V. LIMITS & CONTINUITY OF FUNCTIONS

1. OPEN, CLOSED AND COMPACT SETS

Def. A set $U \subseteq \mathbb{R}$ is called open, when
$\forall x \in U \; \exists r > 0$ s.t. $(x-r, x+r) \subseteq U$.

Prop. (i) $\mathbb{R}$ and $\emptyset$ are open.
(ii) If $\{U_\iota\}_{\iota \in I}$ is a family of open sets, then $\bigcup_{\iota \in I} U_\iota$ is open.
(iii) If $U_1, \dots, U_k$ are open, then $U_1 \cap \dots \cap U_k$ is open.

Pf. (i) $\forall x \in \mathbb{R}$, $(x-1, x+1) \subseteq \mathbb{R}$. ✓
The empty set is open, b/c it's not true that $\exists x \in \emptyset$ s.t. ... ✓
(ii) $x \in \bigcup_\iota U_\iota \Rightarrow \exists \iota \in I$ s.t. $x \in U_\iota \Rightarrow \exists \iota \in I \; \exists r > 0$ s.t. $(x-r, x+r) \subseteq U_\iota \Rightarrow (x-r, x+r) \subseteq \bigcup_\iota U_\iota$.
(iii) Induction on k = exercise. ∎

Def. A set $F \subseteq \mathbb{R}$ is called closed, when $\mathbb{R} \setminus F$ is an open set.

Prop. (i) $\mathbb{R}$ and $\emptyset$ are closed sets.
(ii) If $F_1, \dots, F_k$ are closed, then $F_1 \cup \dots \cup F_k$ is closed.
(iii) If $\{F_\iota\}_{\iota \in I}$ are closed, then so is $\bigcap_{\iota \in I} F_\iota$.

Pf. (i) Clear.
(ii) $F_1, \dots, F_k$ closed $\Rightarrow \mathbb{R} \setminus F_1, \dots, \mathbb{R} \setminus F_k$ open $\Rightarrow \underbrace{(\mathbb{R} \setminus F_1) \cap \dots \cap (\mathbb{R} \setminus F_k)}_{= \mathbb{R} \setminus (F_1 \cup \dots \cup F_k)}$ open.
(iii) Similarly as in (ii), $\bigcap_{\iota \in I} F_\iota = \bigcap \mathbb{R} \setminus (\mathbb{R} \setminus F_\iota)$
$= \mathbb{R} \setminus \bigcup (\mathbb{R} \setminus F_\iota)$. ∎

Example. Intersection of inf. many open sets need not be open (and hence the union of inf. many closed sets need not be closed).

Let $U_n = (-\frac{1}{n}, \frac{1}{n})$ for $n \in \mathbb{Z}_+$. Then, $\bigcap_{n=1}^{\infty} U_n = \{0\}$, which is not open.

Let $F_n = [0, 1 - \frac{1}{n})$ for $n \in \mathbb{Z}_+$. Then, $\bigcup_{n=1}^{\infty} F_n = [0, 1)$, which is not closed.

Thm. A set $X \subseteq \mathbb{R}$ is closed if and only if
$\forall x_0 \in \mathbb{R}$, $[\exists (x_n)_{n=1}^{\infty} \subseteq X$ s.t. $\lim_{n\to\infty} x_n = x_0] \Rightarrow x_0 \in X$.

