**Problem Set 5**
March 8, 2025
**due: March 22, 2025**

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

**1.** For a set $A \subset \mathbb{R}$, the *closure* of $A$, denoted $\overline{A}$, is defined as the intersection of all closed sets in $\mathbb{R}$ containing $A$. The *interior* of $A$, denoted $\mathrm{Int}(A)$, is defined as the union of all open sets in $\mathbb{R}$ contained in $A$.

(a) Prove that $\overline{A}$ is a closed set, for every $A \subset \mathbb{R}$.

(b) Prove that $\overline{A}$ is the smallest closed set containing $A$; i.e., if $E \subset \mathbb{R}$ is closed and $A \subset E$, then $\overline{A} \subset E$.

(c) Prove that $\mathrm{Int}(A)$ is an open set, for every $A \subset \mathbb{R}$.

(d) Prove that $\mathrm{Int}(A)$ is the largest open set contained in $A$.

**2.** (a) Prove that $x \in \overline{A}$ iff there is a sequence $(a_n)$ in $A$ convergent to $x$.

(b) A set $A \subset \mathbb{R}$ is called *dense* (in $\mathbb{R}$), when $A \cap U \neq \varnothing$ for every non-empty open set $U \subset \mathbb{R}$. Prove that $A$ is dense in $\mathbb{R}$ iff $\overline{A} = \mathbb{R}$.

**3.** (a) Show that $\overline{\mathbb{R} \setminus A} = \mathbb{R} \setminus \mathrm{Int}(A)$, and $\mathrm{Int}(\mathbb{R} \setminus A) = \mathbb{R} \setminus \overline{A}$.

(b) Is $\mathrm{Int}(A \cup B) = \mathrm{Int}(A) \cup \mathrm{Int}(B)$ for all $A, B \subset \mathbb{R}$? How about $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$?

(c) Is $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for all $A, B \subset \mathbb{R}$? How about $\overline{A \cap B} = \overline{A} \cap \overline{B}$?

**4.** Prove that, for every non-empty compact set $K \subset \mathbb{R}$, $\inf K \in K$ and $\sup K \in K$.

**5.** We say that a set $B \subset A$ is an *open subset* of $A$, when there exists an open set $U$ in $\mathbb{R}$ such that $B = A \cap U$. We say that $B$ is a *closed subset* of $A$, when there exists a closed set $F$ in $\mathbb{R}$ such that $B = A \cap F$.

(a) Let $f : A \to \mathbb{R}$ be a function. Prove that $f$ is continuous iff $f^{-1}(V)$ is an open subset of $A$ for every open set $V$ in $\mathbb{R}$.

(b) Let $f : A \to \mathbb{R}$ be a function. Prove that $f$ is continuous iff $f^{-1}(G)$ is a closed subset of $A$ for every closed set $G$ in $\mathbb{R}$.

**Practice Problems (not to be submitted):**

**6.** Prove that the intersection of a compact set and a closed set is a compact set.

**7.** Let $A, B \subset \mathbb{R}$, and let $f : A \to B$ be a continuous bijection. Prove that, if $A$ is compact, then $f^{-1} : B \to A$ is also continuous. [Hint: Use Problems 5 and 6.]

**8.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions.

(a) Prove that the set $f^{-1}(0) = \{x \in \mathbb{R} : f(x) = 0\}$ is closed.

(b) Prove that the set $\{x \in \mathbb{R} : f(x) = g(x)\}$ is closed.

**9.** Let $D$ be a dense subset of $\mathbb{R}$, and let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Prove that, if $f(x) = g(x)$ for all $x \in D$, then $f = g$.

**10.** Using the $\varepsilon - \delta$ definition of the functional limit, prove that $\lim_{x \to 1}(x^3 - 1) = 0$.

**11.** Let $f : (-\infty, a) \to \mathbb{R}$ for some $a \in \mathbb{R}$. We say that $L$ is the *limit of $f$ at $-\infty$*, and write $\lim_{x\to-\infty} f(x) = L$, when

$$\forall \varepsilon > 0\ \exists \delta < a\ \forall x \in (-\infty, \delta), |f(x) - L| < \varepsilon\,.$$

Similarly, for $f : (a, \infty) \to \mathbb{R}$, we say that $\lim_{x\to\infty} f(x) = L$, when

$$\forall \varepsilon > 0\ \exists \delta > a\ \forall x \in (\delta, \infty), |f(x) - L| < \varepsilon\,.$$

State and prove analogues of the Algebraic Limit Theorem (for functions) for limits at $-\infty$ and $\infty$.

**12.** Exercise 10.64.

**13.** Exercise 10.66.

**14.** (a) Fix $a \in \mathbb{R}$, and define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = |x - a|$. Prove that $f$ is continuous at every $c \in \mathbb{R}$.

(b) Let $K$ be a non-empty compact subset of $\mathbb{R}$, and let $a \in \mathbb{R}$. Prove that $K$ has a closest point to $a$, that is, prove that there exists $x_0 \in K$ such that $|x_0 - a| \le |x - a|$ for all $x \in K$.