Written Assignment 1

1. Evaluate the following integrals:

(a) $\displaystyle\int (\arcsin x)^2 \, \mathrm{d}x$

(b) $\displaystyle\int \frac{xe^{2x}}{(1+2x)^2} \, \mathrm{d}x$

(c) $\displaystyle\int \frac{(\ln x)^2}{x^3} \, \mathrm{d}x$

(d) $\displaystyle\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} \, \mathrm{d}x \, .$

2. Let f and g be functions of a real variable such that $f(0) = g(0) = 0$ and both f'' and g'' are continuous. Let a be a positive real number. Show that

$$\int_0^a f(x)g''(x) \, \mathrm{d}x = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x) \, \mathrm{d}x \, .$$

3. Let P be a quadratic polynomial such that $P(0) = 1$ and $\displaystyle\int \frac{P(x)}{x^2(x+1)^3} \, \mathrm{d}x$ is a rational function (i.e., a quotient of two polynomials). Find the value of the derivative $P'(0)$. Justify your answer.

[Hints:

- Write $P(x) = ax^2 + bx + c$. Then $P(0)$ and $P'(0)$ can be expressed in terms of the coefficients a, b and c of P. What are they?
- Decompose $P(x)/(x^2(x+1)^3)$ into partial fractions (with some unknown constants for the numerators) and see what is the general form of an antiderivative of each of them.
- These antiderivatives are supposed to sum up to a rational function themselves, which means that no logarithmic terms are allowed. This implies that two of the five unknown constants in the partial fraction decomposition must be zero.
- Finally, convert the above partial fraction decomposition into a polynomial equation and compare the coefficients at the corresponding powers of x on both sides to find the answer.]

4. Find all values of the constant α for which the integral

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{3\alpha}{3x+1} \right) \mathrm{d}x$$

converges. Evaluate the integral for these values of α (as a function of α).

5. For each of the following functions $f(x)$, find all the singularities in the corresponding interval J (i.e., all the points $c \in J$ for which $\lim_{x \to c} f(x)$ is not a real number) and decompose the integral of f along J into a sum of elementary improper integrals. (E.g., if $f(x) = \dfrac{1}{x(x-5)}$ and $J = [0, +\infty)$, then $f(x)$ has singularities at points $c_1 = 0$ and $c_2 = 5$ within J, and

$$\int_0^\infty f(x) \, \mathrm{d}x = \int_0^1 f(x) \, \mathrm{d}x + \int_1^5 f(x) \, \mathrm{d}x + \int_5^{10} f(x) \, \mathrm{d}x + \int_{10}^\infty f(x) \, \mathrm{d}x \, .)$$

Determine which of the integrals in your decomposition converge (you do **not** need to evaluate the integrals).

(a) $f(x) = \dfrac{5x^4 - 3x^2 + 1}{(x^2-1)(2x^2+x-1)}$ and $J = (-\infty, +\infty)$.

(b) $f(x) = \dfrac{1}{x} - \dfrac{1}{x-1}$ and $J = [0, 2]$.

(c) $f(x) = \dfrac{\sin 2x}{2x^2 - \pi x}$ and $J = [0, 2]$.