Pf. ($\Rightarrow$) Suppose X is closed, and let $x_0 \in \mathbb{R}$ and $(x_n)_{n=1}^{\infty} \subseteq X$ be s.t. $\lim_{n\to\infty} x_n = x_0$. Suppose $x_0 \notin X$. Then, x_0 belongs to $\mathbb{R}\setminus X$ which is open, hence can choose $r_0 > 0$ s.t. $(x_0 - r_0, x_0 + r_0) \subseteq \mathbb{R}\setminus X$. It follows that, $\forall n \geq 1$, $|x_0 - x_n| \geq r_0$, which contradicts $\lim_{n\to\infty} x_n = x_0$. ✓

($\Leftarrow$) For a proof by contradiction, suppose there is $x_0 \in \mathbb{R}\setminus X$, s.t. $\forall n \in \mathbb{Z}_+$, $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \nsubseteq \mathbb{R}\setminus X$. Then, $\forall n \in \mathbb{Z}_+$ $\exists x_n \in X$ s.t. $|x_n - x_0| < \frac{1}{n}$. It follows that the sequence $(x_n)_{n=1}^{\infty} \subseteq X$ converges to x_0, and hence $x_0 \in X$, by assumption. ↯ ∎

Examples: 1) Every open interval is an open set.
2) Every closed interval is a closed set.

Pf. 1) Consider an open interval (a,b) for some $a<b$, $a,b \in \mathbb{R}$.
If $x \in (a,b)$, then $a<x<b$. Set $r := \min\{x-a, b-x\}$. Then, $r>0$, and for any $y \in (x-r, x+r)$, one has $a = x-(x-a) \leq x-r < y < x+r < x+(b-x) = b$, hence $y \in (a,b)$. Thus, $(x-r, x+r) \subseteq (a,b)$. ✓
Other types of open intervals = exercise.
2) For any closed interval $[a,b]$, one has $\mathbb{R}\setminus[a,b] = (-\infty, a) \cup (b, \infty)$, which is an open set, by part 1). ∎

Def. A set $K \subseteq \mathbb{R}$ is called compact, when every sequence in K contains a subsequence convergent to an element of K.

Prop. Every closed interval in $\mathbb{R}$ is compact.

Pf. Consider $K = [a,b]$, for some $a<b$, $a,b \in \mathbb{R}$, and let $(x_n)_{n=1}^{\infty}$ be a sequence in K. Since (x_n) is bounded, then there exists a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$, by Bolzano-Weierstrass.
By the above theorem, the limit of (x_{n_k}) is an element of K. ∎

Thm. (Heine-Borel) A subset $K \subseteq \mathbb{R}$ is compact iff K is closed and bounded.

Pf. ($\Leftarrow$) As in the above proposition, every sequence $(x_n)_{n=1}^\infty$ in K is bounded and hence has a convergent subsequence, by B.-W. That subsequence then converges to an element of K, since K is closed. ✓

($\Rightarrow$) First, we show that K is bounded, by contradiction. Suppose, $\forall n \in \mathbb{Z}_+$ $\exists x_n \in K$ s.t. $n < x_n$. Then, $(x_n)_{n=1}^\infty \subseteq K$ has no subsequence convergent to an element of K. Indeed, for if (x_{n_k}) were such a subsequence and $\lim_{k\to\infty} x_{n_k} = x_0$, then, on the one hand, $\exists N_0 \in \mathbb{N}$ s.t. $x_0 < N_0$, and on the other hand, $\exists K_0 \in \mathbb{N}$ s.t. $\forall k \geq K_0$, $x_{n_k} > N_0 + 1$. Hence, $\forall k \geq K_0$, $|x_{n_k} - x_0| > 1$; contradiction. ✓

Now, to show that K is closed, let $(x_n)_{n=1}^\infty \subset K$ be any convergent sequence, and let $x_0 = \lim_{n\to\infty} x_n$. We want to show that $x_0 \in K$. By assumption, $\exists$ subseq. $(x_{n_k})_{k=1}^\infty \subseteq (x_n)_{n=1}^\infty$ convergent to a point $z_0 \in K$. Since (x_n) is convergent to x_0, then each of its subsequences converges to x_0, and thus $z_0 = x_0$. Hence, $x_0 \in K$, as required. ∎

Example: The ternary Cantor set is compact.

Def. A set $U \subseteq \mathbb{R}$ is called an (open) neighbourhood of a point $a \in \mathbb{R}$, when U is open and $a \in U$.

