MIDTERM TEST 1 SOLUTIONS

P.1

Base step: $f_1^2 = 1 = (-1)^{1+1}$. ✓

Inductive Step: Suppose $f_k^2 - f_{k-1} f_{k+1} = (-1)^{k+1}$ for some $k \geq 1$.

Then, $f_{k+1}^2 - f_k f_{k+2} = f_{k+1} \cdot f_{k+1} - f_k \cdot (f_k + f_{k+1}) = f_{k+1} \cdot f_{k+1} - f_k^2 - f_k f_{k+1} =$

$$= f_{k+1} \cdot (\underbrace{f_{k+1} - f_k}_{= f_{k-1}}) - f_k^2 = f_{k+1} \cdot f_{k-1} - f_k^2 = -(f_k^2 - f_{k-1} f_{k+1}) = -((-1)^{k+1}) = (-1)^{k+2}. \blacksquare$$

P.2 All statements are true:

(a) $x \in A \cup (B \cap C) \iff x \in A \vee x \in B \cap C \iff x \in A \vee (x \in B \wedge x \in C) \iff$

$\iff (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \iff x \in A \cup B \wedge x \in A \cup C \iff x \in (A \cup B) \cap (A \cup C)$. ✓

(b) Suppose $A \subset B$, $B \subset C$, and let $x \in A$ be arbitrary. Then, $x \in B$, since $A \subset B$. Hence, $x \in C$, since $B \subset C$. ✓

(c) $x \in A \setminus (B \cup C) \iff x \in A \wedge \neg(x \in B \cup C) \iff x \in A \wedge \neg(x \in B \vee x \in C) \iff$

$\iff x \in A \wedge (x \notin B \wedge x \notin C) \iff (x \in A) \wedge x \notin B) \wedge x \notin C \iff x \in (A \setminus B) \setminus C$. $\blacksquare$

P.3 (a) Let $x \in A$ be arbitrary. Let $y = f(a)$. Then, $y \in f(A)$, and hence $a \in f^{-1}(f(A))$,
as $f^{-1}(f(A)) = \{x \in X \mid f(x) \in f(A)\}$.

(b) $x \in f^{-1}(E \cap F) \iff f(x) \in E \cap F \iff f(x) \in E \wedge f(x) \in F \iff x \in f^{-1}(E) \wedge x \in f^{-1}(F) \iff x \in f^{-1}(E) \cap f^{-1}(F)$.

(c) $x \in f^{-1}(E \setminus F) \iff f(x) \in E \setminus F \iff f(x) \in E \wedge \neg(f(x) \in F) \iff x \in f^{-1}(E) \wedge x \notin f^{-1}(F) \iff$
$\iff x \in f^{-1}(E) \setminus f^{-1}(F)$. $\blacksquare$

P.4 Denote $S = \bigcap\limits_{R \in \mathcal{F}} R$. Then,

(i) For any $x \in X$, $(x,x) \in R$ for all $R \in \mathcal{F}$, and hence $(x,x) \in \bigcap R = S$. Thus,
S is reflexive.

(ii) Let $x, y \in X$ be arbitrary elements such that $(x,y) \in S$.
Then, $(x,y) \in R$, $\forall R \in \mathcal{F}$ and hence $(y,x) \in R$, $\forall R \in \mathcal{F}$. Hence, $(y,x) \in \bigcap R = S$.
Thus, S is symmetric.

(iii) Let $x, y, z \in X$ be arbitrary elements st. $(x,y), (y,z) \in S$.
Then, $\forall R \in \mathcal{F}$, $(x,y) \in R \wedge (y,z) \in R$, hence $(x,z) \in R$. Hence $(x,z) \in \bigcap R = S$.
Thus, S is transitive.

(i), (ii) and (iii) prove that S is an equivalence relation. $\blacksquare$

P.5 For both (a) and (b), let $X=\{1,2\}$, $Y=\{1,2,3\}$, $Z=\{1,2\}$, $f(x)=x$, and $g(1)=g(3)=1$, $g(2)=2$.
Then, $g\circ f$ is bijective, f is injective and not onto, g is onto and not injective. ∎

P.6 Set $Z=f(X)$. Then, $\forall z\in Z\ \exists x\in X$ s.t. $z=f(x)$. Hence, $g: X\to Z$, $g(x):=f(x)$ is onto.
Next, define $h: Z\to Y$ as $h(z)=z$. Since $h(z_1)=h(z_2)\iff z_1=z_2$, then h is injective.
Finally, for every $x\in X$, $(h\circ g)(x)=h(g(x))\underset{\text{def'n of } h}{=}g(x)\underset{\text{def'n of } g}{=}f(x)$. Hence, $h\circ g=f$. ∎

P.7
(a) Clearly, $f(1)=f(1)$, for any $f\in X$, $f(1)=g(1)\iff g(1)=f(1)$ for all $f,g\in X$, and $f(1)=g(1)\wedge g(1)=h(1)\implies f(1)=h(1)$ for all $f,g,h\in X$, which proves that R is an equivalence relation.

