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Journal of Algebra 289 (2005) 148–160
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Flatness testing and torsion freeness of analytic tensor powers

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Received 18 July 2004
Available online 23 March 2005
Communicated by Craig Huneke

Abstract

Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of complex analytic spaces, where X_ξ is reduced of pure dimension and Y_η is smooth of dimension n. We give several sufficient conditions for the following characterization of flatness to hold: f_ξ is flat if and only if the nth analytic tensor power $\underbrace{\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}}_{n \text{ times}}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module.

Keywords: Flatness; Torsion freeness; Vertical components; Gabrielov regularity

1. Introduction and main result

Given a holomorphic mapping $f : X \to Y$ of complex analytic spaces, with $f(\xi) = \eta$, let $f_\xi : X_\xi \to Y_\eta$ denote the germ of f at ξ, and let $f^{\{i\}}_{\xi^{\{i\}}} : X^{\{i\}}_{\xi^{\{i\}}} \to Y_\eta$ be the germ at $\xi^{\{i\}} = (\xi, \ldots, \xi) \in X^i$ of the induced canonical map from the i-fold fibre power of X over Y. The main result of this paper is the following criterion for flatness of analytic morphisms:

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1 The author's research was supported by NSERC Postdoctoral Fellowship PDF-267954-2003.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2005.01.046

Theorem 1.1. *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces, where X_ξ is reduced of pure dimension and Y_η is smooth of dimension n. Suppose that one of the following conditions is satisfied*:

(1) $n < 3$;
(2) $f_\xi : X_\xi \to Y_\eta$ *is a Nash morphism of Nash germs*;
(3) *the singular locus of X_ξ is mapped into a proper analytic subgerm of Y_η*;
(4) *the local ring of the source $\mathcal{O}_{X,\xi}$ is Cohen–Macaulay.*

Then, f_ξ is flat if and only if the nth analytic tensor power $\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module.

Let Ω be an open set in $\mathbb{C}^m$. An analytic function $f \in \mathcal{O}(\Omega)$ is called a *Nash function* if it is algebraic over the ring of regular functions on Ω. An analytic set X is a *Nash set* if it can locally be defined by Nash functions, and *Nash mappings* are analytic mappings whose all components are Nash functions (see Section 2 for details).

Note that in the case of finite modules, flatness is equivalent to freeness. Also, for finite modules M and N over a local analytic algebra R, their analytic tensor product, $M \hat{\otimes}_R N$, equals the ordinary one, $M \otimes_R N$. (By a *local analytic algebra* we mean a ring of the form $\mathbb{C}\{x\}/I$, where $x = (x_1, \ldots, x_m)$ and I is an ideal in $\mathbb{C}\{x\}$.) Thus, the above theorem can be viewed as a generalization to morphisms of local analytic algebras of the following fundamental result of Auslander (the finite case being covered by our condition (3)):

Theorem 1.2 (Auslander [5, Theorem 3.2]). *Let R be an unramified regular local ring of dimension $n > 0$ and let M be a finite R-module. Then M is R-free if and only if the nth tensor power $M^{\otimes n}$ is a torsion-free R-module.*

(Auslander's result was later extended by Lichtenbaum [14] to arbitrary regular local rings.) Theorem 1.1 is a step towards a proof of the following general claim (cf. [3, Conjecture 2.4]):

Conjecture 1.3. *Let $\varphi : R \to A$ be a homomorphism of local analytic $\mathbb{C}$-algebras, where R is regular of dimension n. Then the following conditions are equivalent*:

(i) *A is R-flat*;
(ii) *the nth analytic tensor power $A \hat{\otimes}_R \cdots \hat{\otimes}_R A$ is a torsion-free R-module.*

We believe there are good reasons to expect the conjecture be true, as explained below. Also, it seems plausible that proving Conjecture 1.3 should eventually lead to obtaining the following Galligo–Kwieciński result by algebraic means:

> *Let R be a finitely generated regular $\mathbb{C}$-algebra of dimension n and let A be a finitely generated R-algebra. Then A is R-flat if and only if the nth tensor power $A \otimes_R \cdots \otimes_R A$ is a torsion-free R-module.*

This algebraic generalization of Theorem 1.2 was conjectured by Vasconcelos, who proved it in the case $n = 2$ (see [18, Proposition 6.1]). The result was later proved by Galligo and Kwieciński (see Theorem 2.3 below) for arbitrary n, under the additional hypothesis that the algebra A is equidimensional over $\mathbb{C}$. The Galligo–Kwieciński proof however makes use of transcendental methods that cannot be translated into a purely algebraic argument. (We recall their result and some of the methods in Section 2, as our proof of Theorem 1.1 relies strongly on them.)

The results of this paper arose from our study of the relationship between degeneracies of the family of fibres of an analytic mapping and the existence of *vertical components* in fibre powers of the mapping [1,2]. It thus seems natural and more intuitive to work in this setup, although most of the paper could be as well formulated in the language of local analytic algebras.