A point $a \in \mathbb{R}$ is called a limit point of a set $A \subseteq \mathbb{R}$, when $\forall U$ open nbhd of a $\exists x \in U \cap A \setminus \{a\}$.

Remark: Equivalently, a is a limit point of A, when $\exists (a_n)_{n=1}^\infty \subset A \setminus \{a\}$ s.t. $\lim_{n\to\infty} a_n = a$. (Exercise!)

2. LIMITS & CONTINUITY

Def. Let $A \subseteq \mathbb{R}$ be nonempty, let $f: A \to \mathbb{R}$, and let a be a limit point of A. We say that a number L is the limit of f at a, and write $\lim_{x\to a} f(x) = L$, when

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x \in A$, $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Example. Prove that $\lim_{x\to 1} \frac{1}{x} = 1$.

Here, $f(x) = \frac{1}{x}$ is defined on $A = \mathbb{R}\setminus\{0\}$, so $a=1$ is a limit point of A.
Let $\varepsilon > 0$ be arbitrary. W.l.o.g., assume $\varepsilon \le \frac{1}{2}$.
We want to find a $\delta > 0$ s.t. $0 < |x-1| < \delta \wedge x \neq 0 \Rightarrow \left|\frac{1}{x} - 1\right| < \varepsilon$.
Have, $\left|\frac{1}{x} - 1\right| = \left|\frac{1-x}{x}\right| = \frac{1}{|x|}\cdot|1-x|$.
Suppose that $\delta \le \frac{1}{2}$. Then, $|x-1| < \delta \Rightarrow x > 1 - \frac{1}{2} = \frac{1}{2}$, and hence $|x| = x > \frac{1}{2}$, so that $\frac{1}{|x|} < 2$. Thus, for $0 < |x-1| < \delta$, $\left|\frac{1}{x} - 1\right| < 2\cdot|x-1|$.
Set then $\delta := \frac{\varepsilon}{2}$. Since, by assumption $\varepsilon \le \frac{1}{2}$, then also $\delta \le \frac{1}{2}$, and thus for all x satisfying $0 < |x-1| < \delta$, we have

$$\left|\frac{1}{x} - 1\right| = \frac{1}{|x|}\cdot|1-x| < 2\cdot\delta = 2\cdot\frac{\varepsilon}{2} = \varepsilon, \text{ as required. } \blacksquare$$

__Thm.__ Let $f : A \to \mathbb{R}$ and let $a \in \mathbb{R}$ be a limit point of A. TFAE:

(i) $\lim_{x\to a} f(x) = L$

(ii) For every sequence $(x_n)_{n=1}^{\infty} \subseteq A\setminus\{a\}$ convergent to a, $\lim_{n\to\infty} f(x_n) = L$.

Pf. (i)$\Rightarrow$(ii): Let $(x_n)_{n=1}^{\infty} \subset A\setminus\{a\}$ be s.t. $\lim_{n\to\infty} x_n = a$.
Let $\varepsilon > 0$ be arbitrary, and let $\delta > 0$ be s.t. $\forall x \in A$, $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
Choose $N_0 \in \mathbb{N}$ s.t. $\forall n \ge N_0$, $|x_n - a| < \delta$.
Then, $\forall n \ge N_0$, $|f(x_n) - L| < \varepsilon$, as required. ✓

(ii)$\Rightarrow$(i): For a proof by contradiction, suppose

$$\neg(\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A,\ 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon).$$

Then, we can choose $\varepsilon_0 > 0$ s.t. $\forall n \in \mathbb{Z}_+$ $\exists x_n \in A$ s.t. $0 < |x_n - a| < \frac{1}{n} \wedge |f(x_n) - L| \ge \varepsilon_0$.
But then, the sequence $(x_n)_{n=1}^{\infty} \subseteq A\setminus\{a\}$ converges to a and so $\lim_{n\to\infty} f(x_n) = L$. $\lightning$ $\blacksquare$

