Practice Final Exam

March 25, 2025

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

0. Practice problems from all Problem Sets and Practice Midterms, and unfinished proofs/exercises from lectures.

1. (a) State the Cantor-Schroeder-Bernstein Theorem.

(b) Use only Cantor-Schroeder-Bernstein Theorem to prove that the intervals $(0,1)$ and $(0,1]$ are equinumerous.

(c) Use only Cantor-Schroeder-Bernstein Theorem to prove that the sets $(0,2)\setminus\{1\}$ and $[0,1]$ are equinumerous.

2. (a) State the definition of an equivalence relation on a set A.

(b) Give an example of a reflexive relation on a set A, which is not an equivalence relation. Justify.

(c) Give an example of a symmetric relation on a set A, which is not an equivalence relation. Justify.

3. (a) State the definition of divergence to ∞ and to $-\infty$ (for a sequence of real numbers).

(b) Give an example of an unbounded sequence, which does not diverge to ∞ nor $-\infty$. Justify.

4. (a) State the definitions of supremum and infimum of a non-empty bounded set $A \subset \mathbb{R}$.

(b) Give an example of a bounded set A, for which $\inf(A) \notin A$ and $\sup(A) \in A$. Justify.

(c) Characterize the intervals $I \subset \mathbb{R}$ with the property that $\inf(I), \sup(I) \in I$. Justify.

(d) Give an example of bounded sets $A, B \subset \mathbb{R}$, such that $A \neq \varnothing \neq B$, $A \cap B = \varnothing$, $\inf(A) = \inf(B)$, and $\sup(A) = \sup(B)$. Justify. Could A and/or B be chosen finite? Justify.

5. (a) State the definition of continuity, uniform continuity, and the Lipschitz condition for a function $f : A \to \mathbb{R}$.

(b) State the definitions of pointwise and uniform convergence of a functional sequence (f_n).

(c) State the definitions of pointwise, uniform, and absolute convergence of a functional series $\sum f_n$.

6. (a) State the Extreme Value Theorem and Intermediate Value Theorem.

(b) State the Heine-Borel Theorem.

(c) State the Weierstrass M-Test.

7. For each of the theorems stated in Problem 6, remove one of the assumptions and give a counterexample to a theorem with that assumption missing. Justify. (E.g., give an example of a closed unbounded set, which does not satisfy the definition of a compact set.)

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Use only definitions of continuity, and open and closed sets, to prove that the set $\{x \in \mathbb{R} : f(x) < 0\}$ is open and the set $\{x \in \mathbb{R} : f(x) \geq 0\}$ is closed.

9. (a) State the definition of uniform convergence of a sequence of functions $f_n : A \to \mathbb{R}$.

(b) Let (f_n) be a sequence of functions on $\mathbb{R}$, uniformly convergent to a function $f : \mathbb{R} \to \mathbb{R}$. Suppose that, for each $n \in \mathbb{N}$, there exists $M_n \in \mathbb{R}$ with $f_n(x) > M_n$ for all $x \in \mathbb{R}$. Prove that there exists $M \in \mathbb{R}$ such that $f(x) > M$ for all $x \in \mathbb{R}$.

10. (a) Let $f_n(x) = \dfrac{x}{1+x^n}$, $x \in [0,\infty)$, $n \in \mathbb{N}$. Find the pointwise limit of the sequence (f_n) on $[0,\infty)$. Show that the convergence is not uniform on $[0,\infty)$. Find a smaller set on which the convergence is uniform.

(b) Let $f_n(x) = \dfrac{x}{1+nx^2}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. Find the pointwise limit of the sequence (f_n) on $\mathbb{R}$. Is the convergence uniform? Justify.

(c) Let $f_n(x) = \dfrac{x}{x+n}$, $x \in [0,\infty)$, $n \in \mathbb{N}_+$. Prove that for every $b > 0$ the sequence (f_n) converges uniformly on $[0,b]$.

11. (a) Use the Weierstrass M-Test to prove that the formula $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ defines a continuous function f on the interval $[-1,1]$.

(b) Use the Weierstrass M-Test to prove that the formula $g(x) = \displaystyle\sum_{n=1}^{\infty} \frac{nx}{n^3+x^2}$ defines a continuous function g on $\mathbb{R}$. [Hint: Show that g is continuous on every interval of the form $[-R,R]$ with $R > 0$.]

12. Let (a_n) be a bounded sequence of real numbers such that the series $\sum a_n$ diverges. Prove that the radius of convergence of the power series $\sum a_n x^n$ is 1.