

Problem Set 6

March 22, 2025

due: April 4, 2025

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

1. Exercise 10.74.
2. Define a sequence of polynomials $(P_n)_{n=0}^\infty$ recursively by setting $P_0(x) = 0$, and

$$P_{n+1}(x) = P_n(x) + \frac{1}{2}(x - P_n(x))^2, \quad n \in \mathbb{N}.$$

Prove by induction that the sequence (P_n) converges uniformly to the function \sqrt{x} on the interval $[0, 1]$, by showing that, for all n ,

$$0 \leq \sqrt{x} - P_n(x) \leq \frac{2\sqrt{x}}{2 + n\sqrt{x}},$$

whence $0 \leq \sqrt{x} - P_n(x) \leq 2/n$ for all $x \in [0, 1]$.

3. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two uniformly continuous functions. Prove that the composite function $g \circ f$ is uniformly continuous.
4. Let $A \subset \mathbb{R}$ be a non-empty set.
 - (a) Let $f : A \rightarrow \mathbb{R}$ be a uniformly continuous function. Prove that, for every Cauchy sequence $(x_n) \subset A$, the sequence $(f(x_n))$ is also Cauchy.
 - (b) Give an example of a continuous function $f : A \rightarrow \mathbb{R}$ and a Cauchy sequence $(x_n) \subset A$ such that $(f(x_n))$ is not Cauchy. Justify.
5. Let $f_n(x) = \frac{nx}{1 + n^2x^2}$, for $x \geq 0$, $n \in \mathbb{N}$.
 - (a) Find the pointwise limit $f : [0, \infty) \rightarrow \mathbb{R}$ of the sequence (f_n) .
 - (b) Show that for any $t > 0$, the sequence (f_n) converges uniformly to f on the interval $[t, \infty)$.
 - (c) Show that the sequence (f_n) does not converge uniformly on the interval $[0, \infty)$.

Practice Problems (not to be submitted):

6. A function $f : A \rightarrow \mathbb{R}$ is called *strictly increasing* when $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, and *strictly decreasing* when $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. In either case, we say that f is *strictly monotone*.
Let $I \subset \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be a continuous injection. Prove that f is strictly monotone, and the inverse function $f^{-1} : f(I) \rightarrow I$ is a strictly monotone continuous injection.
7. Exercise 10.23.
8. Exercise 10.25.
9. Exercise 10.26.
10. Exercise 10.53.
11. Exercise 10.56.
12. Exercise 10.73.