__Thm.__ (__Algebraic Limit Thm.__) Let $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$, and let $a \in \mathbb{R}$ be a limit point of A. Suppose $\lim_{x\to a} f(x) = s$, $\lim_{x\to a} g(x) = t$, and $c \in \mathbb{R}$ is a constant. Then:

(i) $\lim_{x\to a} (f \pm g)(x) = s \pm t$

(ii) $\lim_{x\to a} c\cdot f(x) = c\cdot s$

(iii) $\lim_{x\to a} f(x)\cdot g(x) = s\cdot t$

(iv) $\lim_{x\to a} f(x)/g(x) = s/t$, provided $t \neq 0$

Pf. By the above thm., it suffices to show that for every sequence $(x_n)_{n=1}^{\infty} \subseteq A \setminus \{a\}$ convergent to a, the sequences $(f(x_n) \pm g(x_n))$, $(c \cdot f(x_n))$, $(f(x_n) \cdot g(x_n))$, and $(f(x_n)/g(x_n))$ converge to their respective limits. The latter follows from Alg. Limit Thm. for sequences. ∎

Def. A point $a \in \mathbb{R}$ is said to be an isolated point of a set $A \subseteq \mathbb{R}$, when $a \in A$ and $\exists \delta > 0$ s.t. $(a-\delta, a+\delta) \cap A = \{a\}$.

Def. We say that a function $f: A \to \mathbb{R}$ is continuous at a point $a \in A$, when a is an isolated point of A or else $\lim_{x \to a} f(x)$ exist and equals $f(a)$.

Thm. Let $f: A \to \mathbb{R}$ and $a \in A$. TFAE:

(i) f is continuous at a.

(ii) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A,\ |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

(iii) For every sequence $(x_n)_{n=1}^{\infty} \subseteq A$, $\lim_{n \to \infty} x_n = a \Rightarrow \lim_{n \to \infty} f(x_n) = f(a)$

(iv) For every nbhd V of $f(a)$ there is a nbhd U of a s.t. $f(U) \subseteq V$.

Pf. = Exercise (!)

Def. We say that $f: A \to \mathbb{R}$ is continuous, when f is cont's at a for all $a \in A$.

Examples: 1) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Show that $\lim_{x \to a} f(x)$ DNE, $\forall a \in \mathbb{R}$, and hence f is everywhere discont's.

2) Let $g: [0,1] \to \mathbb{R}$ be defined as $g(x) = \begin{cases} 1/q, & x = p/q \text{ in lowest terms} \\ 0, & x \in [0,1] \setminus \mathbb{Q} \end{cases}$

Show that g is continuous at $a \in [0,1]$ iff $a \in [0,1] \setminus \mathbb{Q}$. (Exercise)

Thm. If $f, g: A \to \mathbb{R}$ are continuous at a point $a \in A$, then $f+g$, $f-g$, $f \cdot g$, $c \cdot f$ are cont's at a (where $c \in \mathbb{R}$ is a constant), and f/g is cont's at a provided $g(a) \neq 0$.

Pf. = Exercise (use Alg. Limit Thm.) ∎

Prop. Every polynomial function is continuous on $\mathbb{R}$.
Every rational function $P(x)/Q(x)$ is cont's on $\mathbb{R} \setminus Q^{-1}(0)$.

Pf. By the previous thm., it suffices to show that the constant functions and the identity function $\{x \mapsto x\}$ are continuous. ∎

Thm. Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ be s.t. $f(A) \subseteq B$, f is cont's at $a \in A$, and g is cont's at $f(a)$. Then, $g \circ f$ is cont's at a.

