m}$, for some m, such that $Y \cap U = \pi(X)$, where $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the coordinate projection.

For any $n \in \mathbb{Z}_+$, let $v_n : \mathbb{R}^n \to \mathbb{R}^n$ denote the semialgebraic map

$$(x_1, \dots, x_n) \ \mapsto \ \left(\frac{x_1}{\sqrt{1+x_1^2}}, \dots, \frac{x_n}{\sqrt{1+x_n^2}} \right) .$$

We say that a set $E \subset \mathbb{R}^n$ is *globally subanalytic* if its image $v_n(E)$ is subanalytic in $\mathbb{R}^n$. Since v_n is an analytic isomorphism onto the bounded open set $(-1,1)^n$, it follows that globally subanalytic sets are subanalytic. The importance of the class of globally subanalytic sets stems from the fact that they form an o-minimal structure (see Section 2 for details).

Finally, a set $E \subset \mathbb{R}^n$ is called *arc-symmetric*, when for every real analytic arc $\gamma : (-1,1) \to \mathbb{R}^n$ with $\mathrm{Int}(\gamma^{-1}(E)) \neq \varnothing$, one has $\gamma((-1,1)) \subset E$.

Throughout this paper we shall denote by $\mathscr{AR}(\mathbb{R}^n)$ the class of globally subanalytic arc-symmetric subsets of $\mathbb{R}^n$. This class includes, in particular, the arc-symmetric semialgebraic subsets of $\mathbb{R}^n$ (introduced by Kurdyka [12]). Our main

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goal here is to show that certain fundamental analytic and geometric properties of the latter class can be generalized to $\mathscr{AR}(\mathbb{R}^n)$. This is the content of Sections 4 and 5 of the paper. In fact, as shown in Section 5, many of the arguments of [12] can be easily generalized to the globally subanalytic setting, thanks to a special representation of globally subanalytic sets developed in Section 4.

Our Theorem 4.1 gives a positive answer, for globally subanalytic sets, to a long standing question in real analytic geometry. Namely, it shows that a globally subanalytic set can be realized as the image of a semianalytic set by a finite composite of global blowings-up. This result, in turn, relies on reduction of maximum fibre dimension of a generically finite complex analytic map around a compact set, proved in Section 3. This is a rather delicate point, because as shown by examples of Bierstone and Parusiński [6], in general, an analytic map does not admit such a reduction without the compactness assumption.

In the final section of the paper, we consider the class of functions that are analytic along each analytic arc and have globally subanalytic graphs. Theorem 6.2 gives a positive answer to another long standing question, showing that every such function is blow-analytic in the sense of Kuo [11]. Finally, in Theorem 6.3, we recover – in the globally subanalytic setting – a fundamental observation of Kurdyka that arc-symmetric sets are zero-sets of arc-analytic functions modulo a subset of strictly smaller dimension.

2. Preliminaries

Given the local nature of their definitions, the notions of semianalytic and subanalytic sets can be easily generalized to the setting of real analytic manifolds. We refer the reader to [3] for the basic properties of semianalytic and subanalytic sets (and an extensive bibliography of the subject). Here, let us only recall the statement of the following fundamental result of Hironaka.

Theorem 2.1 (Uniformization theorem)**.** *Let M be a real analytic manifold and let E be a closed subanalytic subset of M. Then, there exists a real analytic manifold N, of dimension* $\dim N = \dim E$*, and a proper real analytic mapping $\varphi : N \to M$, such that $\varphi(N) = E$.*

As mentioned in the Introduction, the globally subanalytic sets form an o-minimal structure $(\mathcal{S}_n)_{n\in\mathbb{N}}$. This means, by definition, that for every $n \in \mathbb{N}$, $\mathcal{S}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$, $X \in \mathcal{S}_n$ implies $X \times \mathbb{R}, \mathbb{R} \times X \in \mathcal{S}_{n+1}$, $\{(x_1, \dots, x_n) : x_1 = x_n\} \in \mathcal{S}_n$, $X \in \mathcal{S}_{n+1}$ implies $\pi(X) \in \mathcal{S}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the coordinate projection, the set $\{(x, y) \in \mathbb{R}^2 : x < y\}$ is in $\mathcal{S}_2$, and the only elements of $\mathcal{S}_1$ are the finite unions of open intervals and singletons. O-minimality is responsible for several finiteness properties that we use in Section 5. For details on o-minimal structures, see [7].

Our proof of Theorem 4.1 involves associating a morphism of complex analytic spaces to a given real analytic map. A natural setting for this construction is that of complex analytic spaces equipped with antiinvolutions. Our main reference here is [9]. Very briefly, an antiinvolution on a complex analytic space $(X, \mathcal{O}_X)$ is a morphism $\sigma : (X, \mathcal{O}_X^{\mathbb{R}}) \to (X, \mathcal{O}_X^{\mathbb{R}})$, such that $\sigma^2 = \mathrm{id}$ and σ induces a morphism of $\mathbb{R}$-ringed spaces $(X, \mathcal{O}_X) \to (X, \overline{\mathcal{O}}_X)$, where $(X, \mathcal{O}_X^{\mathbb{R}})$ is the underlying real analytic space of $(X, \mathcal{O}_X)$ and $\overline{\mathcal{O}}_X$ is the sheaf of antiholomorphic sections.

Given a complex analytic space $(X, \mathcal{O}_X)$ with an antiinvolution σ, the topological space $X^\sigma = \{x \in X : \sigma(x) = x\}$ with the natural structure sheaf $\mathcal{O}_{X^\sigma}$ forms an $\mathbb{R}$-ringed space called the *fixed part space* of (X, σ). By [9, Thm. II.4.10], $(X^\sigma, \mathcal{O}_{X^\sigma})$ is a real analytic space and a closed subspace of $(X, \mathcal{O}_X^{\mathbb{R}})$.

Conversely, let $(X, \mathcal{O}_X)$ denote a real analytic space, and let $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ be its complexification. By [9, Thm. III.3.10], there exists a Stein open neighbourhood $\tilde{Y}$ of X in $\tilde{X}$ and an antiinvolution σ on $(\tilde{Y}, \mathcal{O}_{\tilde{X}}|_{\tilde{Y}})$ (induced by the autoconjugation of $\tilde{X}$), whose fixed part space is $(X, \mathcal{O}_X)$.

3. Reduction of maximum fibre dimension of a generically finite complex analytic map around a compact set

In this section, we prove the key technical ingredient of our global smoothing result, Theorem 3.1 below. Our proof was inspired by [13, Cor. 1.2].

Let $\varphi : X \to Y$ be a morphism of complex analytic spaces, where Y is non-singular of dimension n. Let σ_X and σ_Y be antiinvolutions of X and Y, respectively, which are compatible with φ (i.e., $\sigma_Y \circ \varphi = \varphi \circ \sigma_X$). Let X^{σ_X}, Y^{σ_Y} denote the fixed parts of (X, σ_X) and (Y, σ_Y), respectively, and assume that they are non-empty.

