Problem Set 1
January 8, 2025
due: January 18, 2025

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

1. Construct a truth table for each statement:

(a) $p \Rightarrow \neg q$
(b) $\neg p \vee q$
(c) $[p \wedge (p \Rightarrow q)] \Rightarrow q$
(d) $[p \Rightarrow (q \wedge \neg q)] \Leftrightarrow \neg p$.

2. Use truth tables to determine which of the following statements are tautologies:

(a) $[(p \wedge q) \vee \neg p] \Longleftrightarrow (q \vee \neg p)$.
(b) $[(p \wedge q) \vee (\neg q \Rightarrow p)] \Longleftrightarrow [(q \wedge p) \vee \neg p]$.
(c) $[p \wedge (q \vee r)] \Longleftrightarrow [(p \vee q) \wedge (p \vee r)]$.

3. Define a two-argument logical connective ∇, called *nor*, such that $p \nabla q$ has value T if and only if both p and q are false.

(a) Use a truth table to show that $p \nabla p$ is logically equivalent to $\neg p$.
(b) Construct a truth table for $(p \nabla p) \nabla (q \nabla q)$.
(c) Which of the connectives $p \wedge q, p \vee q, p \Rightarrow q, p \Leftrightarrow q$ is logically equivalent to $(p \nabla p) \nabla (q \nabla q)$?

4. Determine the truth value of the following statement, assuming that x, y and z are real numbers:

$$\forall x \, \exists y \, \forall z, \; (z \leq x + y) \Longrightarrow (z \leq y).$$

5. Recall that an integer m is called *even* when there exists an integer k such that $m = 2k$. Otherwise m is said to be *odd* (i.e., m is odd when it is not true that m is even). Suppose that p, q and r are integers. Give formal proofs of the following statements, by using only the above definitions and the fact that the sum and product of two integers is again an integer.

(a) An integer p is odd if and only if there exists an integer l such that $p = 2l + 1$.
(b) If p is even or q is even or r is even, then pqr is even.
(c) If p^2qr is odd, then p is odd.
(d) If $(p + q)r$ is odd, then precisely one of the integers p and q is even.

6. Exercise 2.5 (a), (d), (e).

7. Exercise 2.7 (a)

Practice Problems (not to be submitted):

8.* Notice that there are many two-argument logical connectives other than the four defined in class (conjunction, disjunction, implication, equivalence). For example, consider a connective ∇ from Problem 3.

(a) Precisely, how many distinct two-argument logical connectives are there?

(b) Use suitable truth tables to show that every two-argument logical connective can be constructed as a combination of negations, conjunctions and disjunctions. (For example, $p \nabla q$ is the same as $\neg(p \vee q)$, which is also the same as $\neg p \wedge (p \vee \neg q)$.)

(c) What is the minimal number of two-argument logical connectives necessary to define all of the others? [Hint: Notice that the negation can be defined by means of two-argument logical connectives; e.g., $\neg p \Leftrightarrow (p \nabla p)$.]

9. Use truth tables to show that disjunction is associative, that is, prove that

$$(p \vee q) \vee r \iff p \vee (q \vee r)\,.$$

10. Which of the following best identifies f as a constant function, where x and y are real numbers?

(a) $\exists x \forall y, f(x) = y$.

(b) $\forall x \exists y$ such that $f(x) = y$.

(c) $\exists y \forall x, f(x) = y$.

(a) $\forall y \exists x$ such that $f(x) = y$.

11. Determine the truth value of each statement, assuming x and y are real numbers:

(a) $\exists x \in [-2, 0]$ such that $x^2 \geq 4$.

(b) $\forall x \in [-2, 0],\ x^2 \geq 1$.

(c) $\exists x$ such that $x - x = 0$.

(d) $\forall x,\ [x - x = 0$ and $\exists y$ such that $x - y > 0]$.

12. Exercise 2.1.

13. Exercise 2.2.

14. Exercise 2.4.

15. Exercise 2.8.

16. Exercise 2.9.

17. Prove or give a counterexample: *The sum of squares of any four consecutive integers is not divisible by* 4.