VI. SEQUENCES & SERIES OF FUNCTIONS

Def. Let $A \subset \mathbb{R}$ and let $\mathcal{F} = \{f : A \to \mathbb{R} \mid f \text{ a function}\}$. A sequence of functions (on A), denoted $(f_n(x))_{n=0}^{\infty}$ is a function $\mathbb{N} \ni n \mapsto f_n(x) \in \mathcal{F}$.

Def. Let $(f_n)_n$ be a sequence of functions on a non-empty set $A \subset \mathbb{R}$.
We say that $(f_n)_n$ is pointwise convergent, when
$\forall a \in A$, $(f_n(a))_n$ is a convergent sequence.
I.e., $\forall a \in A\ \exists L_a \in \mathbb{R}\ \forall \varepsilon > 0\ \exists N_a \in \mathbb{N}\ \forall n \geq N_a,\ |f_n(a) - L_a| < \varepsilon$.

We say that $(f_n)_n$ is uniformly convergent, when there exists a function $f : A \to \mathbb{R}$ s.t.
$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall x \in A,\ |f_n(x) - f(x)| < \varepsilon$. We write $f_n \underset{A}{\rightrightarrows} f$.

We say that $(f_n)_n$ is Cauchy, when
$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > m \geq N\ \forall x \in A,\ |f_m(x) - f_n(x)| < \varepsilon$.

Example. Consider $f_n : [0,1] \to \mathbb{R}$, $x \mapsto x^n$, for $n \in \mathbb{N}$.

Then, $f_n \to f$, where $f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$, but f doesn't converge uniformly on $[0,1]$.
(Indeed, all but finitely many f_n are outside $1/2$-nbhd of f.)

Thm. Let $A \subset \mathbb{R}$ be non-empty, and let $(f_n)_{n=1}^{\infty}$ be a sequence of cont's f'ns on A. If $f_n \underset{A}{\rightrightarrows} f$, then $f : A \to \mathbb{R}$ is continuous.

Pf. Let $a \in A$ be arbitrary. To prove continuity of f at a, let $\varepsilon > 0$ be arbitrary. Choose $N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0\ \forall x \in A$, $|f_n(x) - f(x)| < \varepsilon/3$, and choose $\delta > 0$ s.t. $\forall x \in A$, $|x - a| < \delta \Rightarrow |f_{N_0}(x) - f_{N_0}(a)| < \varepsilon/3$.
Then, $\forall x \in A$, $|x - a| < \delta \Rightarrow |f(x) - f(a)| = |f(x) - f_{N_0}(x) + f_{N_0}(x) - f_{N_0}(a) + f_{N_0}(a) - f(a)|$
$\leq |f(x) - f_{N_0}(x)| + |f_{N_0}(x) - f_{N_0}(a)| + |f_{N_0}(a) - f(a)| < 3 \cdot \varepsilon/3 = \varepsilon$. ∎

Def. Let $A \subset \mathbb{R}$ be non-empty. A series of functions on A, $\sum_{n=0}^{\infty} f_n$, is a pair of sequences $((f_n)_{n=0}^{\infty}, (s_n)_{n=0}^{\infty})$, where the $f_n : A \to \mathbb{R}$ are called the terms of the series, and the sequence of partial sums $(s_n)_{n=0}^{\infty}$ satisfies $s_n(x) = \sum_{k=0}^{n} f_k(x)$ for all $n \in \mathbb{N}$.

We say that the series $\sum f_n$ converges pointwise (resp. uniformly) to a function $f : A \to \mathbb{R}$, when (s_n) converges to f pointwise (resp. uniformly) on A, and write $\sum_{n=0}^{\infty} f_n = f$.

We say that $\sum f_n$ converges absolutely, when $\sum |f_n(x)|$ is convergent, $\forall x \in A$.

Thm. Let $A \subset \mathbb{R}$ be non-empty, and let (f_n) be a sequence of f'ns on A. Then, (f_n) is uniformly convergent on A iff (f_n) is Cauchy on A.