Let K be a compact subanalytic subset of Y^{σ_Y} of positive (real) dimension d, and let U be a relatively compact open neighbourhood of K in Y^{σ_Y}. Let $L = X^{\sigma_X} \cap \varphi^{-1}(K)$ be the fixed part of $\varphi^{-1}(K)$, and suppose that $\dim_{\mathbb{R}} L = \dim_{\mathbb{R}} K$ and $\varphi|_L : L \to K$ is a proper mapping. It follows that the generic fibre dimension of $\varphi|_L$ is zero. Let $k \in \mathbb{N}$ denote the maximum fibre dimension of $\varphi|_L$, and suppose that $k > 0$.

Theorem 3.1. *Under the above assumptions, there are a relatively compact open neighbourhood V of U in Y, a relatively compact open neighbourhood W of L in X, for which $\varphi(W) \subset V$ and the maximum fibre dimension of $\varphi|_W$ is k, and a finite composite of blowings-up of coherent ideal sheaves $\pi : V' \to V$, such that V' is non-singular, π is an isomorphism over the complement of a subanalytic set Σ with $\dim(\Sigma \cap K) < d$, and the strict transform $\varphi' : W' \to V'$ of $\varphi|_W$ has maximum fibre dimension strictly less than k.*

Proof. By Zariski upper semicontinuity of fibre dimension (Cartan-Remmert Theorem), there is a relatively compact open neighbourhood W of L in X, such that the maximum fibre dimension of $\varphi|_W : W \to Y$ is equal to k. Let $Z_k(W) = \{\xi \in W : \mathrm{fbd}_\xi \varphi = k\}$ denote the locus of maximum fibre dimension. Then, $Z_k(W)$ is a complex analytic subset of W and the fibre dimension of φ is constant on $Z_k(W)$.

Suppose for a moment that $Z_k(W)$ is irreducible. By the Remmert Rank Theorem, for every $\eta \in K \cap \varphi(Z_k(W))$ and for every $\xi \in (\varphi|_L)^{-1}(\eta) \cap Z_k(W)$, there are an open neighbourhood $W^{\eta,\xi}$ of ξ in $Z_k(W)$ and an open neighbourhood $V^{\eta,\xi}$ of η in Y, such that $\varphi(W^{\eta,\xi}) \cap V^{\eta,\xi}$ is a complex analytic subset of $V^{\eta,\xi}$ (of dimension $\dim_\xi Z_k(W) - k$). By compactness of the fibre $(\varphi|_L)^{-1}(\eta)$, there are finitely many $\xi_1, \dots, \xi_s \in \varphi^{-1}(\eta)$ such that $W^{\eta,\xi_1} \cup \dots \cup W^{\eta,\xi_s}$ is an open neighbourhood of $(\varphi|_L)^{-1}(\eta) \cap Z_k(W)$ in $Z_k(W)$. Set $V^\eta = V^{\eta,\xi_1} \cap \dots \cap V^{\eta,\xi_s}$. Then, after shrinking W if needed, $\varphi(Z_k(W)) \cap V^\eta = \varphi(W^{\eta,\xi_1} \cup \dots \cup W^{\eta,\xi_s}) \cap V^\eta$ is a complex analytic subset of V^η.

Next, consider an arbitrary $\eta \in K \setminus \varphi(Z_k(W))$. Since the compact sets $L \cap Z_k(W)$ and $(\varphi|_L)^{-1}(\eta)$ are disjoint, there are an open neighbourhood V^η of η in Y and an open neighbourhood W^η of L in W such that $V^\eta \cap \varphi(Z_k(W) \cap W^\eta) = \varnothing$, and so $\varphi(Z_k(W) \cap W^\eta) \cap V^\eta$ is analytic in V^η. Finally, by compactness of L, for every $\eta \in \overline{U} \setminus K$, there are an open neighbourhood V^η of η in Y and an open neighbourhood W^η of L in W such that $V^\eta \cap \varphi(W^\eta) = \varnothing$.

Now, by compactness of $\overline{U}$, there are finitely many $\eta_1, \dots, \eta_s \in U$ such that U is contained in (the relatively compact) $V \coloneqq V^{\eta_1} \cup \dots \cup V^{\eta_s}$. Replacing W with the intersection of the corresponding $W^{\eta_1}, \dots, W^{\eta_s}$, we obtain that $\varphi(W) \subset V$ and $Z = \varphi(Z_k(W))$ is a complex analytic subset of V. Since the generic fibre dimension of $\varphi|_L$ is zero, it follows that $\dim_{\mathbb{R}}(Z \cap K) < d$.

In the general case, by relative compactness of W, the analytic set $Z_k(W)$ has finitely many irreducible components, say $Z_k^1, \dots, Z_k^r$. Shrinking W and V if needed, by the above argument, we may thus assume that each $Z_j \coloneqq \varphi(Z_k^j)$ is a complex analytic subset of V. Set $Z \coloneqq Z_1 \cup \dots \cup Z_r$, and suppose it is defined in V by a coherent ideal sheaf $\mathcal{I}$.

By [9, Thm. IV.1.4], L has in X a fundamental system of Stein σ_X-invariant open neighbourhoods, and so we may assume that $W \subset \mathbb{C}^m$ for some $m \in \mathbb{N}$. Then, by identifying W with the graph of φ, we may further assume that $W \subset V \times \mathbb{C}^m$ and $\varphi : W \to V$ is the coordinate projection.

Since k is the maximum fibre dimension of φ on W, there exists for every $\xi \in W$ an open neighbourhood W^ξ and a generic linear projection $p^\xi : \mathbb{C}^m \to \mathbb{C}^k$ such that the mapping $(\varphi, p^\xi) : W^\xi \to V^\xi \times \Omega^\xi$ is finite, where V^ξ (resp. Ω^ξ) is an open neighbourhood of $\varphi(\xi)$ in V (resp. of $p^\xi(\xi)$ in $\mathbb{C}^k$). By compactness of L, one can choose an open polydisc Λ in $\mathbb{C}^m$ and a single linear projection $p : \mathbb{C}^m \to \mathbb{C}^k$ such that $L \subset W \cap \Lambda$ and $p = p^\xi$ for all $\xi \in W \cap \Lambda$. Then, by the Remmert Proper Mapping Theorem, $\widetilde{W} \coloneqq (\varphi, p)(W \cap \Lambda)$ is an analytic subset of $V \times \Omega$, where $\Omega = p(\Lambda)$.

Let $q : V \times \Omega \to V$ be the projection. We have, for any $\xi \in W$,

$$\varphi(\xi) \in Z \iff \left(q|_{\widetilde{W}}\right)^{-1}(\varphi(\xi)) = \Omega\,, \tag{3.1}$$

and hence

$$\widetilde{W} \supset Z \times \{\tau\},\ \text{for any } \tau \in \Omega\,. \tag{3.2}$$

Now, for $\xi \in W$, let $(\eta, \tau) \in V \times \Omega$ denote the image of ξ by (φ, p) and let J^ξ denote the ideal in $\mathcal{O}_{V,\eta}$ generated by (germs at η of) all the $F_\beta(y)$ over all $F = \sum_{\beta \in \mathbb{N}^k} F_\beta(y)(t - \tau)^\beta$, $(y, t) \in V \times \Omega$, whose germs $F_{(\eta,\tau)}$ vanish on $\widetilde{W}_{(\eta,\tau)}$. Note that $F_\beta(y) = 0$ for all F_β as above if and only if $q^{-1}(y) = \Omega$. By (3.1), the latter implies that $y \in Z$, and hence by Nullstellensatz, $\mathcal{I}_\eta \subset \sqrt{\mathcal{J}^\xi}$.