There are in fact two natural notions of a *vertical component* (and some interesting information about a morphism can be obtained by analyzing the relations between them, see [2]): Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. An irreducible (isolated or embedded) component W_ξ of X_ξ is called *algebraic vertical* if there exists a nonzero element $a \in \mathcal{O}_{Y,\eta}$ such that (the pullback of) a belongs to the associated prime $\mathfrak{p}$ in $\mathcal{O}_{X,\xi}$ corresponding to W_ξ. Equivalently, W_ξ is *algebraic vertical* if an arbitrarily small representative W of W_ξ is mapped into a proper analytic subset of a neighbourhood of η in Y. We say that W_ξ is *geometric vertical* if an arbitrarily small representative of W_ξ is mapped into a nowhere dense subset of a neighbourhood of η in Y. (In the context of Galligo and Kwieciński's [9], this is equivalent to the *hypergerm* $f_\xi(W_\xi)$ having empty interior in Y_η with the transcendental topology.)

The concept of a vertical component (introduced by Kwieciński in [12]) comes up naturally as an equivalent of torsion in algebraic geometry and the two notions of a vertical component coincide in the algebraic case (over an irreducible target). However, it is no longer so in the analytic category. In principle, the existence of the *geometric vertical* components is a weaker condition than the presence of the *algebraic vertical* ones. Indeed, any *algebraic vertical* component (over an irreducible target) is *geometric vertical,* since a proper analytic subset of a locally irreducible analytic set has empty interior. The converse is not true though, as can be seen in the following classical example of Osgood (cf. [10, Kapitel II, §5]):

$$f : \mathbb{C}^2 \ni (x, y) \mapsto \big(x, xy, xy\mathrm{e}^y\big) \in \mathbb{C}^3.$$

Here the image of an arbitrarily small neighbourhood of the origin is nowhere dense in $\mathbb{C}^3$, but its Zariski closure has dimension 3 and therefore the image is not contained in a proper locally analytic subset of the target.

The *geometric vertical* components have proved to be a powerful tool in analytic geometry (see [9,12,13]), allowing for the use of transcendental methods when commutative algebra seemed to fail. On the other hand, the algebraic approach, introduced in [1] and [2], has an advantage of a direct algebraic control over the geometry of analytic morphisms, as all the statements about *algebraic vertical* components (as opposed to *geometric vertical*) can be restated in terms of torsion freeness of the local rings:

Remark 1.4. $f_\xi : X_\xi \to Y_\eta$ has no (isolated or embedded) algebraic vertical components if and only if the local ring $\mathcal{O}_{X,\xi}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module.

(This follows from "prime avoidance," see, e.g., [7, Section 3.2].)

Also, it seems plausible that algebraic properties of analytic morphisms, like flatness, could be controlled by means of *algebraic vertical* components rather than the *geometric vertical* ones. In addition to Theorem 1.1, we present a few more arguments for Conjecture 1.3 below.

1. In [12], Kwieciński showed that flatness of a morphism $f_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces, with Y_η reduced and irreducible, is equivalent to torsion freeness of *all* the analytic tensor powers
$$\underbrace{\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}}_{i \text{ times}} \quad \text{for } i \geqslant 1.$$
2. In fact, as we proved in [1] and [2], for a morphism $f_\xi : X_\xi \to Y_\eta$ and a finite $\mathcal{O}_{X,\xi}$-module M that is not $\mathcal{O}_{Y,\eta}$-flat, already the μth analytic tensor power
$$\underbrace{M \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} M}_{\mu \text{ times}}$$
has nonzero $\mathcal{O}_{Y,\eta}$-torsion, where μ is the length of a minimal set of generators of the flattener ideal of M in $\mathcal{O}_{Y,\eta}$. (See [6, Theorem 7.12] for the definition and universal property of Hironaka's *local flattener*.)
3. The conjecture is true on a "reduced level," under the hypothesis that the domain be of pure dimension. That is, the following theorem holds (cf. [2, Theorem 2.2]):

Theorem 1.5. *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces. Let X_ξ be of pure dimension and let Y_η be reduced and irreducible of dimension n. Then the following conditions are equivalent*:

(i) f_ξ *is open*;
(ii) *the reduced nth analytic tensor power* $(\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi})_{\mathrm{red}}$ *is a torsion-free $\mathcal{O}_{Y,\eta}$-module.*

2. Toolbox

To keep the article self-contained, we gathered in this section most of the local analytic and commutative algebra tools used in the course of the proof of our main result. We start with recalling the Nash category terminology (for a thorough treatment the reader is refered to [17]).

Let Ω be an open subset of $\mathbb{C}^m$, and let $x = (x_1, \ldots, x_m)$ be a system of m complex variables. A function f analytic on Ω is called a *Nash function* at $x_0 \in \Omega$ if there exist an open neighbourhood U of x_0 in Ω and a polynomial $P(x, y) \in \mathbb{C}[x, y]$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. An analytic function is a Nash function on Ω if it is a Nash function at every point of Ω. An analytic mapping $f = (f_1, \dots, f_n) : \Omega \to \mathbb{C}^n$ is a *Nash mapping* if each of its components is a Nash function on Ω.