Pf. Suppose first that $f_n \overset{A}{\rightrightarrows} f$, for some $f : A \to \mathbb{R}$.
Let $\varepsilon > 0$ be arbitrary, and choose $N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0\ \forall x \in A$, $|f_n(x) - f(x)| < \varepsilon/2$.
Then, for all $m, n \geq N_0$, $|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, $\forall x \in A$ ✓

Conversely, suppose (f_n) is Cauchy on A. Then, $\forall x \in A$, $(f_n(x))_n$ is a Cauchy sequence of real numbers, hence convergent to some $f(x) \in \mathbb{R}$. This defines a function $f : A \ni x \mapsto f(x) \in \mathbb{R}$. We'll show that $f_n \overset{A}{\rightrightarrows} f$.
Let $\varepsilon > 0$ be arbitrary, and choose $N_0 \in \mathbb{N}$ s.t. $\forall m, n \geq N_0\ \forall x \in A$, $|f_m(x) - f_n(x)| < \varepsilon/2$.
Fix $x_0 \in A$ and $n_0 \geq N_0$. Then, $\forall m \geq N_0$, $|f_{n_0}(x_0) - f_m(x_0)| < \varepsilon/2$, hence $f_{n_0}(x_0) - \varepsilon/2 < f_m(x_0) < f_{n_0}(x_0) + \varepsilon/2$. Letting $m \to \infty$, by comparison of limits, we get $f_{n_0}(x_0) - \varepsilon/2 \leq f(x_0) \leq f_{n_0}(x_0) + \varepsilon/2$, hence $|f_{n_0}(x_0) - f(x_0)| \leq \varepsilon/2 < \varepsilon$.
Since x_0 and n_0 were arbitrary, we get $|f_n(x) - f(x)| < \varepsilon$, $\forall n \geq N_0\ \forall x \in A$. ∎

Corollary. A series $\sum_{n=0}^{\infty} f_n$ of functions on A is uniformly convergent iff its sequence of partial sums is Cauchy on A.

Thm. (Weierstrass M-Test) Let $A \subset \mathbb{R}$ be non-empty, and let (f_n) be a sequence of functions on A. Suppose that $\forall n \in \mathbb{N}\ \exists M_n \geq 0\ \forall x \in A$, $|f_n(x)| \leq M_n$, and the series $\sum_{n=0}^{\infty} M_n$ is convergent. Then, $\sum_{n=0}^{\infty} f_n$ is absolutely and uniformly convergent on A.

Pf. Let $\sum_0^\infty M_n$ be a given series of upper bounds, and let $(s_n)_{n=0}^\infty$ denote the sequence of partial sums of the series $\sum_0^\infty f_n$.
For every $x \in A$, by Comparison Test $(0 \le \sum_{n=0}^\infty |f_n(x)| \le \sum_{n=0}^\infty M_n)$, the series $\sum |f_n(x)|$ is absolutely convergent, hence can define $f: A \ni x \mapsto \sum_{n=0}^\infty f_n(x) \in \mathbb{R}$.
We'll prove that $\sum f_n$ converges uniformly to f, by showing that (s_n) is Cauchy.
Let $\varepsilon > 0$ be arbitrary, and let $N_0 \in \mathbb{N}$ be s.t. $\sum_{k=m+1}^n M_k < \varepsilon$, $\forall n > m \ge N_0$.
Then, $\forall n > m \ge N_0$ $\forall x \in A$,
$|s_n(x) - s_m(x)| = |\sum_{k=m+1}^n f_k(x)| \le \sum_{k=m+1}^n |f_k(x)| \le \sum_{k=m+1}^n M_k < \varepsilon$. ∎

Def. A <u>power series</u> centered at <u>(centre)</u> $c \in \mathbb{R}$ with <u>coefficients</u> $(a_n)_{n=0}^\infty \subset \mathbb{R}$ is a functional series $\sum_{n=0}^\infty a_n (x-c)^n$.

Thm. Let $\sum_{n=0}^\infty a_n(x-c)^n$ be a power series.
(i) If the series is convergent at $x_0 \in \mathbb{R}$, then it is absolutely and uniformly convergent on $[c-r, c+r]$ for any $0 < r < |c - x_0|$.
(ii) If the series is divergent at $x_0 \in \mathbb{R}$, then it is divergent at every $x \in \mathbb{R}$ with $|x-c| > |x_0 - c|$.