By compactness of L, we can choose finitely many $\xi_1, \dots, \xi_s \in L \cap Z_k(W)$ such that $\bigcup_{j=1}^s W^{\xi_j}$ is an open neighbourhood of $L \cap Z_k(W)$ in W. By (3.2) and coherence of the full sheaf of ideals $\mathcal{I}_{\widetilde{W}}$ in $\mathcal{O}(V \times \Omega)$, J^{ξ_j} extends along $Z \times \{\tau_j\}$ to a coherent ideal $\mathcal{J}_j$ over Z, such that $(\mathcal{J}_j)_{\varphi(\xi)} = J^\xi$ for every $\xi \in Z \times \{\tau_j\}$. Extending $\mathcal{O}_V/\mathcal{J}_j$ by zero outside of Z, we obtain that $\mathcal{J}_j$ is a coherent ideal sheaf in $\mathcal{O}(V)$. Since $\sqrt{(\mathcal{J}_j)_\eta} \supset \mathcal{I}_\eta$, for all $\eta \in V$, it follows by compactness of L that there is an exponent $p_j \in \mathbb{N}$ such that $\mathcal{I}^{p_j} \subset \mathcal{J}_j$ over some relatively compact neighbourhood of U in V.

Let now $\mathcal{J}_Z \coloneqq \prod_{j=1}^{s} \mathcal{J}_j$ and let $\pi' : V' \to V$ be the blowing-up of the ideal $\mathcal{J}_Z$. Note that $\mathcal{I}^{p_1+\cdots+p_s} \subset \mathcal{J}_Z$, whence $\mathcal{I} \subset \sqrt{\mathcal{J}_Z}$ and π' is an isomorphism over $V \setminus Z$. Consider the fibre product diagram

$$\begin{array}{ccc} \widetilde{W} \times_V V' & \longrightarrow & \widetilde{W} \\ \big\downarrow & & \big\downarrow{\scriptstyle q|_{\widetilde{W}}} \\ V' & \xrightarrow{\pi'} & V \end{array}$$

and let $\widetilde{W}' \subset \widetilde{W} \times_V V'$ denote the strict transform of $\widetilde{W}$ by π'. Fix $j \in \{1, \dots, s\}$, let $\eta = \varphi(\xi_j)$ and let η' be an arbitrary point in $\pi'^{-1}(\eta) \cap V'$. Let $\xi' = (\xi_j, \eta') \in \widetilde{W}'$. The ideal $(\pi'^* \mathcal{J}_Z)_{\eta'}$ being invertible, so is $(\pi'^* \mathcal{J}_j)_{\eta'}$. Therefore, $(\pi'^* \mathcal{J}_j)_{\eta'}$ is generated by a single element $F_{\beta_0} \circ \pi'$, where by construction, F_{β_0} is a coefficient in some $F(y,t) = \sum_\beta F_\beta(y)(t-\tau_j)^\beta$ with $F(y,t) \in \mathcal{I}_{\widetilde{W}}(V^{\xi_j} \times \Omega^{\xi_j})$. Now, for y' in an open neighbourhood $V^{\eta'}$ of η',

$$F(\pi'(y'), t) = \sum_\beta F_\beta(\pi'(y'))(t-\tau_j)^\beta = F_{\beta_0}(\pi'(y')) \cdot F'(y', t)\,,$$

where

$$F'(y', t) = \sum_\beta F'_\beta(y')(t-\tau_j)^\beta, \quad \text{and} \quad F'_\beta = \frac{F_\beta \circ \pi'}{F_{\beta_0} \circ \pi'}\,.$$

One can readily see that $\widetilde{W}' \cap (V^{\eta'} \times \Omega^{\xi_j}) \subset F'^{-1}(0)$ and F' doesn't vanish identically on any fibre of the projection $V^{\eta'} \times \Omega^{\xi_j} \to V^{\eta'}$. Let $\varphi' : W' \to V'$ be the strict transform of φ by π', and let $\delta : W' \to W$ be the canonical projection. From the above and since the union of all the $V^{\eta'}$ contains the strict transform of V^{ξ_j}, it follows that the fibre dimension of φ' on the strict transform of W^{ξ_j} is strictly less than k. Since j was arbitrary and the strict transforms of W^{ξ_j} cover $\delta^{-1}(L \cap Z_k(W)) \cap W'$, it follows that the maximum fibre dimension of the whole φ' is strictly less than k in a neighbourhood of $\delta^{-1}(L) \cap W'$.

Finally, if V' were singular, we define $\pi \coloneqq \pi' \circ \pi''$, where $\pi'' : V'' \to V'$ is a resolution of singularities of (a relatively compact open neighbourhood of the strict transform of U in) V'. After shrinking the strict transform W'' of W' by π'', if needed, the strict transform $\varphi'' : W'' \to V''$ of φ' by π'' has the required properties. $\square$

4. Global smoothing of a globally subanalytic arc-symmetric set

As an immediate application of Theorem 3.1, we show that a globally subanalytic set can be realized as the image under a global modification of a semianalytic set, which admits analytic Zariski closure of the same dimension. As an intermediate step we prove the following.

Theorem 4.1. *Let E be a compact subanalytic subset of a real analytic manifold M. Assume $\dim E = d > 0$. Let U be a relatively compact open subanalytic neighbourhood of E in M. Then, there are a closed subanalytic subset Σ of U of dimension less than d, an analytic manifold M', a smooth real analytic subset Z of M' of dimension d, a compact semianalytic subset T of Z of dimension d, and a proper analytic mapping $\pi : M' \to U$, such that π is a composite of finitely* *many (restrictions of) blowings-up of coherent ideals, π is an isomorphism outside $\pi^{-1}(\Sigma)$, and $\pi^{-1}(E) = T$.*

Proof. Given a subanalytic $E \subset M$ as above, there are, by Theorem 2.1, a real analytic manifold N, of dimension $\dim N = d = \dim E$, and a proper real analytic mapping $\psi : N \to M$, such that $\psi(N) = E$. Let X, Y, φ denote the complexifications of N, M, and ψ, respectively. Let σ_X, σ_Y denote the antiinvolutions of X and Y given by the autoconjugations. Then, $\sigma_Y \circ \varphi = \varphi \circ \sigma_X$, and $N = X^{\sigma_X}$, $M = Y^{\sigma_Y}$ are the fixed parts of (X, σ_X) and (Y, σ_Y).

By repetitive use of Theorem 3.1, we obtain a relatively compact open neighbourhood V of U in Y, an open neighbourhood W of N in X, complex analytic spaces W'' and V'', a complex analytic map $\varphi'' : W'' \to V''$, and a finite composite of blowings-up of coherent ideal sheaves $\pi'' : V'' \to V$, all such that V'' is non-singular, π'' is an isomorphism over the complement of a subanalytic set Σ with $\dim(\Sigma \cap E) < d$, φ'' is the strict transform of $\varphi|_W : W \to V$ by π'', and the non-empty fibres of φ'' are zero-dimensional.