A subset X of Ω is called a *Nash subset* of Ω if for every $x_0 \in \Omega$ there exist an open neighbourhood U of x_0 in Ω and Nash functions $f_1, \dots, f_s$ on U, such that $X \cap U = \{x \in U\colon\ f_1(x) = \cdots = f_s(x) = 0\}$. A germ X_ξ at $\xi \in \mathbb{C}^m$ is a *Nash germ* if there exists an open neighbourhood U of ξ in $\mathbb{C}^m$ such that $X \cap U$ is a Nash subset of U. Equivalently, X_ξ is a Nash germ if its defining ideal can be generated by power series algebraic over the polynomial ring $\mathbb{C}[x]$; that is, $\mathcal{O}_{X,\xi} \cong \mathbb{C}\{x\}/(f_1, \dots, f_s)\mathbb{C}\{x\}$ with $f_j \in \mathbb{C}\langle x\rangle$, $j = 1, \dots, s$, where $\mathbb{C}\langle x\rangle$ denotes the algebraic closure of $\mathbb{C}[x]$ in $\mathbb{C}[[x]]$.

The Nash category fits between the algebraic and analytic categories in a way that allows use of transcendental methods to obtain strong algebraic results (like the one we are after). Geometrically, Nash sets are built, locally, from analytic branches of algebraic sets. Moreover, with help of Artin's approximation theorem [4, Theorem 1.7], one easily obtains the following:

Proposition 2.1 [3, Proposition 5.1]. *If W is an* (*isolated or embedded*) *irreducible component of a Nash germ* (*respectively set*), *then W is a Nash germ* (*respectively set*) *itself.*

Next, notice the relationship between the fibre product of analytic mappings and the analytic tensor product. In fact, a reader not familiar with the concept may consider the following a definition of the analytic tensor product (see, e.g., [8, Section 0.28]):

Remark 2.2. Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be holomorphic mappings of analytic spaces, with $f_1(\xi_1) = f_2(\xi_2) = \eta$. Then the local rings $\mathcal{O}_{X_i,\xi_i}$ $(i = 1, 2)$ are $\mathcal{O}_{Y,\eta}$-modules and the local ring at (ξ_1, ξ_2) of the fibre product $Z = X_1 \times_Y X_2$ satisfies the identity

$$\mathcal{O}_{Z,(\xi_1,\xi_2)} = \mathcal{O}_{X_1,\xi_1} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X_2,\xi_2}.$$

In particular, for a morphism $f_\xi : X_\xi \to Y_\eta$, we will tacitly identify the ith analytic tensor power $\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}$ with the local ring of the i-fold fibre power $\mathcal{O}_{X^{\{i\}},\xi^{\{i\}}}$ for $i \geqslant 1$.

Let us now recall some results of Galligo and Kwieciński's [9] that will play an important role in the proof of Theorem 1.1:

Theorem 2.3 (Galligo–Kwieciński [9, Theorem 6.1]). *Let $f_\xi : X_\xi \to Y_\eta$ be a germ of a complex analytic map of germs of complex analytic spaces. Suppose that X_ξ and Y_η are reduced, that X_ξ is of pure dimension, and that Y_η is smooth. Let $n = \dim Y_\eta$. Then the following conditions are equivalent*:

(i) *f_ξ is flat*;
(ii) *the canonical map $f^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ has no* (*isolated or embedded*) *geometric vertical components.*

Let $t = (t_1, \dots, t_n)$ be a system of complex variables. For the rest of the paper, R will stand for the n-dimensional regular local ring $\mathbb{C}\{t\}$. An R-module M is called an *almost finitely generated* R-module if there is a local analytic algebra A over R (i.e., a homomorphism of local analytic algebras $\varphi : R \to A$) such that M is a finite A-module. (For our purposes it is enough to think of modules of the form $\mathcal{O}_{X^{\{i\}},\xi^{\{i\}}}$, where $f_\xi : X_\xi \to Y_\eta$ is a morphism of germs of analytic spaces, with $\mathcal{O}_{Y,\eta} = R = \mathbb{C}\{t\}$.) The following rigidity of the left derived functor of analytic tensor product holds (see [9, Proposition 2.2]):

Proposition 2.4. *Let M and N be almost finitely generated R-modules, and let i_0 be an integer. If $\widehat{\mathrm{Tor}}_{i_0}^R(M, N) = 0$, then $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \geqslant i_0$.*

Define the *flat dimension* of an R-module M, denoted $\mathrm{fd}(M)$, as the length of a shortest R-flat resolution of M (i.e., a resolution by R-flat modules). We have the following fundamental flat dimension additivity formula:

Proposition 2.5 (Galligo–Kwieciński [9, Proposition 2.10]). *Let M and N be almost finitely generated R-modules. If $\widehat{\mathrm{Tor}}_i^R(M, N) = 0$ for all $i \geqslant 1$, then*

$$\mathrm{fd}(M) + \mathrm{fd}(N) = \mathrm{fd}\big(M \hat{\otimes}_R N\big).$$

Remark 2.6. By [9, Theorem 2.7], the flat dimension of an almost finitely generated R-module M satisfies the following Auslander–Buchsbaum-type formula:

$$\mathrm{fd}(M) + \mathrm{depth}(M) = n = \dim R.$$

Hence, for a torsion-free almost finitely generated R-module M, we always have

$$\mathrm{fd}(M) \leqslant n - 1.$$

Indeed, since no nonzero element of R is a zerodivisor of M, then $\mathrm{depth}(M) \geqslant 1$.

Now let $f_\xi : X_\xi \to Y_\eta$ be an *open* morphism of germs of analytic spaces. Let X_ξ be reduced of pure dimension, and let Y_η be smooth of dimension n. Then, the hypotheses of [9, Lemma 5.2] are satisfied and hence we obtain:

Lemma 2.7. *There exist an R-flat almost finitely generated module F and a monomorphism of R-modules $\mathcal{O}_{X,\xi} \to F$.*

Next we observe that *algebraic vertical* components in fibre powers carry over to higher powers. More precisely, for a morphism $f_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces, where Y_η is reduced and irreducible, we have the following:

Remark 2.8. If $\mathcal{O}_{X^{\{k\}},\xi^{\{k\}}}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module, then so are $\mathcal{O}_{X^{\{i\}},\xi^{\{i\}}}$ for $i \leqslant k$. Indeed, for $i < k$, we have a canonical monomorphism of $\mathcal{O}_{Y,\eta}$-modules:

$$\mathcal{O}_{X^{\{i\}},\xi^{\{i\}}} = \underbrace{\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}}_{i \text{ times}} \ni m_1 \hat{\otimes} \cdots \hat{\otimes} m_i$$
$$\mapsto m_1 \hat{\otimes} \cdots \hat{\otimes} m_i \hat{\otimes} 1 \hat{\otimes} \cdots \hat{\otimes} 1 \in \underbrace{\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}}_{k \text{ times}} = \mathcal{O}_{X^{\{k\}},\xi^{\{k\}}}.$$

Hence, the zerodivisors (in R) of $\mathcal{O}_{X^{\{i\}},\xi^{\{i\}}}$ are among those of $\mathcal{O}_{X^{\{k\}},\xi^{\{k\}}}$.

Finally, recall the notion of regularity in the sense of Gabrielov: A morphism $f_\xi : X_\xi \to Y_\eta$ of germs of analytic spaces is called *Gabrielov regular* if, for every isolated irreducible component W_ξ of X_ξ, $\dim_\eta f(W) = \dim_\eta \overline{f(W)}$ for an arbitrarily small representative W of W_ξ, where $\overline{f(W)}$ denotes the Zariski closure of $f(W)$ in a representative of Y at η (see, e.g., [16, Section 1]).

3. Proof of the main result

Suppose first that $f_\xi : X_\xi \to Y_\eta$ is flat. Then so are all of its fibre powers $f^{\{i\}}_{\xi^{\{i\}}} : X^{\{i\}}_{\xi^{\{i\}}} \to Y_\eta$ ($i \geqslant 1$), as flatness is preserved by any base change (see [11, §6, Proposition 8]) and a composition of flat mappings is flat. Hence, in particular (without any extra assumptions), $\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module, by the characterization of flatness in terms of relations (see, e.g., [7, Corollary 6.5]).

For the proof of the other implication, we shall proceed in four cases, according to the conditions in Theorem 1.1. The idea of the proof of cases (1) and (2) is to show that, for every *geometric vertical* component $W_{\xi^{\{n\}}}$ in the n-fold fibre power $X^{\{n\}}_{\xi^{\{n\}}}$, the restriction $f^{\{n\}}_{\xi^{\{n\}}}|W_{\xi^{\{n\}}}$ is Gabrielov regular, and hence in fact $W_{\xi^{\{n\}}}$ is *algebraic vertical*. This reduces the problem to Theorem 2.3. The proof of the third case uses our openness criterion (Theorem 1.5) paired with the techniques of Galligo and Kwieciński outlined in the previous section. The last case is a straightforward consequence of Theorem 1.5.

Case 1

Let $\dim(Y_\eta) = n < 3$ and suppose that the morphism $f_\xi : X_\xi \to Y_\eta$ is not flat. We shall show that there exists an *algebraic vertical* component in the n-fold fibre power $f^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$, which by Remark 1.4 is equivalent to $\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$ having nonzero torsion over $\mathcal{O}_{Y,\eta}$.

As noted above, the problem can be reduced to Theorem 2.3 by proving that every *geometric vertical* component $W_{\xi^{\{n\}}}$ in $X^{\{n\}}_{\xi^{\{n\}}}$ is *algebraic vertical*. This is so because of the following simple argument.

