**Problem Set 2**

January 16, 2025

**due: February 1, 2025**

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

**1.** Exercise 2.23.

**2.** Prove that the following hold for arbitrary sets $A, B$ and $C$, or give an explicit counterexample.

(a) $(A \subset B) \iff [B = A \cup (B \setminus A)]$.

(b) $(A \subset B) \iff [(C \setminus A) \cap (C \setminus B) = C \setminus B]$.

**3.** Let $A$ be a nonempty set. A relation $R$ on a $A$ is called an *equivalence relation*, when it satisfies the following axioms:

(ER1) $\forall a \in A,\ aRa$ (reflexivity)

(ER2) $\forall a, b \in A,\ aRb \Rightarrow bRa$ (symmetry)

(ER3) $\forall a, b, c \in A,\ (aRb \wedge bRc) \Rightarrow aRc$ (transitivity).

Given an equivalence relation $R$ on $A$, one defines an *equivalence class* of an element $a \in A$ as

$$[a]_R = \{b \in A :\ bRa\}.$$

For an equivalence class $E \subset A$ (with respect to $R$), an element $a \in A$ is called a *representative* of $E$, when $a \in E$.

(a) Prove that, for all $a, b \in A$, either $[a]_R = [b]_R$ or else $[a]_R \cap [b]_R = \varnothing$.

(b) Prove that if $E$ is an equivalence class with respect to $R$ then, for all $a \in A$, $a \in E \Rightarrow E = [a]_R$.

(c) Let $\Delta_A := \{(a, a) : a \in A\}$. Prove that every equivalence relation $S$ on $A$ satisfies $\Delta_A \subset S$.

(d) Let $\mathcal{F}$ be a nonempty set of equivalence relations on $A$. Prove that $\bigcap_{S \in \mathcal{F}} S$ is an equivalence relation on $A$.

(e) Let $\mathcal{E}$ denote the set of *all* equivalence relations on $A$. Prove that $\Delta_A = \bigcap_{S \in \mathcal{E}} S$. [See Problem 9 below for notation.]

**4. Construction of $\mathbb{Z}$:** Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ be the set of natural numbers. Define a relation $R$ on $\mathbb{N} \times \mathbb{N}$ by

$$(a, b)\, R\, (c, d) \ :\iff \ a + d = c + b\,.$$

(a) Use only the laws of commutativity ($a + b = b + a$), associativity ($a + (b + c) = (a + b) + c$) and cancellation ($a + c = b + c \ \Rightarrow \ a = b$) of addition on $\mathbb{N}$ to prove that $R$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

(b) Denote by $\mathbb{Z}$ the set of equivalence classes $\{[(a, b)]_R : a, b \in \mathbb{N}\}$. Define the operations of addition and subtraction in $\mathbb{Z}$ by

$$[(a, b)]_R +_{\mathbb{Z}} [(c, d)]_R := [(a + c, b + d)]_R, \qquad [(a, b)]_R -_{\mathbb{Z}} [(c, d)]_R := [(a + d, c + b)]_R.$$

Prove that these operation are well-defined, that is, independent of the choices of representatives of the equivalence classes.

(c) Prove that, for all $a, b, c, d, e, f \in \mathbb{N}$,

$$[(a,b)]_R -_{\mathbb{Z}} [(c,d)]_R = [(e,f)]_R \iff [(a,b)]_R = [(c,d)]_R +_{\mathbb{Z}} [(e,f)]_R.$$

(d) Prove that the functions $\varphi : \mathbb{N} \to \mathbb{Z}$ and $\psi : \mathbb{N} \to \mathbb{Z}$ defined as $\varphi(n) = [(n,0)]_R$, $\psi(n) = [(0,n)]_R$ are injective, $\mathbb{Z} = \varphi(\mathbb{N}) \cup \psi(\mathbb{N})$, and $\varphi(\mathbb{N}) \cap \psi(\mathbb{N}) = \{[(0,0)]_R\}$.

(e) Prove that, for all $m, n \in \mathbb{N}$, $\varphi(m) +_{\mathbb{Z}} \varphi(n) = \varphi(m+n)$, $\psi(m) +_{\mathbb{Z}} \psi(n) = \psi(m+n)$, and $\varphi(n) +_{\mathbb{Z}} \psi(n) = [(0,0)]_R$.

From now on, we shall identify the set $\mathbb{N}$ with the subset $\varphi(\mathbb{N})$ of $\mathbb{Z}$, and write $n$ for $\varphi(n)$ in $\mathbb{Z}$. We shall also write $-n$ for $\psi(n)$.

**5.** Let $f : X \to Y$ be a function.

(a) Prove that, if $A = f^{-1}(f(A))$ for all $A \subset X$, then $f$ is injective.

(b) Prove that, if $f(f^{-1}(E)) = E$ for all $E \subset Y$, then $f$ is surjective.

(c) Prove that, if $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subset X$, then $f$ is injective.

**Practice Problems (not to be submitted):**

**6.** Complete the proofs of all the theorems stated in class thus far (cf. Part I of the Lecture Notes).

**7.** Let $A$, $B$ and $C$ be subsets of a universal set $U$. Define the *symmetric difference* of $A$ and $B$ by

$$A \Delta B := (A \setminus B) \cup (B \setminus A).$$

(a) Draw a Venn diagram for $A\Delta B$.

(b) What is $A\Delta A$?

