Problem Set 4
February 17, 2023
due: March 8, 2025

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

1. Let $(a_n)_{n=1}^{\infty}$ be a sequence defined recursively as follows: $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2 + a_n}$, for all $n \geq 1$. Prove that the sequence converges and find its limit.

2. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence, and let S denote the set of all *subsequential limits* of $(a_n)_{n=1}^{\infty}$, that is, all real numbers s such that there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ with $\lim_{k\to\infty} a_{n_k} = s$. One defines *limit superior* of $(a_n)_{n=1}^{\infty}$, denoted $\limsup a_n$, as $\sup S$, and *limit inferior* of $(a_n)_{n=1}^{\infty}$, denoted $\liminf a_n$, as $\inf S$. Prove the following:

(a) $\limsup a_n = \lim\limits_{N\to\infty} \sup\{a_n : n \geq N\}$

(b) $\liminf a_n = \lim\limits_{N\to\infty} \inf\{a_n : n \geq N\}$

(c) For every $s_0 \in \mathbb{R}$, if there exists a sequence $(s_k)_{k=1}^{\infty}$ with values in S, such that $\lim_{k\to\infty} s_k = s_0$, then $s_0 \in S$.

3. Construction of $\mathbb{R}$: Let $\mathbb{Q}$ denote the ordered field of of rational numbers, and let $\mathcal{C}$ be the set of all Cauchy sequences with values in $\mathbb{Q}$. More precisely, $(a_n)_{n=1}^{\infty} \in \mathcal{C}$ iff $a_n \in \mathbb{Q}$ for all $n \in \mathbb{Z}_+$, and

$$\forall \varepsilon \in \mathbb{Q}, \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall m, n \geq N, \ -\varepsilon < a_m - a_n < \varepsilon \,.$$

(a) Define a relation R on $\mathcal{C}$ by setting $(a_n)R(b_n)$ iff $\lim_{n\to\infty}(a_n - b_n) = 0$. Prove that R is an equivalence relation on $\mathcal{C}$.

(b) Let $\mathbb{R}$ denote the set of equivalence classes in $\mathcal{C}$ modulo R. Define

$$[(a_n)] + [(b_n)] \coloneqq [(a_n + b_n)], \qquad [(a_n)] \cdot [(b_n)] \coloneqq [(a_n b_n)]\,,$$

for any $[(a_n)], [(b_n)] \in \mathbb{R}$. Prove that the above operations of addition and multiplication are well defined (i.e., independent of the choices of representatives of equivalence classes).

(c) Let $\varphi : \mathbb{Q} \to \mathbb{R}$ be defined as $\varphi(q) = [(\underline{q})]$, where $(\underline{q})$ denotes the constant sequence with all terms equal to q. Prove that φ is an injection, which preserves the field operations (i.e., $\varphi(q_1 + q_2) = \varphi(q_1) + \varphi(q_2)$ and $\varphi(q_1 q_2) = \varphi(q_1) \cdot \varphi(q_2)$ for all $q_1, q_2 \in \mathbb{Q}$.)

(d) For $[(a_n)], [(b_n)] \in \mathbb{R}$, we say that $[(a_n)] < [(b_n)]$ iff $\neg([(a_n)] = [(b_n)])$ and there exists $N \in \mathbb{Z}_+$ such that $a_n < b_n$ for all $n \geq N$. Prove that $\mathbb{R}$ with so-defined addition, multiplication, and ordering satisfies the axioms of ordered field, in which $0 = [(\underline{0})]$ and $1 = [(\underline{1})]$. Show that $q_1 < q_2 \Leftrightarrow \varphi(q_1) < \varphi(q_2)$ for any $q_1, q_2 \in \mathbb{Q}$ (whence $\mathbb{R}$ contains $\mathbb{Q}$ as an ordered subfield).

(e) Prove that $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$, that is, show that for all $[(a_n)], [(b_n)] \in \mathbb{R}$, if $[(a_n)] < [(b_n)]$ then there exists $q \in \mathbb{Q}$ such that $[(a_n)] < [(\underline{q})] < [(b_n)]$, in the above sense.

(f) **Bonus**: Prove that so-constructed $\mathbb{R}$ is complete. That is, prove that for every non-empty bounded above set $X \subset \mathbb{R}$, there exists $[(c_n)] \in \mathbb{R}$ such that

$$\forall [(a_n)] \in X, \ [(a_n)] < [(c_n)]$$

and

$$\forall [(b_n)] \in \mathbb{R} \setminus \{[(c_n)]\}, \quad (\forall [(a_n)] \in X, \ [(a_n)] < [(b_n)]) \Longrightarrow [(c_n)] < [(b_n)]\,.$$

Practice Problems (not to be submitted):

4. Prove that, if $P(x)$ and $Q(x)$ are polynomials of positive degrees, then the sequence $a_n = \sqrt[n]{\left|\frac{P(n)}{Q(n)}\right|}$ converges to 1. [Hint: You may apply the Algebraic Limit Theorem and Squeeze Theorem, as well as other results proved in class, as needed.]

5. Exercise 5.17.

6. Exercise 5.18.

7. Exercise 6.8.

8. Exercise 6.11.

9. Exercise 6.13.

10. Exercise 6.19.

11. Let $(a_n)_{n=1}^{\infty}$ be a sequence defined recursively as follows: $a_1 = 2$, and $a_{n+1} = 2 - \frac{1}{a_n}$, for all $n \geq 1$. Prove that the sequence converges and find its limit.

12. Let $(a_n)_{n=1}^{\infty}$ be a sequence defined recursively as follows: $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 \cdot a_n}$, for all $n \geq 1$. Prove that the sequence converges and find its limit.

13. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence, and suppose that $\liminf a_n = \limsup a_n$. Prove that $(a_n)_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} a_n = \liminf a_n$.

14. Let (a_n) and (b_n) be two bounded sequences.

(a) Prove that $\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n) \leq \limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.

(b) Give an example of sequences (a_n) and (b_n) for which the leftmost inequality in part (a) is strict.

(c) Give an example of sequences (a_n) and (b_n) for which the rightmost inequality in part (a) is strict.