Let E'' be the real part of $V'' \cap \pi''^{-1}(E)$ and let N'' be the real part of $W'' \cap \pi''^{-1}(N)$. Then, E'' is a compact subanalytic subset of the real part of V'', N'' is a real analytic subspace of the real part of W'', and $E'' = \varphi''(N'')$. Note that E'' is, in fact, semianalytic, as the image of a finite morphism of real analytic spaces ([10, Lem. 7.3.6]).

By the Remmert Rank Theorem, every $\xi \in W''$ has an open neighbourhood, whose image by φ'' is an analytic subset of an open neighbourhood of $\varphi''(\xi)$. By relative compactness of $U'' \coloneqq \pi''^{-1}(U) \cap V''$, as in the first part of the proof of Theorem 3.1, we may assume that $Z'' \coloneqq \varphi''(W'')$ is a complex analytic subset of V'' (after shrinking V'' around U'' and W'' around N'' if needed). Clearly, $E'' \subset Z''$, and $\dim_{\mathbb{R}} E'' = \dim_{\mathbb{R}} N'' = \dim_{\mathbb{C}} W'' = \dim_{\mathbb{C}} Z''$.

By resolution of singularities (e.g., [5, Thm. 1.6]), there is a finite composite of blowings-up with smooth centers $\pi' : V' \to V''$ resolving the singularities of Z'' in V''. Moreover, π' can be chosen such that it commutes with the autoconjugations ([10, pp. 4.24-4.28]). Let M' be the real part of V', Z the real part of the strict transform Z' of Z'', let $T \coloneqq Z \cap \pi'^{-1}(E'')$, and let $\pi : M' \to U$ be the real part of $\pi'' \circ \pi'$. Then, M', Z, T, and π have all the required properties. □

We are now ready to prove our global smoothing of pure-dimensional globally subanalytic arc-symmetric sets.

Corollary 4.2. *Suppose $E \in \mathscr{AR}(\mathbb{R}^n)$ is of dimension $d > 0$. Then, there exist a proper modification $\sigma : M' \to \mathbb{R}^n$ of $\mathbb{R}^n$ in a closed subanalytic set Σ of dimension less than d, a smooth d-dimensional real analytic set $Z \subset M'$, and a semianalytic subset S of Z of dimension d, such that $\sigma = v_n^{-1} \circ \pi$, π is a composite of finitely many (restrictions of) blowings-up of coherent ideals, π is an isomorphism outside $\pi^{-1}(v_n(\Sigma))$, and $S = \sigma^{-1}(E)$. If E is pure-dimensional, then S is non-singular. Moreover, $E \cap \Sigma \in \mathscr{AR}(\mathbb{R}^n)$.*

Proof. Given E as above, the set $\overline{v_n(E)} \subset [-1,1]^n$ is a compact subanalytic set in $\mathbb{R}^n$, of positive dimension d. Let U be a relatively compact open neighbourhood of $[-1,1]^n$ in $\mathbb{R}^n$. By Theorem 4.1, there are an analytic manifold M'', a smooth real analytic subset Z' of M'' of dimension d, a compact semianalytic subset T of Z' of dimension d, and a proper analytic mapping $\pi : M'' \to U$, such that π is a composite of finitely many (restrictions of) blowings-up of coherent ideals, and $T = \pi^{-1}(\overline{v_n(E)})$.

Since the frontier $F \coloneqq \overline{v_n(E)} \setminus v_n(E)$ is a closed subanalytic set of dimension strictly less than d and $F \subset [-1,1]^n \setminus (-1,1)^n$, then $Z \coloneqq Z' \setminus \pi^{-1}(F)$ is a smooth d-dimensional real analytic subset of the manifold $M' \coloneqq M'' \setminus \pi^{-1}(F)$. (Note that up to this point one does not need the arc-symmetry of E.)

Set $S \coloneqq T \setminus \pi^{-1}(F)$. We claim that, if E is pure-dimensional, then S is the union of certain connected components of Z. Indeed, for any connected component $\tilde{S}$ of S there is a connected component $\tilde{Z}$ of Z such that $\tilde{S} \subset \tilde{Z}$. Now, $\dim \tilde{S} = d = \dim \tilde{Z}$ and $\tilde{S}$ is subanalytic and arc-symmetric. Hence $\tilde{S} = \tilde{Z}$, by Lemma 5.3 below.

To prove the last claim of the corollary, by induction, it suffices to show that if $\pi : M' \to M$ is a blowing-up with subanalytic arc-symmetric center $C \subset M$, and X is an arc-symmetric subset of M', then $\pi(X) \cup C$ is arc-symmetric. Let then $\gamma : (-1,1) \to M$ be an analytic arc such that $\mathrm{Int}(\gamma^{-1}(\pi(X) \cup C)) \neq \varnothing$. If $\mathrm{Int}(\gamma^{-1}(C)) \neq \varnothing$, then $\gamma((-1,1)) \subset C$, by arc-symmetry. Suppose then that $\gamma((-1,1)) \cap C$ consists of isolated points (which then must be the case, by subanalyticity of $\gamma^{-1}(C)$). Let $\tilde{\gamma} : (-1,1) \to M'$ be the lifting of γ (i.e., $\gamma(t) = \pi(\tilde{\gamma}(t))$ for $t \in (-1,1)$). Then, $\mathrm{Int}(\tilde{\gamma}^{-1}(X)) \neq \varnothing$, and hence $\tilde{\gamma}((-1,1)) \subset X$, by arc-symmetry. Consequently, $\gamma((-1,1)) = \pi(\tilde{\gamma}((-1,1))) \subset \pi(X)$, which completes the proof. □

5. Arc-symmetric globally subanalytic sets

We now turn to the study of the class of arc-symmetric sets within the family of globally subanalytic subsets of $\mathbb{R}^n$. Let us begin with a few immediate observations.

Remark 5.1.

(1) Every $E \in \mathscr{AR}(\mathbb{R}^n)$ is a closed set (in the Euclidean topology on $\mathbb{R}^n$). This follows from the subanalytic Curve Selection Lemma (see, e.g., [8, 1.17]).
(2) As mentioned in Section 1, $\mathscr{AR}(\mathbb{R}^n)$ contains all arc-symmetric semialgebraic subsets of $\mathbb{R}^n$.
(3) $\mathscr{AR}(\mathbb{R}^n)$ contains globally subanalytic real analytic sets in $\mathbb{R}^n$. Indeed, real analytic sets are arc-symmetric.

Theorem 5.2. *There exists a noetherian topology on $\mathbb{R}^n$, whose closed sets are precisely the elements of $\mathscr{AR}(\mathbb{R}^n)$.*

The theorem follows easily from the two lemmas below. We shall call the above noetherian topology the *$\mathscr{AR}$-topology* on $\mathbb{R}^n$. The elements of $\mathscr{AR}(\mathbb{R}^n)$ will henceforth be called $\mathscr{AR}$-closed sets.

Lemma 5.3. *Let Γ be a connected, smooth, subanalytic subset of $\mathbb{R}^n$, and let $E \subset \mathbb{R}^n$ be subanalytic and arc-symmetric. Then*

$$\Gamma \not\subset E \implies \dim(\Gamma \cap E) < \dim \Gamma\,.$$

The proof of Lemma 5.3 is identical to that of [12, 1.6], as it only relies on basic topological properties of o-minimal structures. We include it for the reader's convenience.

