Practice Midterm Test 2

March 4, 2025

All numbered exercises are from the textbook *Real Analysis, Foundations and Functions of One Variable*, by Laczkovich and Sos.

0. Practice problems from Problem Sets 3 and 4, and unfinished proofs/exercises from lectures.

1. Let A and B be non-empty subsets of $\mathbb{R}$, bounded above, and such that for every $a \in A$ there exists $b \in B$ with $a \leq b$. Prove that $\sup A \leq \sup B$.

2. Let $A \subset \mathbb{R}$ be non-empty and bounded below. Let B be the set of lower bounds of A. Show that B is bounded above. What is $\sup B$ in terms of A? Justify.

3. Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence.

(a) Prove that, if $\limsup a_n < M$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n < M$.

(b) Prove that, if $\liminf a_n > m$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > m$.

[Hint: Argue by contradiction.]

4. Give an example of a bounded sequence (a_n), for which $\liminf a_n < \limsup a_n$. Find the $\liminf a_n$ and $\limsup a_n$ for your sequence. Justify.

5. Give an example of a Cauchy sequence of rational numbers, which does not converge to a rational number. Justify.

6. Exercise 7.12. You may assume that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

7. Exercise 7.13.

8. Prove the following Ratio Test: *Let $\sum_{n=0}^{\infty} a_n$ be a series with non-zero terms.*

(a) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges absolutely.

(b) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series diverges.

[Hint: Let $L = \limsup \left|\frac{a_{n+1}}{a_n}\right|$. If $L < 1$, then choose $r \in \mathbb{R}$ such that $L < r < 1$. Then, by Problem 3, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_{n+1}| < r \cdot |a_n|$, and the geometric series $\sum_n r^n$ is convergent.]

9. (a) Give an example of a divergent series $\sum_n a_n$, with $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 1$. Justify.

(b) Give an example of a divergent series $\sum_n a_n$, with $\liminf \left|\frac{a_{n+1}}{a_n}\right| < \limsup \left|\frac{a_{n+1}}{a_n}\right|$, and $\liminf \left|\frac{a_{n+1}}{a_n}\right| = 1$. Justify.

10. Prove the following Root Test: *Let $\sum_{n=0}^{\infty} a_n$ be a series with non-zero terms.*

(a) If $\limsup \sqrt[n]{|a_n|} < 1$, then the series converges absolutely.

(b) If $\limsup \sqrt[n]{|a_n|} > 1$, then the series diverges.

[Hint: For part (b), prove that the sequence (a_n) has infinitely many terms satisfying $a_n \geq 1$.]

11. (a) State the Monotone Convergence and Bolzano-Weierstrass Theorems.

(b) Give an example of a series $\sum_n a_n$ such that its sequence of partial sums (s_n) contains no convergent subsequence. Justify.

12. Let $\sum_n a_n$ and $\sum_n b_n$ be two series with positive terms, and suppose that the sequence (a_n/b_n) converges to a non-zero real number. Prove that $\sum_n a_n$ converges iff $\sum_n b_n$ converges.