Pf. (i) Suppose $\sum_0^\infty a_n(x-c)^n$ is convergent at $x_0 \neq c$, and let $0 < r < |c-x_0|$ be arbitrary. Then, $\frac{r}{|c-x_0|} \in (0,1)$ and hence $\sum_{n=0}^\infty \left(\frac{r}{|c-x_0|}\right)^n$ is convergent.

Moreover, by Divergence Test, $\lim_{n\to\infty} a_n(x_0-c)^n = 0$, and so there is $M > 0$ s.t. $|a_n(x_0-c)^n| \le M$ for all $n \in \mathbb{N}$.
Now, let $\varepsilon > 0$ be arbitrary, and choose $N_0 \in \mathbb{N}$ s.t. $\forall n > m \ge N_0$, $M \cdot \sum_{k=m+1}^n \left(\frac{r}{|c-x_0|}\right)^n < \varepsilon$. Then, for all $x \in [c-r, c+r]$, and all $n > m \ge N_0$, we have:
$|\sum_{k=m+1}^n a_n(x-c)^n| \le \sum_{k=m+1}^n |a_n(x-c)^n| \le \sum_{k=m+1}^n |a_n(x_0-c)^n| \cdot \left(\frac{r}{|x_0-c|}\right)^n \le M \cdot \sum_{k=m+1}^n \left(\frac{r}{|x_0-c|}\right)^n < \varepsilon$,

which proves that $\sum_0^\infty a_n(x-c)^n$ is uniformly (and absolutely) convergent on $[c-r, c+r]$. ✓

(ii) For a proof by contradiction, suppose there is $x_1 \in \mathbb{R}$, with $|x_1 - c| > |x_0 - c|$, and s.t. $\sum a_n(x_1-c)^n$ is convergent. Choose $r > 0$ s.t. $|x_0 - c| < r < |x_1 - c|$. Then, by part (i), $\sum a_n(x-c)^n$ is absolutely convergent $\forall x \in [c-r, c+r]$. But $x_0 \in [c-r, c+r]$. ↯ ∎

Def. For a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, one defines its <u>radius of convergence</u> as

$$R = \begin{cases} 0, & \text{if the series converges only at } x_0 = c \\ \infty, & \text{if the series converges at every } x \in \mathbb{R} \\ \sup\{r > 0 \mid \text{the series conv. on } [c-r', c+r'], \forall r' \in (0,r)\}, & \text{otherwise.} \end{cases}$$

Def. $e^x = \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\sin(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$$

$$\cos(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot x^{2n} \quad , \text{for all } x \in \mathbb{R}.$$

Prop. The radii of convergence of the above series are ∞.

Pf. By comparison, it suffices to show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is abs. conv. $\forall x \in \mathbb{R}$. Fix $x_0 \in \mathbb{R}$, and denote $a_n := \frac{x_0^n}{n!}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x_0|^{n+1}}{n!(n+1)} \cdot \frac{n!}{|x_0|^n} = |x_0|/_{n+1} \xrightarrow[n\to\infty]{} 0, \text{ hence } \sum a_n \text{ is abs. conv.}$$

by Ratio Test. ∎

Def. Let $U \neq \emptyset$ be an open set. A function $f: U \to \mathbb{R}$ is called <u>(real) analytic</u> on U, when $\forall c \in U \ \exists r > 0 \ \exists (a_n)_{n=0}^{\infty} \subset \mathbb{R}$ s.t.

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \quad \text{for all } x \in (c-r, c+r).$$

Thm. (Stone-Weierstrass) Let $K \subset \mathbb{R}$ be a non-empty compact set, and let $f: K \to \mathbb{R}$ be a continuous function. Then, there exists a sequence $(P_n)_n$ of polynomials s.t. $P_n \underset{K}{\rightrightarrows} f$.

Def. An <u>open cover</u> for a set $A \subset \mathbb{R}$ is any family $\{U_i\}_{i\in I}$ of open sets, s.t. $A \subset \bigcup_{i \in I} U_i$.

Lemma. If $K \subset \mathbb{R}$ is compact, then any open cover of K admits a finite subcover.

