Practice Midterm Test 1

January 30, 2023

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

0. Practice problems from Problem Sets 1 and 2.

1. Let $X = \{f : [0,1] \to \mathbb{R}\}$ be the set of all real-valued functions on the closed interval $[0,1]$. Define a relation R on X by setting

$$fRg \iff f(1) = g(1)\,.$$

(a) Prove that R is an equivalence relation on X.

(b) For a function $f \in X$, let $[f]_R$ denote the equivalence class of f with respect to R. Find the cardinality of the set $\{[f]_R : f \in X\}$ of all equivalence classes modulo R. Justify your answer.

2. Prove that the Cartesian product $\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$ is uncountable.

3. Let X, Y be sets. Prove that $|X| \le |Y|$ if and only if $|\mathcal{P}(X)| \le |\mathcal{P}(Y)|$.

4. Show that the following pairs of sets $X, Y \subset \mathbb{R}$ are equinumerous by finding a specific bijection between the sets in each pair.

(a) $X = [1,2)$ and $Y = (1,2)$.

(b) $X = [0,1]$ and $Y = \mathbb{R}$.

5. Let α and β be transfinite cardinals. Prove that

(a) For all $n \in \mathbb{N}$, $(n + \alpha = n + \beta) \iff \alpha = \beta$.

(b) For all $n \in \mathbb{N}$, $(n + \alpha = \aleph_0 + \beta) \iff \alpha = \beta$.

(c) $(\aleph_0 + \alpha = \aleph_0 + \beta) \iff \alpha = \beta$.

6. Exercise 8.2.

7. Exercise 8.4.

8. Exercise 8.7.

9. Exercise 8.8.

10. Use only the Cantor-Bernstein theorem to prove that the following pairs of sets X, Y are equinumerous.

(a) $X = [0,1]$ and $Y = \displaystyle\bigcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k} \right)$.

(b) $X = \mathbb{R}$ and $Y = \mathbb{R} \setminus \mathbb{Q}$.

11. (a) Prove that $\aleph_0 \cdot \aleph_0 = \aleph_0$.

(b) Prove that $\aleph_0 \cdot \mathfrak{c} = \mathfrak{c}$.

(c) Prove that $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$.