Problem Set 3

February 6, 2023

due: February 27, 2023

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

1. A real number is said to be *algebraic* if it is a root of a polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with integer coefficients $a_0, \ldots, a_n$ (for some $n \in \mathbb{N}$), where $a_0 \neq 0$.

(a) Prove that the set of polynomials with integer coefficients is countable.

(b) Prove that the set of algebraic numbers is countable.

2. Let $P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ be a polynomial of degree $n \in \mathbb{Z}_+$, with $a_1, \ldots, a_n \in \mathbb{R}$.

(a) Prove that, if z is a root of P (i.e., $P(z) = 0$) then $|z^n| = |a_1 z^{n-1} + \cdots + a_{n-1}z + a_n|$.

(b) Prove that, if z is a root of P then

$$|z| \leq 2 \cdot \max\{\sqrt[d]{|a_d|} : 1 \leq d \leq n\}\,.$$

[Hint: Try a proof by contradiction, using part (a).]

3. Define the following operations of addition and multiplication on $\mathbb{R} \times \mathbb{R}$

$$(x, y) + (u, v) := (x + u, y + v)\,, \qquad (x, y) \cdot (u, v) := (xu - yv, xv + uy)\,.$$

(a) Verify that $\mathbb{R} \times \mathbb{R}$ with so-defined addition and multiplication satisfies the axioms of a field.

(b) Identify the multiplicative identity 1 and its additive inverse -1. Show that there exists an element $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ with $(x_0, y_0) \cdot (x_0, y_0) = -1$.

(c) Prove that there exists an injection $\varphi : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that the field structure of $\mathbb{R}$ is induced (via φ) by that of $\mathbb{R} \times \mathbb{R}$. That is, find an injection $\varphi : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that, for all $x, y \in \mathbb{R}$, $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ (where the addition and multiplication on the right is that in $\mathbb{R} \times \mathbb{R}$).

(d) Fix an injective $\varphi : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ with the above properties. Use the fact that, in $\mathbb{R}$, we have $x^2 > 0$ for every $x \neq 0$, to prove that there is no real number s such that $\varphi(s) = (x_0, y_0)$ (where (x_0, y_0) is the element from part (b)).

4. (a) Prove that in any ordered field $(\mathbb{F}, +, \cdot, \prec)$, we have $0 \prec a^2 + 1$ for all $a \in \mathbb{F}$.

(b) Use part (a) to prove that, if $(\mathbb{F}, +, \cdot)$ is a field in which the equation $x^2 + 1 = 0$ has a solution, then the field $\mathbb{F}$ cannot be ordered (i.e., there exists no ordering "$\prec$" on $\mathbb{F}$ that would satisfy axioms $O1 - O4$). Conclude that the field $\mathbb{R} \times \mathbb{R}$ from Problem 3 cannot be ordered.

5. Let S be a nonempty bounded subset of $\mathbb{R}$ and let $a \in \mathbb{R}$. Define $aS := \{a \cdot x : x \in S\}$. Prove that, if $a < 0$, then $\sup(aS) = a \cdot \inf S$ and $\inf(aS) = a \cdot \sup S$.

6. Let $f : [0, +\infty) \to \mathbb{R}$ be a function such that, for all $x > 0$, $f(x) = \sup\{f(z) : z \in [0, x)\}$. Prove that f is increasing (i.e., for all x and y from $[0, +\infty)$, if $x \leq y$ then $f(x) \leq f(y)$).

Practice Problems (not to be submitted):

7. Let $f : [0, +\infty) \to \mathbb{R}$ be a function such that, for all $x > 0$,

$$\sup\{f(z) : z \in [0, x)\} \;=\; f(x) \;=\; \inf\{f(y) : y \in (x, +\infty)\}\,.$$

Prove that $f([0, +\infty))$ is an interval (i.e., if real numbers u and w are both contained in $f([0, +\infty))$ and $u < w$, then $v \in f([0, +\infty))$ for every $v \in (u, w)$), by following these steps: For a proof by contradiction, suppose that there are $u, v, w \in \mathbb{R}$ such that $u < v < w$, $u, w \in f([0, +\infty))$ and $v \notin f([0, +\infty))$.

(a) Let then x_u and x_w be such that $f(x_u) = u$ and $f(x_w) = w$. Show that $x_u < x_w$.

(b) Define sets $A := \{x \in [x_u, x_w] : f(x) < v\}$ and $B := \{x \in [x_u, x_w] : f(x) > v\}$. Show that A and B are nonempty and bounded. Let $\alpha := \sup A$ and $\beta := \inf B$. Prove that $\alpha = \beta$.

(c) Prove that $f(\alpha) = \inf f(B)$.

(d) Prove that $f(\alpha) = v$, thus contradicting the hypothesis that $v \notin f([0, +\infty))$.

8. Exercise 4.6.

9. Exercise 4.8.

10. Exercise 4.9.

11. Exercise 4.10.

12. Exercise 4.12.

13. Exercise 4.13.

14. Exercise 4.16.

15. Exercise 4.20.

16. Exercise 4.25.