Practice Term Test 1

October 17, 2021

All numbered exercises are from the textbook *Lectures on Real Analysis*, by F. Larusson.

0. Exercises from Problem Sets 1–5.

1. Suppose that P is a polynomial of degree $2n+1$, such that $P(x)+c$ has precisely one real root for every $c \in \mathbb{R}$. Prove that the function P is strictly increasing.

2. State and prove Rolle's Theorem.

3. State and prove the Mean Value Theorem.

4. Prove that a bounded function $f : [a,b] \to \mathbb{R}$ is integrable if and only if for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

5. (a) Give an example of a bounded non-integrable function $f : [0,1] \to \mathbb{R}$. Justify.

(b) Give an example of a pointwise convergent sequence (f_n) of integrable functions on $[0,1]$, such that $\lim_{n\to\infty} \int_0^1 f_n$ exists, but (f_n) does not converge uniformly to any function $f : [0,1] \to \mathbb{R}$. Justify.

6. (a) Prove that if $f_n \rightrightarrows f$ on A and each f_n is bounded on A, then f is bounded on A.

(b) Give an example of a pointwise convergent sequence (f_n) of bounded functions such that $\lim_{n\to\infty} f_n$ is unbounded. Justify.

(c) Give an example of a pointwise convergent sequence (f_n) of bounded differentiable functions on $[0,1]$ such that the sequence (f_n') is unbounded. Justify.

7. Exercise 8.1.

8. Exercise 8.12.

9. (a) State definitions of equiboundedness and equicontinuity of sequences of functions.

(b) Give an example of a sequence (f_n) of equibounded continuous functions on $[0,1]$, which does not contain a uniformly convergent subsequence. Justify.

(c) Let (f_n) be a sequence of differentiable functions on $[0,1]$, such that $f_n(0) = 0$ for all n and the sequence (f_n') is uniformly convergent on $[0,1]$. Prove that the sequence (f_n) is equibounded and equicontinuous.

10. State the Cauchy Criterion for convergence of functional series.

11. (a) Give an example of an absolutely convergent series $\sum_n f_n$, which is not uniformly convergent. Justify.

(b) Give an example of an absolutely convergent series $\sum_n f_n$ of integrable functions on $[0,1]$, such that $\int_0^1 \sum_n f_n \neq \sum_n \int_0^1 f_n$. Justify.