Let $W_{\xi^{\{n\}}}$ be an (isolated or embedded) irreducible component of $X^{\{n\}}_{\xi^{\{n\}}}$. Then $f^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ can be extended to a holomorphic mapping $f^{\{n\}} : X^{\{n\}} \to Y$ (of the n-fold fibre power of a reduced purely-dimensional X over a smooth n-dimensional Y) so that $W_{\xi^{\{n\}}}$ extends to a component W of $X^{\{n\}}$. Denote by $\mathrm{nR}(f^{\{n\}}|W)$ the set of points where

$f^{\{n\}}|W$ is not regular in the sense of Gabrielov (see [16, Section 1]). Then, by Remmert's Rank Theorem, $\mathrm{nR}(f^{\{n\}}|W)$ is a subset of the locus of nongeneric fibre dimension in W, and thus the image of $\mathrm{nR}(f^{\{n\}}|W)$ is of codimension (at least) two in the image $f^{\{n\}}(W)$. If $W_{\xi^{\{n\}}}$ is *geometric vertical*, then the image $f^{\{n\}}(W)$ is already of codimension (at least) one with respect to Y, and hence

$$\dim f^{\{n\}}\big(\mathrm{nR}\big(f^{\{n\}}|W\big)\big) \leqslant \dim Y - 1 - 2 \leqslant -1,$$

i.e., $f^{\{n\}}(\mathrm{nR}(f^{\{n\}}|W)) = \emptyset$. Thus, $f^{\{n\}}|W$ is Gabrielov regular, so that $W_{\xi^{\{n\}}}$ is an *algebraic vertical* component.

Case 2

Suppose that the Nash morphism $f_\xi : X_\xi \to Y_\eta$ of Nash germs is not flat, and let $W_{\xi^{\{n\}}}$ be a *geometric vertical* component in $f^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$, which exists by Theorem 2.3. Since X_ξ is a Nash germ, then obviously so are all its fibre powers $X^{\{i\}}_{\xi^{\{i\}}}$ ($i \geqslant 1$). Hence, by Proposition 2.1, our component $W_{\xi^{\{n\}}}$ is a Nash germ.

Consider the morphism $f^{\{n\}}_{\xi^{\{n\}}}|W_{\xi^{\{n\}}} : W_{\xi^{\{n\}}} \to Y_\eta$ of Nash germs. By passing to the graph of f, we can assume that

$$\xi^{\{n\}} = (0, \eta), \qquad W_{\xi^{\{n\}}} \subset \big(\mathbb{C}^{mn} \times Y\big)_{\xi^{\{n\}}},$$

and $f^{\{n\}}_{\xi^{\{n\}}}|W_{\xi^{\{n\}}}$ is a germ at $\xi^{\{n\}}$ of the canonical projection $\pi : \mathbb{C}^{mn} \times Y \to Y$. This makes $f^{\{n\}}_{\xi^{\{n\}}}$ a germ of a polynomial mapping. Next, observe that $W_{\xi^{\{n\}}}$ being Nash, there exists a germ of an algebraic set $Z_{\xi^{\{n\}}}$ in $(\mathbb{C}^{mn} \times Y)_{\xi^{\{n\}}}$ such that

$$W_{\xi^{\{n\}}} \subset Z_{\xi^{\{n\}}} \quad \text{and} \quad \dim Z_{\xi^{\{n\}}} = \dim W_{\xi^{\{n\}}}$$

(cf. [17, Theorem 2.10]). By Chevalley's Theorem [15, Chapter 7, §8.3], the image $f^{\{n\}}(Z)$ of an arbitrarily small representative Z of $Z_{\xi^{\{n\}}}$ is algebraic constructible, and hence

$$\dim_\eta \overline{f^{\{n\}}(W)} \leqslant \dim_\eta \overline{f^{\{n\}}(Z)} = \dim_\eta f^{\{n\}}(Z) = \dim_\eta f^{\{n\}}(W),$$

which shows that $W_{\xi^{\{n\}}}$ is *algebraic vertical*.

Remark 3.1. Note that the above argument cannot be extended beyond the Nash category. In general, a fibre power of a Gabrielov regular morphism of germs of analytic spaces need not be regular itself: Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces with X_ξ of pure dimension and Y_η irreducible of dimension n. Let Y be a locally irreducible representative of Y_η and let X be a pure-dimensional representative of X_ξ such that $f(X) \subset Y$. Define $S = \{y \in Y\colon \dim f^{-1}(y) > l\}$, where l is the minimal fibre dimension of f on X, and suppose that $\dim_\eta \overline{S} = n$, where $\overline{S}$ denotes the Zariski closure of S in Y. Then the top fibre power $X^{\{n\}}_{\xi^{\{n\}}}$ contains an isolated *geometric vertical* component W which is not *algebraic vertical*. In particular, $f^{\{n\}}_{\xi^{\{n\}}}$ is not Gabrielov regular (see [2, Proposition 3.1] and [2, Example 3.3]).