Proof. Suppose that $\dim \Gamma \cap E = \dim \Gamma = k$. Then, $\mathrm{Int}_\Gamma(\Gamma \cap E) \neq \varnothing$, so one can pick a point $a \in \overline{\mathrm{Int}_\Gamma(\Gamma \cap E)}$. Let then U be an open chart around a in Γ and let $\varphi : U \to \mathbb{B}^k$ be an analytic isomorphism onto the open unit ball in $\mathbb{R}^k$ such that $\varphi(a) = 0$. We have $\varphi(\mathrm{Int}_\Gamma(\Gamma \cap E)) \cap \mathbb{B}^k \neq \varnothing$, and hence can pick a

$b \in \varphi(\mathrm{Int}_\Gamma(\Gamma \cap E)) \cap \mathbb{B}^k$. Let now $x \in \mathbb{B}^k$ be arbitrary and let $\tilde{\gamma} : [-1,1] \to \mathbb{B}^k$ be an analytic arc with $\tilde{\gamma}(-1) = b$, $\tilde{\gamma}(1) = x$. Set $\gamma \coloneqq \varphi^{-1} \circ \tilde{\gamma}$. Then, $\mathrm{Int}(\gamma^{-1}(E)) \neq \varnothing$, and hence by arc-symmetry of E, $\gamma^{-1}(E) = [-1,1]$. In particular, $\varphi^{-1}(x) \in E$. Since x was arbitrary, we have $U \subset \mathrm{Int}_\Gamma(\Gamma \cap E)$, and so $a \in \mathrm{Int}_\Gamma(\Gamma \cap E)$. Since a was arbitrary, this proves $\overline{\mathrm{Int}_\Gamma(\Gamma \cap E)} = \mathrm{Int}_\Gamma(\Gamma \cap E)$, and thus $\Gamma \cap E = \Gamma$, by connectedness of Γ. □

Lemma 5.4. *Let Γ be a globally subanalytic, smooth, connected subset of $\mathbb{R}^n$, and let $\{E_i\}_{i \in I} \subset \mathscr{AR}(\mathbb{R}^n)$. Then, there exist $i_1, \dots, i_s \in I$ such that*

$$\Gamma \cap \bigcap_{i \in I} E_i \;=\; \Gamma \cap E_{i_1} \cap \dots \cap E_{i_s} \,.$$

The proof, again, is virtually identical to that of [12, Lem. 1.5]. We include it for the reader's convenience.

Proof. We proceed by induction on $k = \dim \Gamma$. If $k = 0$, then Γ is a singleton and there is nothing to show. Suppose then that $k \geq 1$ and the claim holds for all globally subanalytic, smooth, connected subsets of $\mathbb{R}^n$ of dimensions less than k. If $\Gamma \subset E_i$ for all $i \in I$, then again there is nothing to show, so let $i_0 \in I$ be such that $\Gamma \cap E_{i_0} \not\supset \Gamma$. By Lemma 5.3, the globally subanalytic set $\Gamma \cap E_{i_0}$ is then of dimension less than or equal to $k-1$. By o-minimality, $\Gamma \cap E_{i_0}$ is a finite union of connected, smooth, globally subanalytic sets $\Gamma_1, \dots, \Gamma_s$. By induction, for each $j = 1, \dots, s$, there exists a finite index subset $I_j \subset I$ such that $\Gamma_j \cap \bigcap_{i \in I} E_i = \Gamma_j \cap \bigcap_{i \in I_j} E_i$. Then,

$$\Gamma \cap \bigcap_{i \in I} E_i \;=\; (\Gamma_1 \cup \dots \cup \Gamma_s) \cap \bigcap_{i \in I} E_i \;=\; \bigcup_{j=1}^{s} (\Gamma_j \cap \bigcap_{i \in I_j} E_i) \;=\; \Gamma \cap \bigcap_{i \in I_1 \cup \dots \cup I_s} E_i \,.$$

□

Proof of Theorem 5.2. By Lemma 5.4, letting $\Gamma = \mathbb{R}^n$, intersection of an arbitrary family of $\mathscr{AR}$-closed sets is an $\mathscr{AR}$-closed set. Clearly, finite unions of arc-symmetric sets are also arc-symmetric. So are the empty set $\varnothing$ and $\mathbb{R}^n$. Noetherianity of the $\mathscr{AR}$-topology follows from Lemma 5.4 again, since every decreasing sequence of $\mathscr{AR}$-closed sets stabilizes. □

Given a set $E \in \mathscr{AR}(\mathbb{R}^n)$, we will say that E is *$\mathscr{AR}$-irreducible* if E cannot be expressed as a union of two proper $\mathscr{AR}$-closed subsets. By noetherianity of $\mathscr{AR}$-topology, every $\mathscr{AR}$-closed set E can be uniquely expressed as a finite union of $\mathscr{AR}$-irreducible sets

$$E = E_1 \cup \dots \cup E_s \,, \quad \text{where } E_i \not\subset \bigcup_{j \neq i} E_j \text{ for all } i = 1, \dots, s \,.$$

The sets $E_1, \dots, E_s$ are called the *$\mathscr{AR}$-irreducible components* of E.

By noetherianity of $\mathscr{AR}$-topology, one can also define the *$\mathscr{AR}$-closure* of an arbitrary set $S \subset \mathbb{R}^n$, denoted $\overline{S}^{\mathscr{AR}}$, as the smallest (with respect to inclusion) $\mathscr{AR}$-closed subset of $\mathbb{R}^n$ that contains S.

Remark 5.5. Unfortunately, in the subanalytic context, the $\mathscr{AR}$-closure behaves in a much less controlled way than in the semialgebraic setting of [12]. In particular, for an arbitrary globally subanalytic set S, one may have $\dim \overline{S}^{\mathscr{AR}} > \dim S$. Indeed, consider, for example, $S = \{(x,y) \in \mathbb{R}^2 \;:\; y = \sin(x),\; -1 \leq x \leq 1\}$. Then, S

is globally subanalytic in $\mathbb{R}^2$ as a bounded subanalytic set, however by analytic continuation any arc-symmetric set in $\mathbb{R}^2$ containing S must contain the whole graph of the sine function as well. Thus, $\overline{S}^{\mathscr{AR}} = \mathbb{R}^2$.

Nonetheless, the topological dimension (as a subanalytic set) of any $\mathscr{AR}$-closed set coincides with its $\mathscr{AR}$-Krull dimension, as shown below. For a non-empty $\mathscr{AR}$-closed set E we define its *Krull dimension* as

$$\dim_K E = \sup\{l \in \mathbb{N} \,:\, \exists E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_l \subset E, \ \text{ with } E_j \ \mathscr{AR}\text{−irreducible}\}\,.$$

Theorem 5.6. *If E is a non-empty $\mathscr{AR}$-closed set in $\mathbb{R}^n$, then*

$$\dim_K E \;=\; \dim E\,,$$

where $\dim E$ is the supremum of dimensions of real analytic submanifolds of E.