Pf. Let $(x_n)_{n=1}^{\infty} \subset A$ be any sequence convergent to a. Then, by continuity of f, $(f(x_n))_{n=1}^{\infty}$ is convergent to $f(a)$, and hence by continuity of g, $(g(f(x_n)))_{n=1}^{\infty}$ is convergent to $g(f(a))$. ∎

Thm. Let $f: A \to \mathbb{R}$ be continuous. If $K \subseteq A$ is compact, then $f(K)$ is compact.

Pf. Let $(y_n)_{n=1}^{\infty}$ be an arbitrary sequence in $f(K)$. For every $n \in \mathbb{Z}_+$, choose $x_n \in K$ s.t. $f(x_n) = y_n$. Then, the sequence $(x_n)_1^{\infty} \subset K$ has a subsequence $(x_{n_k})_{k=1}^{\infty}$ convergent to $x_0 \in K$. By continuity of f, we get $\lim_{k \to \infty} f(x_{n_k}) = f(x_0)$; i.e., the subsequence $(f(x_{n_k}))_{k=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ converges to a point of $f(K)$, as required. ∎

Corollary. (Extreme Value Theorem) A continuous function $f: K \to \mathbb{R}$ on a compact set admits a maximum and minimum value.

Pf. By above, $f(K)$ is compact, and hence closed and bounded. Thus, $\inf f(K)$ and $\sup f(K)$ exist and belong to $f(K)$. ∎

Def. A function $f: A \to \mathbb{R}$ is uniformly continuous, when
$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x_1, x_2 \in A, \ |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$.

Thm. Let $f: A \to \mathbb{R}$ be continuous. If A is compact, then f is uniformly continuous.

Pf. For a proof by contradiction, suppose there is $\varepsilon_0 > 0$ s.t. $\forall n \in \mathbb{Z}_+$ $\exists x_n^1, x_n^2 \in A$ s.t. $|x_n^1 - x_n^2| < \frac{1}{n} \wedge |f(x_n^1) - f(x_n^2)| \geq \varepsilon_0$. Consider the sequence $(x_n^1)_{n=1}^\infty$. By compactness of A, there is a subsequence $(x_{n_k}^1)_{k=1}^\infty$ convergent to $x_0 \in A$. Then, $(x_{n_k}^2)_{k=1}^\infty$ is also convergent to x_0, since $\lim_{k\to\infty} |x_{n_k}^1 - x_{n_k}^2| = \lim_{n\to\infty} |x_n^1 - x_n^2| = 0$, by Squeeze Thm. By continuity of f, $\lim_{k\to\infty} f(x_{n_k}^1) = f(x_0) = \lim_{k\to\infty} f(x_{n_k}^2)$, which contradicts $|f(x_{n_k}^1) - f(x_{n_k}^2)| \geq \varepsilon_0$, $\forall k \in \mathbb{Z}_+$. ∎

Example - Warning: Both closedness & boundedness of A are necessary for uniform continuity, in general.

1) Consider $f(x) = x^2$ on $[0, \infty)$. Suppose f is uniformly cont's. Then, for $\varepsilon = 1$, there is $\delta_0 > 0$ s.t. $\forall\, 0 \leq x_1 < x_2$, $x_2 - x_1 < \delta_0 \Rightarrow x_2^2 - x_1^2 < 1$. But, for $x_1 = \frac{2}{\delta_0}$, $x_2 = \frac{2}{\delta_0} + \frac{\delta_0}{2}$, we have $x_2 - x_1 = \frac{\delta_0}{2} < \delta_0$, while $x_2^2 - x_1^2 = (x_2 - x_1)\cdot(x_2 + x_1) = \frac{\delta_0}{2}\cdot\left(\frac{2}{\delta_0} + \frac{2}{\delta_0} + \frac{\delta_0}{2}\right) > \frac{\delta_0}{2}\cdot\frac{4}{\delta_0} = 2 > 1$. ↯