(c) What is $A\Delta\varnothing$?

(d) What is $A\Delta U$?

(e) Prove that $A\Delta(B\Delta C) = (A\Delta B)\Delta C$.

(f) Exercise 2.22.

**8.** Let $A, B, C$ and $D$ be subsets of a universal set $U$. For each of the following, prove the equality of sets or give a counterexample. [Hint: It might be helpful to draw Venn diagrams first.]

(a) $(A \setminus B) \cup C = [(A \cup C) \setminus B] \cup (B \cap C)$.

(b) $A \cup (B \setminus C) = [(A \cup B) \setminus C] \cup (A \cap C)$.

(c) $(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cap D)$.

(d) $A \setminus (B \cup C) = (A \setminus B) \setminus C$.

(e) $(A \cup B \cup C) \setminus (A \cup B) = C$.

(f) $A \setminus [B \setminus (C \setminus D)] = (A \setminus B) \cup [(A \cap C) \setminus D]$.

(g) $A \cup B \cup C \cup D = (A \setminus B) \cup (B \setminus C) \cup (C \setminus D) \cup (D \setminus A) \cup (A \cap B \cap C \cap D)$.

**9.** Let $\mathcal{B}$ be a nonempty set of sets. One defines

$$\bigcup \mathcal{B} = \bigcup_{B \in \mathcal{B}} B := \{x : \exists B \in \mathcal{B} \text{ s.t. } x \in B\} \quad \text{and} \quad \bigcap \mathcal{B} = \bigcap_{B \in \mathcal{B}} B := \{x : \forall B \in \mathcal{B}, x \in B\}.$$

Let $A$ be a set and let $\mathcal{B}$ be a nonempty set of sets. Prove the following distributive and de Morgan laws:

(a) $A \cup \left( \bigcap_{B \in \mathcal{B}} B \right) = \bigcap_{B \in \mathcal{B}} (A \cup B)$ (b) $A \cap \left( \bigcup_{B \in \mathcal{B}} B \right) = \bigcup_{B \in \mathcal{B}} (A \cap B)$

(c) $A \setminus \left( \bigcup_{B \in \mathcal{B}} B \right) = \bigcap_{B \in \mathcal{B}} (A \setminus B)$ (d) $A \setminus \left( \bigcap_{B \in \mathcal{B}} B \right) = \bigcup_{B \in \mathcal{B}} (A \setminus B)$.

**10. Construction of $\mathbb{N}$:** Define the set $\mathbb{N}$ recursively by assuming that $\varnothing \in \mathbb{N}$, and for every $n \in \mathbb{N}$, $n \cup \{n\} \in \mathbb{N}$. Thus, $\mathbb{N}$ contains the elements $\varnothing$, $\{\varnothing\}$, $\{\varnothing, \{\varnothing\}\}$, $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$, etc. For an element $n \in \mathbb{N}$, denote by $n+1$ the element $n \cup \{n\}$ of $\mathbb{N}$. Prove that the inclusion "$\subseteq$" defines a linear order relation on $\mathbb{N}$, with respect to which $n < n+1$ for all $n \in \mathbb{N}$.

**11.** Let $X$ be a nonempty set. Prove or give a counterexample for each of the following:

(a) If $R$ and $S$ are transitive relations on $X$, then so is $R \cap S$.

(b) If $R$ and $S$ are transitive relations on $X$, then so is $R \cup S$.

(c) If $\mathcal{F}$ is a nonempty family of equivalence relations on $X$, then $\bigcap_{R \in \mathcal{F}} R$ is an equivalence relation on $X$.

**12.** A relation $R$ on a nonempty set $A$ is called *antisymmetric* if, for all $a, b \in A$,

$$[aRb \wedge bRa] \implies a = b\,.$$

(a) Give an example of an antisymmetric equivalence relation on $\mathbb{R}$.

(b) Can you give any other? If so go ahead, otherwise explain why not.

(c) Prove or give a counterexample: For every nonempty set $A$, there is precisely one antisymmetric equivalence relation on $A$.

**13.** (a) Give an example of functions $f : X \to Y$ and $g : Y \to Z$ such that $f$ and $g \circ f$ are both injective, but $g$ is not injective.

(b) Give an example of functions $f : X \to Y$ and $g : Y \to Z$ such that $g$ and $g \circ f$ are both surjective, but $f$ is not surjective.

**14.** Let $f : X \to X$ be a function. Define $f^0 := \mathrm{id}_X$ and $f^{k+1} := f \circ f^k$ for all $k \in \mathbb{N}$. Prove the following statement:

$$(\exists\, n \geq 1 \text{ such that } f^n \text{ is bijective}) \implies f \text{ is bijective}\,.$$

**15.** Suppose that $f : X \to Y$ and $g : Y \to X$ are functions such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Prove that $f$ and $g$ are bijective and are the inverses of one another, that is, $f^{-1} = g$ and $g^{-1} = f$.

**16.** Prove that every function $f : X \to Y$ can be written as a composite $f = h \circ g$, where $g$ is a surjection and $h$ is an injection. [Hint: Consider a relation $R$ on $X$ given by $x_1 R x_2$ iff $f(x_1) = f(x_2)$. Show that $R$ is an equivalence relation. Let $\mathcal{E}$ be the set of equivalence classes in $X$ relative to $R$. Define $g$ to be an appropriate surjection from $X$ to $\mathcal{E}$.]