(b) For every equivalence class $E\in\{[f]_R \mid f\in X\}$ there is a real number c_E such that the constant function $\bar{c}_E$ (i.e., $\bar{c}_E: [0,1]\ni x\mapsto c_E\in\mathbb{R}$) satisfies $\bar{c}_E\in E$ (or $E=[\bar{c}_E]_R$).
Hence, the function $\varphi: \mathbb{R}\ni c_E\mapsto[\bar{c}_E]_R\in\{[f]_R \mid f\in X\}$ is a surjection.
On the other hand, if $c_1\neq c_2$ are two real numbers then $\bar{c}_1(1)=c_1\neq c_2=\bar{c}_2(1)$, and hence $[\bar{c}_1]_R\neq[\bar{c}_2]_R$, which proves that φ is injective. Thus, φ is a bijection, and so $|\{[f]_R \mid f\in X\}|=|\mathbb{R}|=\mathfrak{c}$. ∎

P.8 Assume $|X| \leq |Y|$, and let $\varphi : X \hookrightarrow Y$ be an injection.

Define $\psi : \mathcal{P}(X) \to \mathcal{P}(Y)$ as $\psi(A) = \varphi(A)$ for any $A \subseteq X$.

Then, for any two sets $A, B \subseteq X$, if $A \neq B$, then $\exists x \in A \setminus B$ or else $\exists x \in B \setminus A$.

If $x_0 \in A \setminus B$, then by injectivity of φ, $\forall x_1 \in B$, $\varphi(x_0) \neq \varphi(x_1)$. Hence, $\varphi(x_0) \notin \varphi(B)$, and so $\varphi(x_0) \in \varphi(A) \setminus \varphi(B)$, which proves that $\varphi(A) \neq \varphi(B)$.

Similarly, if $x_0 \in B \setminus A$, then as above, $\forall x_1 \in A$, $\varphi(x_0) \neq \varphi(x_1)$. Hence, $\varphi(x_0) \notin \varphi(A)$, and so $\varphi(x_0) \in \varphi(B) \setminus \varphi(A)$, which proves that $\varphi(B) \neq \varphi(A)$.

In any case, we proved the injectivity of ψ, hence $|\mathcal{P}(X)| \leq |\mathcal{P}(Y)|$. ∎

P.9 First note that the identity $\mathbb{R} \setminus \mathbb{Q} \ni x \mapsto x \in \mathbb{R}$ defines an injection. It thus suffices to find an injection $\mathbb{R} \hookrightarrow \mathbb{R} \setminus \mathbb{Q}$. We'll define it as a composite function.

Set $h : \mathbb{R} \to (0,1)$ to be $h(x) = \frac{1}{2} + \frac{\arctan(x)}{\pi}$. It is injective, by injectivity of arctan.

Next, define $g : (0,1) \to \mathbb{R} \setminus \mathbb{Q}$ as $g(p/q) = 2^p \cdot 3^q \cdot \sqrt{5}$, if $x = p/q$ with $p, q \in \mathbb{N}_+$ in lowest terms, and $g(x) = x$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Then, g is injective, b/c 2 and 3 are relatively prime.

Finally, define $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R} \setminus \mathbb{Q}$ as $f\left(2^p 3^q \cdot \frac{\sqrt{5}}{5^\ell}\right) = 2^p 3^q \cdot \frac{\sqrt{5}}{5^{\ell+1}}$ for all $p, q \in \mathbb{N}_+$ and $\ell \in \mathbb{N}$, and $f(x) = x$ for all $x \notin \left\{2^p 3^q \cdot \frac{\sqrt{5}}{5^\ell} \mid p, q \in \mathbb{N}_+, \ell \in \mathbb{N}\right\}$. Then, f is injective, b/c 2, 3 and 5 are all relatively prime (easy check). Now, $f \circ g \circ h$ is our injection.

By Cantor-Bernstein, $\mathbb{R} \setminus \mathbb{Q} \sim \mathbb{R}$. ∎

P.10 Let A denote the set of all finite sequences with integer terms.

Then, $A = \bigcup_{k=1}^{\infty} A_k$, where $A_k = \{\text{sequences with } k \text{ integer terms}\} = \{(a_1, a_2, \dots, a_k) \mid a_i \in \mathbb{Z}\}$.

Thus, A is a countable union of sets A_k, and so by a theorem from class, it suffices to show that the sets A_k are all countable. Fix $k \geq 1$. It is easy to see that $A_k = \underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{k \text{ times}}$.

To prove that A_k is countable, by another theorem from class, it suffices to find an injection $A_k \hookrightarrow \mathbb{N}_+$. Let then $p_1, \dots, p_k$ be k pairwise distinct prime numbers, and define $\varphi: A_k \to \mathbb{N}_+$ as $\varphi((a_1, \dots, a_k)) = p_1^{a_1} \cdot \dots \cdot p_k^{a_k}$. By relative primeness of the p_i, it follows that $\varphi((a_1, \dots, a_k)) = \varphi((b_1, \dots, b_k)) \iff a_1 = b_1 \wedge \dots \wedge a_k = b_k \iff (a_1, \dots, a_k) = (b_1, \dots, b_k)$. ∎

Bonus. To see that f is surjective, consider $A = S$. By assumption, there is n_S s.t. $f^{n_S}(S) = S$, i.e., $f^{n_S} = f \circ \underbrace{(f \circ \dots \circ f)}_{n_S - 1 \text{ times}}$ is surjective. The composite $f \circ$ sth. being surjective implies that f is so.

Next, for a proof by contradiction, suppose that f is not injective. Choose then $x_1 \neq x_2$ s.t. $f(x_1) = f(x_2)$, and consider the set $A = \{x_1, x_2\}$. By assumption, $|A| = 2$. On the other hand, $|f(A)| = 1$ and hence, for any $n \geq 1$, $|f^n(A)| = 1$. Therefore, there is no $n \geq 1$ s.t. $f^n(A) = A$, b/c equal sets have equal cardinalities. A contradiction. ∎