2) Consider $f(x) = 1/x$ on $(0,1)$. Suppose f is uniformly cont's. Then, for $\varepsilon = 1$, there is $0 < \delta_0 \leq 1$ s.t. $\forall\, 0 < x_1 < x_2 < 1$, $x_2 - x_1 < \delta_0 \Rightarrow \frac{1}{x_1} - \frac{1}{x_2} < 1$. But, for $x_1 = \frac{\delta_0}{4}$, $x_2 = \frac{\delta_0}{2}$, have $x_2 - x_1 = \frac{\delta_0}{4} < \delta_0$, while $\frac{1}{x_1} - \frac{1}{x_2} = \frac{x_2 - x_1}{x_1 x_2} = \frac{\delta_0}{4}\cdot\frac{8}{\delta_0^2} = \frac{2}{\delta_0} \geq \frac{2}{1} > 1$. ↯

Thm. (Intermediate Value Thm.) If I is an interval in $\mathbb{R}$, and $f: I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval.

Pf. Let $u, v, w \in \mathbb{R}$ be s.t. $u < v < w \wedge u \in f(I) \wedge w \in f(I)$. We want $v \in f(I)$. Choose $x_u, x_w \in I$ s.t. $f(x_u) = u$, $f(x_w) = w$. W.l.o.g., suppose that $x_u < x_w$. Set $A = \{x \in [x_u, x_w] : f(x) < v\}$. Then, $A \neq \emptyset$ as $x_u \in A$, and A is bdd, so $\alpha := \sup A$ exists. Let $(x_n)_{n=1}^\infty \subset A$ be a sequence convergent to α. By continuity of f, $f(\alpha) = \lim_{n\to\infty} f(x_n)$, and hence $f(\alpha) \leq v$. Since $\alpha + \frac{1}{n} \notin A$, $\forall n \in \mathbb{Z}_+$, the sequence $z_n := \alpha + \frac{1}{n}$ satisfies $z_n \xrightarrow[n\to\infty]{} \alpha$ and $f(z_n) \geq v$ for all n. By continuity of f at α, again, $f(\alpha) = \lim_{n\to\infty} f(z_n)$, and hence $f(\alpha) \geq v$. Thus, $f(\alpha) = v$, and so $v \in f(I)$. ∎

Corollary. A continuous function maps closed intervals onto closed intervals.

Pf. If $I=[a,b]$, then I is closed & bdd, hence compact. By EVT, $f(I)$ is closed & bdd, and by IVT, $f(I)$ is an interval. Thus, if $\alpha=\inf f(I)$, $\beta=\sup f(I)$, it follows that $f(I)=[\alpha,\beta]$. ∎

Example: Let $g:[0,1]\to[0,1]$ be continuous. Show that there is $a\in[0,1]$ s.t. $g(a)+2a^5=3a^7$.

Consider $f(x):=g(x)+2x^5-3x^7$, for $x\in[0,1]$. Then, f is cont's on $[0,1]$. Have $f(0)=g(0)+0\geq 0$ and $f(1)=g(1)+2-3=g(1)-1\leq 0$. Hence, either $f(0)=0$, or $f(1)=0$, or $f(0)>0 \wedge f(1)<0$ and the claim follows from IVT. ∎

Def. A function $f:A\to\mathbb{R}$ is called:

- increasing, when $\forall x_1,x_2\in A$, $x_1<x_2 \Rightarrow f(x_1)\leq f(x_2)$
- strictly increasing, when $\forall x_1,x_2\in A$, $x_1<x_2 \Rightarrow f(x_1)<f(x_2)$
- decreasing, when $\forall x_1,x_2\in A$, $x_1<x_2 \Rightarrow f(x_1)\geq f(x_2)$
- strictly decreasing, when $\forall x_1,x_2\in A$, $x_1<x_2 \Rightarrow f(x_1)>f(x_2)$

Thm. Let I be an interval and let $f:I\to\mathbb{R}$ be a continuous injection. Then:

(i) f is strictly monotone

(ii) $f^{-1}:f(I)\to I$ is a (strictly monotone) continuous injection.