Proof. By Proposition 5.8 below, we have $\dim_K E \leq \dim E$. For the proof of the other inequality, we proceed by induction on $d = \dim E$. The base case being clear, assume $d \geq 1$. By the Good Directions Lemma in o-minimal structures (see [8, 4.9] or [7, Thm. VII.4.2]), there is a d-dimensional linear subspace U of $\mathbb{R}^n$ such that the orthogonal projection $\pi : \mathbb{R}^n \to U$ has finite fibres when restricted to E. Suppose U is spanned by vectors $u_1, \dots, u_d$ in $\mathbb{R}^n$, and let $V = \operatorname{span}\{u_2, \dots, u_d\}$. Then, the set $F = E \cap \pi^{-1}(V)$ is $\mathscr{AR}$-closed as the intersection of two $\mathscr{AR}$-closed sets, and of dimension $d-1$. By the finiteness of decomposition into $\mathscr{AR}$-irreducible components, at least one such component of F is of dimension $d-1$. □

For the next result recall that, by Corollary 4.2, for any $E \in \mathscr{AR}(\mathbb{R}^n)$ of dimension $d > 0$, there are a proper modification $\sigma : M' \to \mathbb{R}^n$ of $\mathbb{R}^n$ in a closed subanalytic set Σ of dimension less than d, a d-dimensional smooth analytic set $Z \subset M'$, and a semianalytic subset S of Z of dimension d, such that $\sigma = v_n^{-1} \circ \pi$, π is a composite of finitely many blowings-up of coherent ideals, π is an isomorphism outside $\pi^{-1}(v_n(\Sigma))$, $S = \sigma^{-1}(E)$, and $E \cap \Sigma \in \mathscr{AR}(\mathbb{R}^n)$. We denote by $\operatorname{Reg}_d E$ the locus of smooth points of E of dimension d.

Proposition 5.7. *Let $E \subset \mathbb{R}^n$ be an $\mathscr{AR}$-irreducible set of dimension $d > 0$. Let $\sigma : M' \to \mathbb{R}^n$ and $Z \subset M'$ be as in Corollary 4.2. Then, there is a connected component $\tilde{Z}$ of Z such that*

$$\sigma(\tilde{Z}) \supset \overline{\operatorname{Reg}_d E}\,.$$

Proof. Let $\sigma^{-1}(E) = E_1 \cup \cdots \cup E_s$ be the decomposition into $\mathscr{AR}$-irreducible components of the $\mathscr{AR}$-closed set $\sigma^{-1}(E)$. Assume, without loss of generality, that $\dim E_1 = d$. Let $\tilde{Z}$ be the connected component of Z, for which $E_1 \subset \tilde{Z}$. Since $\dim \tilde{Z} = \dim E_1$, then $\tilde{Z} = E_1$, by Lemma 5.3. We shall show that $\sigma(\tilde{Z})$ contains a non-empty open subset of every connected component of $\operatorname{Reg}_d E \setminus \Sigma$. To this end, let us first prove the following:

Claim 1. $\overline{\sigma(\tilde{Z})}^{\mathscr{AR}} \subset \sigma(\tilde{Z}) \cup (E \cap \Sigma)\,.$

It suffices to show that $F \coloneqq \sigma(\tilde{Z}) \cup (E \cap \Sigma)$ is arc-symmetric. Let $\gamma : (-1,1) \to \mathbb{R}^n$ be an analytic arc such that $\operatorname{Int}(\gamma^{-1}(F)) \neq \varnothing$. If $\operatorname{Int}(\gamma^{-1}(E \cap \Sigma)) \neq \varnothing$, then $\gamma((-1,1)) \subset E \cap \Sigma$, by arc-symmetry. Suppose then that $\operatorname{Int}(\gamma^{-1}(E \cap \Sigma)) = \varnothing$. Then, γ can be lifted to $\tilde{\gamma} : (-1,1) \to M'$ (so that $\gamma(t) = \sigma(\tilde{\gamma}(t))$, for all $t \in (-1,1)$). It follows that $\operatorname{Int}(\tilde{\gamma}^{-1}(\tilde{Z})) \neq \varnothing$, and hence $\tilde{\gamma}((-1,1)) \subset \tilde{Z}$, by arc-symmetry. Thus, $\gamma((-1,1)) = \sigma(\tilde{\gamma}((-1,1))) \subset \sigma(\tilde{Z})$, which completes the proof of Claim 1.

Suppose now that there exists a connected component of $\mathrm{Reg}_d E \setminus \Sigma$ such that $\sigma(\tilde{Z})$ does not contain its non-empty open subset. Let $\mathcal{C}$ be the union of all such connected components of $\mathrm{Reg}_d E \setminus \Sigma$. Then, $\dim(\sigma(\tilde{Z}) \cap \mathcal{C}) < d$. Since σ is an isomorphism over $\mathrm{Reg}_d E \setminus \Sigma$ and $\dim(E \cap \Sigma) < d$, it follows that for every connected component C of $\mathrm{Reg}_d E \setminus \Sigma$ there exists precisely one component of Z whose image by σ contains an open subset of C. Let $Z' := Z \setminus \tilde{Z}$ be the union of the remaining components of Z. By Claim 1, it now follows that $E = \overline{\sigma(\tilde{Z})}^{\mathscr{AR}} \cup \overline{\sigma(Z')}^{\mathscr{AR}}$ is a decomposition of E into proper $\mathscr{AR}$-closed subsets.

The above contradiction proves that $\sigma(\tilde{Z}) \cap C$ contains a non-empty open subset of C, for every connected component C of $\mathrm{Reg}_d E \setminus \Sigma$. On the other hand, $\sigma(\tilde{Z})$ is a closed subset of E, since σ is proper and $\tilde{Z} = E_1$ is closed. Therefore, by connectedness, $\sigma(\tilde{Z}) \cap C = C$ for every C as above, which completes the proof of the proposition. □

Proposition 5.8. *Let $E, F \in \mathscr{AR}(\mathbb{R}^n)$, $F \subsetneq E$, and suppose that E is $\mathscr{AR}$-irreducible of positive dimension. Then,* $\dim F < \dim E$.

Proof. For a proof by contradiction, suppose that $\dim F = \dim E = d$. Then, there exists a connected component C of $\mathrm{Reg}_d E$ such that $\dim(F \cap C) = d$. By Lemma 5.3, $F \cap C = C$. Let σ and Z be as in Proposition 5.7. As in the proof of Proposition 5.7, there is a component $\tilde{Z}$ of Z, such that $\tilde{Z}$ coincides with an $\mathscr{AR}$-irreducible component of $\sigma^{-1}(F)$ and $\sigma(\tilde{Z}) \supset C$. Then, $F \supset \sigma(\tilde{Z}) \supset \overline{\mathrm{Reg}_d E}$.

