Problem Set 6

March 26, 2023

due: April 10, 2023

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

1. Exercise 10.74.

2. Define a sequence of polynomials $(P_n)_{n=0}^{\infty}$ recursively by setting $P_0(x) = 0$, and

$$P_{n+1}(x) = P_n(x) + \frac{1}{2}(x - P_n(x)^2), \quad n \in \mathbb{N}.$$

Prove by induction that the sequence (P_n) converges uniformly to the function $\sqrt{x}$ on the interval $[0, 1]$, by showing that, for all n,

$$0 \ \leq \ \sqrt{x} - P_n(x) \ \leq \ \frac{2\sqrt{x}}{2 + n\sqrt{x}},$$

whence $0 \leq \sqrt{x} - P_n(x) \leq 2/n$ for all $x \in [0, 1]$.

3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two uniformly continuous functions. Prove that the composite function $g \circ f$ is uniformly continuous.

4. Let $A \subset \mathbb{R}$ be a non-empty set.

(a) Let $f : A \to \mathbb{R}$ be a uniformly continuous function. Prove that, for every Cauchy sequence $(x_n) \subset A$, the sequence $(f(x_n))$ is also Cauchy.

(b) Give an example of a continuous function $f : A \to \mathbb{R}$ and a Cauchy sequence $(x_n) \subset A$ such that $(f(x_n))$ is not Cauchy. Justify.

5. Let $f_n(x) = \dfrac{nx}{1 + n^2x^2}$, for $x \geq 0$, $n \in \mathbb{N}$.

(a) Find the pointwise limit $f : [0, \infty) \to \mathbb{R}$ of the sequence (f_n).

(b) Show that for any $t > 0$, the sequence (f_n) converges uniformly to f on the interval $[t, \infty)$.

(c) Show that the sequence (f_n) does not converge uniformly on the interval $[0, \infty)$.

Practice Problems (not to be submitted):

6. Exercise 10.23.

7. Exercise 10.25.

8. Exercise 10.26.

9. Exercise 10.53.

10. Exercise 10.56.

11. Exercise 10.73.