Pf. (i) First observe that, if $a<b<c$ are points of I, then one cannot have $f(a)<f(b)>f(c)$ or $f(a)>f(b)<f(c)$, by IVT and injectivity of f. Hence, for any $x\in I$, for any $u,v\in I$ s.t. $u<x<v$, either (a) $f(u)<f(x)<f(v)$ or else (b) $f(u)>f(x)>f(v)$. Suppose there exists $x_1\in I$ s.t. (a) holds at x_1.

Claim: Then, (a) holds at every $x\in I$.

For a proof by contradiction, suppose $\exists x_2\in I$ s.t. (b) holds at x_2.

If $x_1<x_2$, then $f(x_1)<f(x_2)$ (by (a) at x_1) $\wedge$ $f(x_1)>f(x_2)$ (by (b) at x_2). ↯

If $x_2<x_1$, then $f(x_2)<f(x_1)$ (by (a) at x_1) $\wedge$ $f(x_2)>f(x_1)$ (by (b) at x_2). ↯

Thus, f is strictly increasing on I.

If, symmetrically, there is $x_1 \in I$ at which (6) holds, we show that (6) holds at all $x \in I$, and thus f is strictly decreasing. ✓

(ii) Suppose w.l.o.g. that f is strictly increasing. Then, so is $g := f^{-1} : f(I) \to I$. Pick $y_0 \in f(I)$, and let $\varepsilon > 0$ be arbitrary. Suppose y_0 is not an endpt of $f(I)$. Let $x_0 \in I$ be s.t. $f(x_0) = I$, so $x_0 = g(y_0)$. Set $x_1 = x_0 - \varepsilon$, $x_2 = x_0 + \varepsilon$, and let $y_1 = f(x_1)$, $y_2 = f(x_2)$, and $\delta = \min\{|y_1 - y_0|, |y_2 - y_0|\}$. Then, $\forall y \in f(I)$,

$$|y - y_0| < \delta \Rightarrow y \in (y_1, y_2) = (f(x_1), f(x_2)) \Rightarrow g(y) \in (g(f(x_1)), g(f(x_2))) = (x_0 - \varepsilon, x_0 + \varepsilon).$$

∎

Def. A function $f : A \to \mathbb{R}$ is called *Lipschitz*, when $\exists L \in \mathbb{R}$ s.t. $\forall x_1, x_2 \in A$, $|f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$.
L is then called a *Lipschitz constant* of f.

Def. $f : A \to \mathbb{R}$ is called a *contraction*, when f is Lipschitz with a Lip. constant < 1. That is, when $\exists L \in [0,1)$ $\forall x_1, x_2 \in A$, $|f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$

Remark: Every Lipschitz function is uniformly continuous.

Example - Warning: Not the other way around!

Consider $f(x) = \begin{cases} x \cdot \sin(\pi/x^2), & x \in (0,1] \\ 0, & x = 0 \end{cases}$. Then, since $\lim_{x \to 0^+} f(x) = 0$,

f is continuous on $[0,1]$, and hence uniformly continuous.
But f is not Lipschitz.
Indeed, for $n \in \mathbb{Z}_+$, define $x_n := \frac{1}{\sqrt{2n}}$, $y_n := \frac{1}{\sqrt{2n + \frac{1}{2}}}$. Then,

$\forall n \geq 1$, $f(x_n) = x_n \cdot \sin(2n\pi) = 0$, $f(y_n) = y_n \cdot \sin(2n\pi + \frac{\pi}{2}) = y_n = \frac{1}{\sqrt{2n + 1/2}}$,

and $|x_n - y_n| = \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n + \frac{1}{2}}} = \frac{\sqrt{2n + \frac{1}{2}} - \sqrt{2n}}{\sqrt{2n} \cdot \sqrt{2n + \frac{1}{2}}}$.