Pf. (of Lemma):

Claim 1: For any $r>0$, there is a finite number of open intervals (a_n-r, a_n+r) whose union covers K.

Indeed, since K is bounded, then $K\subset[-R,R]$ for some $R>0$. Given arbitrary $r>0$, let then $a_0=-R$, $a_1=-R+r, \dots, a_n=-R+n\cdot r, \dots$. By the Archimedean Principle, there exists $N\in\mathbb{N}$ s.t. $-R+N\cdot r\geq R$. Then, $K\subset\bigcup_{n=0}^{N}(a_n-r, a_n+r)$. ✓

Now, let $\{U_\lambda\}_{\lambda\in J}$ be an arbitrary open cover of K, and suppose it admits no finite subcover. We shall construct a nested sequence of compact sets $\{K_n\}$ in K as follows: By Claim 1, $\exists a_1\in\mathbb{R}$ s.t. $K_1:=K\cap[a_1-1, a_1+1]$ cannot be covered by fin. many U_λ. Pick $x_1\in K_1$. Inductively, having chosen $K_1,\dots,K_k$ and $x_1,\dots,x_k$, by Claim 1, $\exists a_{k+1}\in\mathbb{R}$ s.t. $K_{k+1}:=K_k\cap[a_{k+1}-\frac{1}{2^{k+1}}, a_{k+1}+\frac{1}{2^{k+1}}]$ cannot be covered by fin. many U_λ, and pick $x_{k+1}\in K_{k+1}$. Now, by construction, the sequence (x_n) is Cauchy, since $|x_{n+1}-x_n|\leq\frac{1}{2^{n-1}}$, $\forall n\geq 1$. Since K is closed, it follows that $p:=\lim_{n\to\infty}x_n\in K$. Since, $\forall n$, $a_n-\frac{1}{2^n}\leq x_n\leq a_n+\frac{1}{2^n}$, it follows that $a_n-\frac{1}{2^n}\leq p\leq a_n+\frac{1}{2^n}$, and so $p\in\bigcap_{n\geq1}K_n$. Let U_{λ_p} be s.t. $p\in U_{\lambda_p}$. By openness, $\exists r>0$ s.t. $(p-r, p+r)\subset U_{\lambda_p}$. But then $K_n\subset U_{\lambda_p}$ for all n s.t. $2\cdot\frac{1}{2^n}<r$. ↯ ∎

Proof of S-W: Let $A:=\{f\in C(K;\mathbb{R}) \mid \forall\varepsilon>0\ \exists P\in\mathbb{R}[x]\ \forall x\in K, |f(x)-P(x)|<\varepsilon\}$.

We want to show that $A=C(K;\mathbb{R})$.

First, we'll establish a few properties of A.

1) Given $f\in C(K;\mathbb{R})$, if $\forall\varepsilon>0\ \exists g\in A\ \forall x\in K, |f(x)-g(x)|<\varepsilon$, then $f\in A$.

Pf. For $\varepsilon>0$, choose $g\in A$ s.t. $|f-g|<\varepsilon/2$ and $P\in\mathbb{R}[x]$ s.t. $|g-P|<\varepsilon/2$. Then, $|f-P|<\varepsilon$. ✓

2) $f,g\in A \Rightarrow f\cdot g\in A$.

Pf. Let $M>0$ be s.t. $|f(x)|\leq M \wedge |g(x)|\leq M$, $\forall x\in K$. Given $\varepsilon>0$, choose $P,Q\in\mathbb{R}[x]$ s.t. $|Q(x)-g(x)|<1$, $\forall x$, and $|g(x)-Q(x)|, |f(x)-P(x)|<\frac{\varepsilon}{2(\max_{x\in K}|g(x)|+\max_{x\in K}|f(x)|+1)}$, $\forall x\in K$.

Then, $\forall x\in K$,

$$|fg(x)-PQ(x)|=|fg(x)-fQ(x)+fQ(x)-PQ(x)|\leq|f(x)|\cdot|g(x)-Q(x)|+|Q(x)|\cdot|f(x)-P(x)|<\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}=\varepsilon.$$ ✓

3) $f,g\in A \Rightarrow f+g\in A$. (Exercise!)