In the next section we give a characterization of analytic morphisms that are Gabrielov regular together with all their fibre powers.

Case 3

Let $Z_\eta \subset Y_\eta$ be a proper analytic subgerm such that the singular locus of X_ξ is mapped into Z_η (i.e., the Galligo–Kwieciński hypergerm $f_\xi(\operatorname{Sing} X_\xi)$ is contained in Z_η).

Since, by assumption, the nth analytic tensor power $\mathcal{O}_{X,\xi} \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \cdots \hat{\otimes}_{\mathcal{O}_{Y,\eta}} \mathcal{O}_{X,\xi}$ is a torsion-free $\mathcal{O}_{Y,\eta}$-module, then (Remarks 1.4 and 2.2) the n-fold fibre power $f^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ has no *algebraic vertical* components. In particular, there are no *isolated* algebraic vertical components in $X^{\{n\}}_{\xi^{\{n\}}}$, and hence f_ξ is open, by Theorem 1.5.

Openness being an open condition, we can extend $f_\xi : X_\xi \to Y_\eta$ to an open analytic mapping $f : X \to Y$ of reduced analytic spaces, where X is of pure dimension and Y is smooth of dimension n. Moreover, this can be done so that Z_η extends to a proper analytic subset Z of Y with $f(\operatorname{Sing} X) \subset Z$. We may now conclude that f is flat over $Y \setminus Z$, as for a mapping of smooth spaces openness is equivalent to flatness (cf. [8, Proposition 3.20]). Hence also $f^{\{k\}} : X^{\{k\}} \to Y$ is flat over $Y \setminus Z$ for every $k \geqslant 1$, because this is so locally.

Fix $k \in \{1, \dots, n-1\}$. We will now show that $\widehat{\operatorname{Tor}}_i^{\mathcal{O}_{Y,\eta}}(\mathcal{O}_{X,\xi}, \mathcal{O}_{X^{\{k\}},\xi^{\{k\}}}) = 0$ for all positive integers i. For simplicity of notation, let $R = \mathcal{O}_{Y,\eta}$, $M = \mathcal{O}_{X,\xi}$, and $N = \mathcal{O}_{X^{\{k\}},\xi^{\{k\}}}$. By Lemma 2.7, we have an exact sequence of almost finitely generated R-modules

$$0 \to M \to F \to F/M \to 0,$$

where F is R-flat. Thus, after tensoring with N, we get an exact sequence

$$0 \to \widehat{\operatorname{Tor}}_1^R(F/M, N) \xrightarrow{\lambda} M \hat{\otimes}_R N \to F \hat{\otimes}_R N \to F/M \hat{\otimes}_R N \to 0$$

and isomorphisms

$$\widehat{\operatorname{Tor}}_{i+1}^R(F/M, N) \cong \widehat{\operatorname{Tor}}_i^R(M, N) \quad \text{for all } i \geqslant 1.$$

Pick any $m \in \widehat{\operatorname{Tor}}_1^R(F/M, N)$. There is a nonzero $r \in R$ such that $rm = 0$. In fact, the flatness of the restriction $f^{\{k\}}|(f^{\{k\}})^{-1}(Y \setminus Z)$ implies that any r with $\{r = 0\}_\eta \supset Z_\eta$ will do. Therefore, for some nonzero $r \in R$, $r \cdot \lambda(m) = 0$ in $M \hat{\otimes}_R N \cong \mathcal{O}_{X^{\{k+1\}},\xi^{\{k+1\}}}$, and hence either $\lambda(m) = 0$ or else $\mathcal{O}_{X^{\{k+1\}},\xi^{\{k+1\}}}$ has nontrivial torsion over $\mathcal{O}_{Y,\eta}$. The latter is impossible though, by our assumptions and Remark 2.8, as $k + 1 \leqslant n$. Thus, by injectivity of λ, $m = 0$, whence

$$\widehat{\operatorname{Tor}}_1^R(F/M, N) = 0.$$

The rigidity of $\widehat{\mathrm{Tor}}^R$ (Proposition 2.4) now implies that

$$\widehat{\mathrm{Tor}}^R_{i+1}(F/M, N) = 0$$

and hence

$$\widehat{\mathrm{Tor}}^R_i(M, N) = 0 \quad \text{for all } i \geqslant 1,$$

as required.

Finally, by the flat dimension formula (Proposition 2.5) and torsion freeness of $\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$ (see Remark 2.6), we obtain

$$n - 1 \geqslant \mathrm{fd}(\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}) = \mathrm{fd}(\mathcal{O}_{X,\xi}) + \mathrm{fd}(\mathcal{O}_{X^{\{n-1\}},\xi^{\{n-1\}}}) = \cdots = n \cdot \mathrm{fd}(\mathcal{O}_{X,\xi}).$$

Hence $\mathrm{fd}(\mathcal{O}_{X,\xi}) = 0$, so that $\mathcal{O}_{X,\xi}$ is $\mathcal{O}_{Y,\eta}$-flat.