Hence, $\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|} = \frac{1}{\cancel{\sqrt{2n + \frac{1}{2}}}} \cdot \frac{\sqrt{2n} \cdot \cancel{\sqrt{2n + \frac{1}{2}}}}{\sqrt{2n + \frac{1}{2}} - \sqrt{2n}} = \frac{\sqrt{2n} \cdot (\sqrt{2n + \frac{1}{2}} + \sqrt{2n})}{2n + \frac{1}{2} - 2n} =$

$$= 2\sqrt{2n} \cdot (\sqrt{2n + \tfrac{1}{2}} + \sqrt{2n}) \xrightarrow[n \to \infty]{} \infty.$$

Thus, there's no $L>0$ s.t. $|f(x_n)-f(y_n)|\le L\cdot|x_n-y_n|$ for all $n\ge 1$. ∎

Thm. (Fixed Point Theorem) Let $A\subseteq\mathbb{R}$ be a non-empty closed set, and let $f:A\to A$ be a contraction. Then, there exists a unique $p\in A$ s.t. $f(p)=p$.

Pf. Let $x_1\in A$ be arbitrary. Define a sequence $(x_n)_{n=1}^{\infty}\subset A$ by setting $x_{k+1}:=f(x_k)$ for all $k\ge 1$. We claim that (x_n) is Cauchy.

Indeed, let $\alpha\in[0,1)$ be s.t. $|f(x)-f(y)|\le\alpha\cdot|x-y|$ for all $x,y\in A$.

We first show, by induction on $k\ge 2$, that $|x_{k+1}-x_k|\le\alpha^{k-1}|x_2-x_1|$.

For $k=2$, by def'n $|x_3-x_2|=|f(x_2)-f(x_1)|\le\alpha\cdot|x_2-x_1|$ ✓

If $|x_{k+1}-x_k|\le\alpha^{k-1}\cdot|x_2-x_1|$, then $|x_{k+2}-x_{k+1}|=|f(x_{k+1})-f(x_k)|\le\alpha\cdot|x_{k+1}-x_k|\le\alpha^k\cdot|x_2-x_1|$ ✓

Now, let $\varepsilon>0$ be arbitrary. Choose $N_0\in\mathbb{N}$ s.t. $\frac{\alpha^{N_0-1}}{1-\alpha}\cdot|x_2-x_1|<\varepsilon$ (which exists, since $\alpha^n\xrightarrow[n\to\infty]{}0$).

Then, $\forall n>m\ge N_0$,

$$|x_n-x_m|\le|x_n-x_{n-1}|+|x_{n-1}-x_{n-2}|+\dots+|x_{m+1}-x_m|\le$$
$$\le(\alpha^{n-2}+\alpha^{n-3}+\dots+\alpha^{m-1})\cdot|x_2-x_1|=\alpha^{m-1}\cdot(1+\alpha+\alpha^2+\dots+\alpha^{n-m-1})|x_2-x_1|\le$$
$$\le\alpha^{N_0-1}\cdot\sum_{n=0}^{\infty}\alpha^n\cdot|x_2-x_1|=\frac{\alpha^{N_0-1}}{1-\alpha}\cdot|x_2-x_1|<\varepsilon,$$ which proves that (x_n) is Cauchy.

By completeness of $\mathbb{R}$, $\exists p\in\mathbb{R}$ s.t. $\lim_{n\to\infty}x_n=p$.

Since $(x_n)\subset A$ and A is closed, then $p\in A$.

Now, by continuity of f at p, $f(p)=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}x_n=p$, so p is a fixed point.

Finally, for the proof of uniqueness, suppose $q\in A$ is s.t. $f(q)=q$. Then, $|p-q|=|f(p)-f(q)|\le\alpha\cdot|p-q|$, which is only possible when $|p-q|=0$; i.e., $p=q$. ∎