Suppose that $G := E \setminus F \neq \varnothing$. We claim that then $\overline{G}$ is $\mathscr{AR}$-closed. Indeed, for any analytic arc $\gamma : (-1, 1) \to \mathbb{R}^n$ with $\mathrm{Int}(\gamma^{-1}(\overline{G})) \neq \varnothing$, one has $\mathrm{Int}(\gamma^{-1}(F)) = \varnothing$ for else $\gamma((-1, 1)) \subset F$, by arc-symmetry. Therefore, $\gamma((-1, 1))$ intersects F (at most) at isolated points, and hence by continuity $\gamma((-1, 1)) \subset \overline{G}$. Consequently, $E = F \cup \overline{G}$ is a union of two proper $\mathscr{AR}$-closed subsets, contradicting $\mathscr{AR}$-irreducibility of E. Thus, one must have $G = \varnothing$, that is, $E = F$, which again contradicts the assumptions of the proposition. □

Theorem 5.9. *For every $E \in \mathscr{AR}(\mathbb{R}^n)$ of dimension $d > 0$, there exists $F \in \mathscr{AR}(\mathbb{R}^n)$ such that* $\dim(E \cap F) < d$ *and $E \setminus F$ is a d-dimensional manifold.*

Proof. Assume without loss of generality that E is $\mathscr{AR}$-irreducible. Let $\sigma : M' \to \mathbb{R}^n$, $\Sigma \subset \mathbb{R}^n$, and $Z \subset M'$ be as in the paragraph preceding Proposition 5.7. Recall that, by Claim 1 in the proof of Proposition 5.7, for every connected component $\tilde{Z}$ of Z, we have $\overline{\sigma(\tilde{Z})}^{\mathscr{AR}} \subset \sigma(\tilde{Z}) \cup (E \cap \Sigma)$.

If $E \setminus \Sigma \subset \mathrm{Reg}_d E$, then there is nothing to show. Suppose then that $(E \setminus \Sigma) \setminus \mathrm{Reg}_d E \neq \varnothing$. Let $\mathrm{Reg}_{<d} E$ denote the locus of smooth points of E of dimension(s) less than d. For every connected component C of $\mathrm{Reg}_{<d} E \setminus \Sigma$, there is precisely one connected component $\tilde{Z}_C$ of Z such that $\sigma(\tilde{Z}_C)$ contains a non-empty open subset of C, and hence $\sigma(\tilde{Z}_C) \supset C$. For such $\tilde{Z}_C$ one has $\dim \sigma(\tilde{Z}_C) < d$, for else $\sigma(\tilde{Z}_C) \supset \overline{\mathrm{Reg}_d E}$, as in the proof of Proposition 5.7; a contradiction. It follows that

$$\overline{\sigma(\bigcup_C \tilde{Z}_C)}^{\mathscr{AR}} \subset \bigcup_C \sigma(\tilde{Z}_C) \cup (E \cap \Sigma)$$

is of dimension strictly less than d, where the union is over all connected components C of $\mathrm{Reg}_{<d}E \setminus \Sigma$. Therefore, the set

$$F := (E \cap \Sigma) \cup \overline{\sigma(\bigcup_C \tilde{Z}_C)}^{\mathscr{AR}}$$

has all the required properties. □

Remark 5.10. Note that, in general, one cannot expect that $\mathrm{Reg}_d E = E \setminus F$ for some $\mathscr{AR}$-closed set F. Indeed, this may not be true even if E is real algebraic.

6. Globally subanalytic arc-analytic functions

Let $E \in \mathscr{AR}(\mathbb{R}^n)$ be non-empty. A function $f : E \to \mathbb{R}$ is called *arc-analytic*, when $f \circ \gamma$ is an analytic function for every analytic arc $\gamma : (-1, 1) \to E$. It is called globally subanalytic, when the graph Γ_f of f is a globally subanalytic set in $\mathbb{R}^{n+1}$. Following [12], we will denote by $\mathscr{A}_a(E)$ the ring of all arc-symmetric globally subanalytic functions on E.

It is well known that every globally subanalytic arc-analytic function is continuous in the Euclidean topology (see, e.g., [4, Lem. 6.8]). Moreover, by a straightforward adaptation of [12, Prop. 5.1], one has the following.

Remark 6.1. Let $E \in \mathscr{AR}(\mathbb{R}^n)$ be non-empty, and let $f : E \to \mathbb{R}^m$ be a globally subanalytic function whose all components are arc-analytic. Then

(i) $\Gamma_f \in \mathscr{AR}(\mathbb{R}^n \times \mathbb{R}^m)$.
(ii) If $Z \in \mathscr{AR}(\mathbb{R}^m)$, then $f^{-1}(Z) \in \mathscr{AR}(\mathbb{R}^n)$.

Let now M be a real analytic manifold, and let $f : M \to \mathbb{R}$ be a function. We will say that f is a *globally subanalytic function on M*, if M admits a closed embedding into $\mathbb{R}^n$ for some $n > 0$ such that the graph Γ_f is a globally subanalytic set in $\mathbb{R}^{n+1}$. The following theorem shows that globally subanalytic arc-analytic functions are blow-analytic in the sense of Kuo [11].

Theorem 6.2. *Let M be a real analytic manifold and let $f : M \to \mathbb{R}$ be a globally subanalytic arc-analytic function on M. Then, there are a real analytic manifold $\widetilde{M}$ and a proper modification $\sigma : \widetilde{M} \to M$, such that σ is a composite of finitely many blowings-up of coherent ideal sheaves and $f \circ \sigma$ is real analytic.*

Proof. Without loss of generality, suppose that M is a closed real analytic submanifold of $\mathbb{R}^n$, of dimension $\dim M = d > 0$, and $f : M \to \mathbb{R}$ is an arc-analytic function whose graph Γ_f is globally subanalytic in $\mathbb{R}^{n+1}$. Then, $\Gamma_f \in \mathscr{AR}(\mathbb{R}^{n+1})$ and Γ_f is of pure dimension d. Consider a compact subanalytic set $E \subset \mathbb{R}^{n+1}$ defined as $E := \overline{v_{n+1}(\Gamma_f)}$. By Theorem 2.1, there are a real analytic manifold N, of dimension $\dim N = d$, and a proper real analytic mapping $\psi : N \to \mathbb{R}^{n+1}$, such that $\psi(N) = E$. Let $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection onto the first n coordinates. Then, $p \circ \psi$ is proper and generically finite, and $p(E)$ is compact and subanalytic in $\mathbb{R}^n$. Let U be a relatively compact open subanalytic neighbourhood of $[-1, 1]^n$ in $\mathbb{R}^n$. Let X, Y, and φ denote the complexifications of N, $\mathbb{R}^n$, and $p \circ \psi$, respectively. We may assume that $Y = \mathbb{C}^n$ and φ factors through the projection $Y \times \mathbb{C} \to Y$ onto the first n coordinates.

As in the proof of Theorem 4.1, one can now find a relatively compact open neighbourhood V of U in Y, an open neighbourhood W of N in X, complex analytic spaces V' and W', a complex analytic map $\varphi' : W' \to V'$, and a finite composite of blowings-up of coherent ideal sheaves $\pi' : V' \to V$, all such that:

(i) V' is non-singular
(ii) π' is an isomorphism over the complement of a subanalytic set Σ with $\dim(p(E) \cap \Sigma) < d$
(iii) $\varphi' : W' \to V'$ is the strict transform of $\varphi : W \to V$ by π', and the non-empty fibres of φ' are zero-dimensional
(iv) the image $Z' = \varphi'(W')$ is a smooth complex analytic subset of V', of dimension $\dim(Z') = d$.