Case 4

Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces, where $\mathcal{O}_{X,\xi}$ is Cohen–Macaulay and $\mathcal{O}_{Y,\eta}$ is regular of dimension n. Then, by [8, Proposition 3.20], f_ξ is flat if and only if it is open. Hence, if there are no *algebraic vertical* components in $\mathcal{O}_{X^{\{n\}},\xi^{\{n\}}}$, the flatness of f_ξ follows from Theorem 1.5.

4. Fibre powers and Gabrielov regularity

As pointed out in Remark 3.1, in general a fibre power of a Gabrielov regular mapping need not be regular itself. This is possible even in the case of mappings between smooth spaces, as shown in [2, Example 3.3].

It is interesting to know how to avoid such "hidden irregularity" phenomena. Ideally, one would like to have a condition on a morphism $f_\xi : X_\xi \to Y_\eta$, which would force all fibre powers $f^{\{i\}}_{\xi^{\{i\}}} : X^{\{i\}}_{\xi^{\{i\}}} \to Y_\eta$ ($i \geqslant 1$) to behave regularly in the sense of Gabrielov on both the isolated and embedded components. (Note that this is a stronger property than having only dominating or *algebraic vertical* components in the $X^{\{i\}}_{\xi^{\{i\}}}$ for $i \geqslant 1$.) Such a criterion would automatically yield Conjecture 1.3.

For the time being, we are only able to address this problem in the reduced case (Proposition 4.2 below), although it seems plausible that one could resolve the general problem along the lines of Propositions 4.1 and 4.2.

Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces, with Y_η irreducible of dimension n. Let Y be an n-dimensional irreducible representative of Y_η, and let X be a representative of X_ξ such that the components of X are precisely the representatives in X of the components of X_ξ and $f(X) \subset Y$ (where f represents the germ f_ξ). Furthermore let

$$\mathrm{fbd}_x f = \dim_x f^{-1}\big(f(x)\big)$$

be the *fibre dimension* of f at a point x.

In the remainder of this section we will use the following notation of [2]: $l = \min\{\mathrm{fbd}_x f\colon\ x \in X\}$, $k = \max\{\mathrm{fbd}_x f\colon\ x \in X\}$, and $A_j = \{x \in X\colon\ \mathrm{fbd}_x f \geqslant j\}$ for $l \leqslant j \leqslant k$. We then have $X = A_l \supset A_{l+1} \supset \cdots \supset A_k$, and by the Cartan–Remmert Theorem (see [15]), the A_j are analytic in X. Define $B_j = f(A_j) = \{y \in Y\colon\ \dim f^{-1}(y) \geqslant j\}$ for $l \leqslant j \leqslant k$. The upper semi-continuity of $\mathrm{fbd}_x f$ (as a function of x) implies that the germs $(A_j)_\xi$ and $(B_j)_\eta$ are independent of the choices of representatives made above.

Note that, except for B_k (cf. proof of Proposition 4.1 below), the B_j may not even be semianalytic in general. Nonetheless, there is an interesting connection between the filtration of the target by fibre dimension $Y \supset B_l \supset B_{l+1} \supset \cdots \supset B_k$ and the isolated irreducible components of the n-fold fibre power $X^{\{n\}}$ that we describe below.

Proposition 4.1. *Under the above assumptions, let $X^{\{n\}} = \bigcup_{i\in I} W_i$ be the decomposition into finitely many isolated irreducible components through $\xi^{\{n\}}$. Then*

(a) *For each $j = l, \ldots, k$, there exist components $W_{i_j,1}, \ldots, W_{i_j,p_j}$ of $X^{\{n\}}$ such that*

$$B_j = \bigcup_{q=1}^{p_j} f^{\{n\}}(W_{i_j,q}).$$

(b) *If $y \in B_j$ with $\dim f^{-1}(y) = s$ $(s \geqslant j)$, Z is an irreducible component of the fibre $(f^{\{n\}})^{-1}(y)$ of dimension ns, and W is an irreducible component of $X^{\{n\}}$ containing Z, then $f^{\{n\}}(W) \subset B_j$.*

Proof. Fix $j \geqslant l + 1$ (the statement is trivial for $j = l$ as $B_l = f(X)$). Pick any $y \in B_j$. Then $\dim f^{-1}(y) = s$ for some $s \geqslant j$. Let Z be an irreducible component of the fibre $(f^{\{n\}})^{-1}(y)$ of dimension ns, and let W be an irreducible component of $X^{\{n\}}$ containing Z. We will show that $f^{\{n\}}(W) \subset B_j$.

