Problem Set 3

January 30, 2025

due: February 14, 2025

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

1. A real number is said to be *algebraic* if it is a root of a polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with integer coefficients $a_0, \dots, a_n$ (for some $n \in \mathbb{N}$), where $a_0 \neq 0$.

(a) Prove that the set of polynomials with integer coefficients is countable.

(b) Prove that the set of algebraic numbers is countable.

2. Define the following operations of addition and multiplication on $\mathbb{R} \times \mathbb{R}$

$$(x, y) + (u, v) := (x + u, y + v), \qquad (x, y) \cdot (u, v) := (xu - yv, xv + uy).$$

(a) Verify that $\mathbb{R} \times \mathbb{R}$ with so-defined addition and multiplication satisfies the axioms of a field.

(b) Identify the multiplicative identity 1 and its additive inverse -1. Show that there exists an element $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ with $(x_0, y_0) \cdot (x_0, y_0) = -1$.

(c) Prove that there exists an injection $\varphi : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that the field structure of $\mathbb{R}$ is induced (via φ) by that of $\mathbb{R} \times \mathbb{R}$. That is, find an injection $\varphi : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that, for all $x, y \in \mathbb{R}$, $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ (where the addition and multiplication on the right is that in $\mathbb{R} \times \mathbb{R}$).

(d) Fix an injective $\varphi : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ with the above properties. Use the fact that, in $\mathbb{R}$, we have $x^2 > 0$ for every $x \neq 0$, to prove that there is no real number s such that $\varphi(s) = (x_0, y_0)$ (where (x_0, y_0) is the element from part (b)).

3. (a) Prove that in any ordered field $(\mathbb{F}, +, \cdot, \prec)$, we have $0 \prec a^2 + 1$ for all $a \in \mathbb{F}$.

(b) Use part (a) to prove that, if $(\mathbb{F}, +, \cdot)$ is a field in which the equation $x^2 + 1 = 0$ has a solution, then the field $\mathbb{F}$ cannot be ordered (i.e., there exists no ordering "$\prec$" on $\mathbb{F}$ that would satisfy axioms $O1 - O4$). Conclude that the field $\mathbb{R} \times \mathbb{R}$ from Problem 3 cannot be ordered.

4. Let S be a nonempty bounded subset of $\mathbb{R}$ and let $a \in \mathbb{R}$. Define $aS := \{a \cdot x : x \in S\}$. Prove that, if $a < 0$, then $\sup(aS) = a \cdot \inf S$ and $\inf(aS) = a \cdot \sup S$.

5. Let $f : [0, +\infty) \to \mathbb{R}$ be a function such that, for all $x > 0$, $f(x) = \sup\{f(z) : z \in [0, x)\}$. Prove that f is increasing (i.e., for all x and y from $[0, +\infty)$, if $x \leq y$ then $f(x) \leq f(y)$).

Practice Problems (not to be submitted):

6. Let $P(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$ be a polynomial of degree $n \in \mathbb{Z}_+$, with $a_1, \dots, a_n \in \mathbb{R}$.

(a) Prove that, if z is a root of P (i.e., $P(z) = 0$) then $|z^n| = |a_1 z^{n-1} + \cdots + a_{n-1}z + a_n|$.

(b) Prove that, if z is a root of P then

$$|z| \leq 2 \cdot \max\{\sqrt[d]{|a_d|} : 1 \leq d \leq n\}.$$

[Hint: Try a proof by contradiction, using part (a).]

7. Let $f : [0, +\infty) \to \mathbb{R}$ be a function such that, for all $x > 0$,

$$\sup\{f(z) : z \in [0, x)\} = f(x) = \inf\{f(y) : y \in (x, +\infty)\}.$$

Prove that $f([0, +\infty))$ is an interval (i.e., if real numbers u and w are both contained in $f([0, +\infty))$ and $u < w$, then $v \in f([0, +\infty))$ for every $v \in (u, w)$), by following these steps: For a proof by contradiction, suppose that there are $u, v, w \in \mathbb{R}$ such that $u < v < w$, $u, w \in f([0, +\infty))$ and $v \notin f([0, +\infty))$.

(a) Let then x_u and x_w be such that $f(x_u) = u$ and $f(x_w) = w$. Show that $x_u < x_w$.

(b) Define sets $A := \{x \in [x_u, x_w] : f(x) < v\}$ and $B := \{x \in [x_u, x_w] : f(x) > v\}$. Show that A and B are nonempty and bounded. Let $\alpha := \sup A$ and $\beta := \inf B$. Prove that $\alpha = \beta$.

(c) Prove that $f(\alpha) = \inf f(B)$.

(d) Prove that $f(\alpha) = v$, thus contradicting the hypothesis that $v \notin f([0, +\infty))$.

8. Exercise 4.6.

9. Exercise 4.8.

10. Exercise 4.9.

11. Exercise 4.10.

12. Exercise 4.12.

13. Exercise 4.13.

14. Exercise 4.16.

15. Exercise 4.20.

16. Exercise 4.25.