Moreover, by uniqueness of fibre product, $V' \times_V W \cong (V' \times \mathbb{C}) \times_{V \times \mathbb{C}} W$, and hence φ' factors as $\varphi' = q' \circ \psi'$, where $q' : V' \times \mathbb{C} \to V'$ is the projection and $\psi' : W' \to V' \times \mathbb{C}$ is a complex analytic map. Clearly, the non-empty fibres of ψ' are zero-dimensional.

Let $A \coloneqq \psi'(W')$. By relative compactness of $U' \coloneqq \pi'^{-1}(U)$, as in the proof of Theorem 4.1, we may assume that A is analytic in $V' \times \mathbb{C}$. The projection $q'|_A : A \to Z'$ being proper, A is a subset of the zero-set of a non-zero polynomial $P(y', z) \in \mathcal{O}(V')[z]$.

Let Z be the real part of Z', and let π be the real part of π'. Further, let $F \coloneqq p(E) \setminus p(v_{n+1}(\Gamma_f)) = p(E) \setminus v_n(M)$. Since M is closed in $\mathbb{R}^n$, then $F \subset [-1,1]^n \setminus (-1,1)^n$. By Proposition 5.7, there is a connected component M' of the real analytic manifold $Z \setminus \pi^{-1}(F)$, such that $(v_n^{-1} \circ \pi)(M') = M$. The function $f \circ v_n^{-1} \circ \pi$ is arc-analytic and satisfies the equation $Q(x, (f \circ v_n^{-1} \circ \pi)(x)) = 0$, where $Q(x,t)$ is the real part of $P(y', z)$. Therefore, by [4, Thm. 1.1], there is a (finite, by relative compactness of M') composite of blowings-up $\tau : \widetilde{M} \to M'$ such that $f \circ v_n^{-1} \circ \pi \circ \tau : \widetilde{M} \to \mathbb{R}$ is real analytic. □

We conclude this paper with a brief discussion of the relationship between the $\mathscr{AR}$-closed sets and the globally subanalytic arc-analytic functions on $\mathbb{R}^n$. We believe that, as in the semialgebraic setting, the $\mathscr{AR}$-closed sets are precisely the zero-loci of globally subanalytic arc-analytic functions (Conjecture 6.4 below). It appears, however, that to establish such a relationship one would need a subanalytic analogue of the Efroymson Extension Theorem, which is currently unavailable. As a partial evidence for the conjecture, we recover below a fundamental observation of Kurdyka that arc-symmetric sets are zero-sets of arc-analytic functions modulo a subset of strictly smaller dimension (cf. [12, Thm. 6.2]).

Theorem 6.3. *Let $E \subset \mathbb{R}^n$ be a non-empty $\mathscr{AR}$-closed set of dimension $\dim E = d$. Then, there exists a function $f \in \mathscr{A}_a(\mathbb{R}^n)$ such that*

$$E \subset f^{-1}(0), \quad \text{and} \quad \dim(f^{-1}(0) \setminus E) < d\,.$$

Proof. We proceed by indunction on d. If $d = 0$, then E is finite and hence a zero-set of a polynomial. Suppose then that $d \geq 1$ and the theorem holds for all arc-symmetric globally subanalytic sets of dimensions less than d.

Given E as above, the set $\overline{v_n(E)} \subset [-1,1]^n$ is a compact subanalytic set in $\mathbb{R}^n$, of dimension d. Let U be a relatively compact open subanalytic neighbourhood of $[-1,1]^n$ in $\mathbb{R}^n$. By Theorem 4.1, there are a closed subanalytic set $\Sigma \subset U$ of dimension less than d, a real analytic manifold M', a smooth real analytic subset Z of M', of dimension d, and a finite composite of blowings-up $\pi : M' \to U$, such that π is an isomorphism outside $\pi^{-1}(\Sigma)$ and $\pi^{-1}(E) \subset Z$.

Let $E_1, \dots, E_s$ be the d-dimensional $\mathscr{AR}$-irreducible components of E. By Proposition 5.7, there are connected components $Z_1, \dots, Z_t$ of Z, each a manifold of dimension d, such that

$$\pi(Z_1 \cup \dots \cup Z_t) \supset v_n(\mathrm{Reg}_d E_1 \cup \dots \cup \mathrm{Reg}_d E_s) \supset v_n(\mathrm{Reg}_d E)\,.$$

Let Z' denote the real analytic submanifold $Z_1 \cup \dots \cup Z_t$ in M'. By [9, Thm. IV.2.1], there is a real analytic function $g : M' \to \mathbb{R}$ such that $g^{-1}(0) = Z'$.

Next, consider $F \coloneqq v_n^{-1}(\Sigma)$. We have $F \in \mathscr{AR}(\mathbb{R}^n)$ and $\dim F < d$. Hence, by the inductive hypothesis, there is a function $h \in \mathscr{A}_a(\mathbb{R}^n)$ such that $h^{-1}(0) \supset F$. We may now define a subanalytic function $f_1 : \mathbb{R}^n \to \mathbb{R}$ by

$$f_1(x) = \begin{cases} (g \circ \pi^{-1} \circ v_n)(x) \cdot h(x), & x \in \mathbb{R}^n \setminus F \\ 0, & x \in F\,. \end{cases}$$

Note first that f_1 is, in fact, globally subanalytic (since $g \circ \pi^{-1}|_{v_n(\mathbb{R}^n)}$ is a restriction of a subanalytic function on U). Moreover, we claim that f_1 is arc-analytic. Indeed, let $\gamma : (-1,1) \to \mathbb{R}^n$ be an arbitrary analytic arc. If $\mathrm{Int}(\gamma^{-1}(F)) \neq \varnothing$, then $\gamma((-1,1)) \subset F$, by arc-symmetry of F, and so $f_1 \circ \gamma \equiv 0$. If, in turn, $\mathrm{Int}(\gamma^{-1}(F)) = \varnothing$, then γ lifts by $v_n^{-1} \circ \pi$ to a unique analytic arc $\tilde{\gamma} : (-1,1) \to M'$ such that $\gamma = v_n^{-1} \circ \pi \circ \tilde{\gamma}$. We have then

$$(f_1 \circ \gamma)(t) = (g \circ \tilde{\gamma})(t) \cdot h(\gamma(t)), \quad t \in (-1,1)\,,$$

which is analytic. Therefore, $f_1 \in \mathscr{A}_a(\mathbb{R}^n)$.

By construction, $\mathrm{Reg}_d E \subset f_1^{-1}(0)$. Hence, by Theorem 5.9, there exists $G \in \mathscr{AR}(\mathbb{R}^n)$ such that

$$\dim G < d, \quad \text{and} \quad E \setminus f_1^{-1}(0) \subset G\,.$$

By the inductive hypothesis, there exists $f_2 \in \mathscr{A}_a(\mathbb{R}^n)$ such that $f_2^{-1}(0) \supset G$. It follows that the function $f \coloneqq f_1 \cdot f_2$ has the required properties. □

Conjecture 6.4. *We conjecture that, as in the semialgebraic setting (*[1, 2]*), every $\mathscr{AR}$-closed set is precisely the zero locus of a globally subanalytic arc-analytic function, and every globally subanalytic arc-analytic function on an $\mathscr{AR}$-closed set $E \subset \mathbb{R}^n$ is a restriction to E of a globally subanalytic arc-analytic function on $\mathbb{R}^n$.*

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