Suppose to the contrary that $W \cap (X^{\{n\}} \setminus (f^{\{n\}})^{-1}(B_j)) \neq \emptyset$, that is, suppose that there exists $z = (x_1, \ldots, x_n) \in W$ such that $f(x_i) \in Y \setminus B_j$ for $i = 1, \ldots, n$. Then $\mathrm{fbd}_{x_i} f \leqslant j - 1$, $i = 1, \ldots, n$, and hence $\mathrm{fbd}_z f^{\{n\}} \leqslant n(j-1) = nj - n$. In particular, the generic fibre dimension of $f^{\{n\}}|W$ is not greater than $nj - n$. Since $\mathrm{rank}(f^{\{n\}}|W) \leqslant \dim Y = n$, then $\dim W \leqslant (nj - n) + n = nj$.

Now we have: $W \supset Z$, $\dim W \leqslant nj$, $\dim Z = ns \geqslant nj$, and both W and Z irreducible. This is only possible when $W = Z$, and hence $f^{\{n\}}(W) = f^{\{n\}}(Z) = \{y\} \subset B_j$, a contradiction. Therefore $f^{\{n\}}(W) \subset B_j$, which completes the proof of part (b) of our proposition.

Part (a) follows immediately, since for any $y \in B_j$ and any irreducible component Z of $(f^{\{n\}})^{-1}(y)$ of the highest dimension, there exists an isolated irreducible component W of $X^{\{n\}}$ that contains Z. □

We can now establish a criterion for an analytic morphism to be Gabrielov regular together with all of its fibre powers:

Proposition 4.2. *Let $f_\xi : X_\xi \to Y_\eta$ be a morphism of germs of analytic spaces, with Y_η irreducible of dimension n. The following conditions are equivalent*:

(i) $f^{\{i\}}_{\xi^{\{i\}}} : X^{\{i\}}_{\xi^{\{i\}}} \to Y_\eta$ *is Gabrielov regular for all* $i \geqslant 1$;
(ii) *all the restrictions* $f|A_j$ *are Gabrielov regular* ($j = l, \dots, k$).

Proof. Suppose first that $f|A_j$ ($j = l, \dots, k$) are regular. Fix a positive integer i and let W be an isolated irreducible component of $X^{\{i\}}$. Since the components of $X^{\{i\}}$ are precisely the representatives of those of $X^{\{i\}}_{\xi^{\{i\}}}$, it suffices to show that $f^{\{i\}}|W$ is regular.

Let q be the greatest integer for which the generic fibre $F = F_1 \times \cdots \times F_i$ of $f^{\{i\}}|W$ contains a component F_m of dimension q. Then $f^{\{i\}}(W) \subset B_q = f(A_q)$. The property of being a fibre of dimension q is an open condition on A_q. Hence W is induced by an irreducible component V of A_q with the generic fibre dimension of $f|V$ equal q, in the sense that there exists a component V of A_q such that $f^{\{i\}}(W) = f(V)$. By assumption, $\dim_\eta \overline{f(V)} = \dim_\eta f(V)$ (where closure is in the Zariski topology in Y), hence also $\dim_\eta \overline{f^{\{i\}}(W)} = \dim_\eta f^{\{i\}}(W)$; i.e., $f^{\{i\}}|W$ is Gabrielov regular.

Suppose now that there exists $j \in \{l, \dots, k\}$ for which $f|A_j$ is *not* regular. We shall show that then regularity of $f^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ fails, where $n = \dim Y$.

Fix $j \in \{l, \dots, k\}$ such that $f|A_j$ is not regular. Pick $y \in B_j$ with $\dim f^{-1}(y) = j$, and let Z be an isolated irreducible component of the fibre $(f^{\{n\}})^{-1}(y)$ of dimension nj. Let W be an isolated irreducible component of $X^{\{n\}}$ containing Z. Then $f^{\{n\}}(W) \subset B_j$, by Proposition 4.1(b). Moreover, $f^{\{n\}}|W$ has no fibres of dimension less than or equal to $n(j-1)$. Indeed, otherwise the generic fibre dimension of $f^{\{n\}}|W$ would be at most $n(j-1)$, so that $\dim W \leqslant n(j-1) + n = nj = \dim Z$, and hence $W = Z$, a contradiction (see the proof of Proposition 4.1). Thus, the generic fibre $F = F_1 \times \cdots \times F_n$ of $f^{\{n\}}|W$ contains a component F_m of dimension j.

Now, there is an isolated irreducible component V of A_j such that $\dim_\eta \overline{f(V)} > \dim_\eta f(V)$ and the generic fibre dimension of $f|V$ is j. Our $y \in B_j$ can then be chosen from $f(V)$, and Z a component of $((f|V)^{-1}(y))^n$. Since being a j-dimensional fibre is an open condition on A_j, then (as in the first part of the proof) we find that $f^{\{n\}}(W) = f(V)$, so that $\dim_\eta \overline{f^{\{n\}}(W)} > \dim_\eta f^{\{n\}}(W)$. Thus $f^{\{n\}}_{\xi^{\{n\}}} : X^{\{n\}}_{\xi^{\{n\}}} \to Y_\eta$ is not regular. □

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