4) $f\in A, c\in\mathbb{R} \to c\cdot f\in A$. (Exercise)

5) $f \in \mathcal{A} \Rightarrow P(f) \in \mathcal{A}$, $\forall P \in \mathbb{R}[x]$.

Pf. Follows from 2), 3), 4). ✓

6) $f \in \mathcal{A} \Rightarrow |f| \in \mathcal{A}$.

Pf. By 2), $f^2 \in \mathcal{A}$. Note that $|f| = \sqrt{f^2}$. Hence, if $\max_{x\in K}|f(x)| \le 1$, then by PS6-P.2, $\forall \varepsilon > 0\ \exists P \in \mathbb{R}[x]\ \forall x \in K$, $|\sqrt{f^2(x)} - P(f^2(x))| < \varepsilon$. Since $P(f^2) \in \mathcal{A}$, then $|f| \in \mathcal{A}$, by 1). If $\max_{x\in K}|f(x)| =: M > 1$, then by above $\left|\frac{f}{M}\right| = \frac{1}{M}\cdot|f| \in \mathcal{A}$, and hence $|f| \in \mathcal{A}$, by 4). ✓

7) $f, g \in \mathcal{A} \Rightarrow \max\{f,g\}, \min\{f,g\} \in \mathcal{A}$.

Pf. Note that, $\forall x \in K$, $\max\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$ and $\min\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$. Thus, the claim follows from 3)+4)+6). ✓

8) $\forall x_1 \ne x_2 \in K\ \forall \alpha, \beta \in \mathbb{R}\ \exists h \in \mathcal{A}$ s.t. $h(x_1) = \alpha \wedge h(x_2) = \beta$.

Pf. Let $P \in \mathbb{R}[x]$ be s.t. $P(x_1) \ne P(x_2)$, and set $h(x) = \alpha + (\beta - \alpha)\cdot\frac{P(x) - P(x_1)}{P(x_2) - P(x_1)}$. ✓

Let now $f \in \mathcal{C}(K;\mathbb{R})$, and $\varepsilon > 0$ be arbitrary.

We shall construct a function $g \in \mathcal{A}$ s.t. $\forall x \in K$, $|f(x) - g(x)| < \varepsilon$, whence $f \in \mathcal{A}$, by 1).

Step 1: For all $x, y \in K$, choose a function $h_{x,y} \in \mathcal{A}$ s.t. $h_{x,y}(x) = f(x) \wedge h_{x,y}(y) = f(y)$. (exists, by 8))

Step 2: Fix $x_0 \in K$. For every $y \in K$, since $f(y) - h_{x_0 y}(y) = 0$, then by continuity of $f - h_{x_0 y}$, can choose an open nbhd U_y of y s.t. $f(z) - h_{x_0 y}(z) > -\varepsilon$ for all $z \in U_y$. The open cover $\{U_y\}_{y\in K}$ of K admits a finite subcover, say, $\{U_{y_1}, \dots, U_{y_n}\}$. Set $h_{x_0} := \min\{h_{x_0 y_1}, \dots, h_{x_0 y_n}\}$. Then, $h_{x_0} \in \mathcal{A}$, and $\forall z \in K$, $f(z) + \varepsilon > h_{x_0}(z)$, since $f(z) + \varepsilon > h_{x_0 y_i}(z)$, where i is s.t. $z \in U_{y_i}$.
Note that $h_{x_0}(x_0) = f(x_0)$.

Step 3: By continuity of the functions $h_x - f$, and since $h_x(x) - f(x) = 0$, for every $x \in K$, can choose an open nbhd V_x of x s.t. $h_x(z) - f(z) > -\varepsilon$ for all $z \in V_x$. The open cover $\{V_x\}_{x\in K}$ of K admits a finite subcover, say, $\{V_{x_1}, \dots, V_{x_m}\}$. Set $g := \max\{h_{x_1}, \dots, h_{x_m}\}$.
Then, $g \in \mathcal{A}$, and $\forall z \in K$, $g(z) > f(z) - \varepsilon$, since $h_{x_i}(z) > f(z) - \varepsilon$, where i is s.t. $z \in V_{x_i}$.

We thus have $f(z) - \varepsilon < g(z) < f(z) + \varepsilon$, $\forall z \in K$, as required.

∎