# Practice Term Test 2

November 14, 2021

All numbered exercises are from the textbook *Lectures on Real Analysis*, by F. Larusson.

**0.** Exercises from Problem Sets 6 and 7.

**1.** Find the radi of convergence of the power series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1}$ and $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k}$. Conclude that the functions $\sin(x)$ and $\cos(x)$ defined as the sums of the above two power series (respectively) are infinitely differentiable on $\mathbb{R}$.

**2.** Let $c \neq 0$ be arbitrary. Use the result of Exercise 8.2 and the Maclaurin series from Problem 1 above to find the Taylor series of $\sin(x)$ and $\cos(x)$ centered at $c$.

**3.** Let $c \neq 0$ be arbitrary. Use the formula $\exp(x+y) = \exp(x)\cdot\exp(y)$ to find the Taylor series of $\exp(x)$ centered at $c$.

**4.** Let $f(x) = x+2x^2+3x^3+\cdots+2021x^{2021}$. Find the Maclaurin series of $f$, find its radius of convergence, and determine if the function equals the sum of its series in some neighbourhood of zero. Justify.

**5.** (a) Prove that a closed ball is a closed set in any metric space.

(b) Give an example of a metric space $(X, d)$ and an infinite collection of closed sets in $X$ whose union is not closed.

(c) Give an example of a metric space $(X, d)$, in which the intersection of any family of open sets is open and the union of any family of closed sets in closed.

**6.** Given a metric space $(X, d)$ and a set $A \subset X$, prove that

(a) $A = \overline{A}$ iff $A$ is closed;

(b) $A = \mathrm{Int}(A)$ iff $A$ is open.

**7.** Recall that a sequence in a metric space is said to converge to a point $a$, when every open neighbourhood of $a$ contains all but finitely many terms of that sequence. Let $(X, d)$ be a metric space, let $(x_n)$ be a sequence in $X$, and let $a, b \in X$ be such that $(x_n)$ is convergent to $a$ *and* convergent to $b$. Prove that $a = b$.

**8.** Recall that a point $a$ in a metric space $(X, d)$ is said to be a limit point of a set $A \subset X$, when for every open neighbourhood $U$ of $a$ one has $A \cap U \setminus \{a\} \neq \varnothing$. We shall denote the set of limit points of $A$ by $A'$. Given a metric space $(X, d)$ and a set $A \subset X$, prove that

(a) $\overline{A} = A \cup A'$;

(b) $A$ is closed iff $A \supset A'$.

**9.** Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space $(X, d)$, and let $a \in X$. Prove that $\lim_{n\to\infty} x_n = a$ iff $a$ is the only limit point of the set $\{x_n : n \in \mathbb{Z}_+\}$.

**10.** Exercise 9.9 (a)–(f).

**11.** Two metrics $d, \rho$ on a set $X$ are called *(metric) equivalent*, when there exist constants $m, M > 0$ such that
$$m \cdot d(x,y) \leq \rho(x,y) \leq M \cdot d(x,y)\,,$$
for all $x, y \in X$. Let $X = \mathbb{R}^2$, let $d_H$ denote the "hub" metric from Problem 8, PS. 7, let $d_R$ denote the "river" metric from Problem 9, PS. 7, and let $d_2$ be the Euclidean metric on $X$. Are any two of these metrics equivalent? Prove or give counterexamples.

**12.** Prove that, if $d, \rho$ are metric equivalent on $X$, then they are topologically equivalent (i.e., a set $U \subset X$ is open with respect to $d$ iff it is open with respect to $\rho$).

**13.** Let $X$ be the set of all real-valued sequences, and let
$$X_1 := \{(x_n) \in X \,:\, (x_n) \text{ is bounded}\}\,,$$
$$X_2 := \{(x_n) \in X \,:\, (x_n) \text{ is convergent}\}\,,$$
$$X_3 := \{(x_n) \in X \,:\, \lim_{n\to\infty} x_n = 0\}\,.$$

For $(x_n), (y_n) \in X$, define
$$d_\infty((x_n),(y_n)) := \sup\{|x_n - y_n| : n \in \mathbb{N}\}\,.$$

Show that $(X_j, d_\infty)$ is a metric space for $j = 1, 2, 3$, but $(X, d_\infty)$ is not.

**14.** Let $X_1, X_2, X_3$ and $d_\infty$ be as in Problem 13.

(a) Prove that $X_2$ is a closed set in the metric space $(X_1, d_\infty)$.
[Hint: A sequence $(x_n) \subset \mathbb{R}$ is divergent iff $\liminf x_n < \limsup x_n$.]

(b) Prove that $X_3$ is a closed set in the metric space $(X_2, d_\infty)$.

**15.** Let $X$ be the set of continuous real-valued functions on the interval $[0,1]$ and let $d$ be the supremum metric on $X$. Suppose that $f, g \in X$ satisfy $f(x) < g(x)$ for all $x \in [0,1]$. Is the set
$$\{h \in X : f(x) < h(x) < g(x) \text{ for all } x \in [0,1]\}$$
an open set in $X$? Is it an open ball? Justify your answers.

**16.** Let $(X,d)$ be a metric space and let $Y \subset X$. Recall that a function $d_Y$ defined as
$$d_Y := d|_{Y\times Y} : Y \times Y \ni (x,y) \mapsto d(x,y) \in \mathbb{R}$$
is called the induced metric on $Y$ and the pair $(Y, d_Y)$ is called a metric subspace of $(X,d)$.

(a) Let $(Y, d_Y)$ be a subspace of a metric space $(X,d)$. Prove that $V \subset Y$ is open in $(Y, d_Y)$ if and only if there exists $U \subset X$ open in $(X,d)$ such that $V = U \cap Y$. Prove that $G \subset Y$ is closed in $(Y, d_Y)$ if and only if there exists $F \subset X$ closed in $(X,d)$ such that $G = F \cap Y$.

(b) Let $(X,d)$ be a metric space, let $Y \subset X$ and $Z \subset Y$. Prove that, if $Z$ is open (resp. closed) in $(Y, d_Y)$ and $Y$ is open (resp. closed) in $(X,d)$, then $Z$ is open (resp. closed) in $(X,d)$.

**17.** Prove that the following is a necessary and sufficient condition for a metric space $(X,d)$ to be compact: *If $\{F_i\}_{i\in I}$ is any family of closed subsets of $X$ such that $\bigcap_{i\in K} F_i \neq \varnothing$ for any finite subset $K \subset I$, then $\bigcap_{i\in I} F_i \neq \varnothing$.*

**18.** Give an example of a family $\{F_i\}_{i\in\mathbb{Z}_+}$ of closed sets in $(\mathbb{R}, |\cdot|)$, such that $\bigcap_{i\in K} F_i \neq \varnothing$ for any finite subset $K \subset \mathbb{Z}_+$ but $\bigcap_{i\in\mathbb{Z}_+} F_i = \varnothing$.

**19.** Recall that the *Cantor set* is a subset of $[0,1]$ defined as $C := \bigcap_{n=1}^{\infty} C_n$, where $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, and $C_{i+1}$ is obtained by removing the open middle thirds from all the maximal closed intervals of $C_i$, for $i \geq 1$.

(a) Prove that, for every $n \in \mathbb{Z}_+$, $C$ contains all the endpoints of the maximal closed intervals in $C_n$.

(b) Prove that $a \in C$ iff $a$ admits a ternary expansion (possibly infinite) of the form $a = 0.a_1a_2a_3\ldots$ with $a_i \in \{0, 2\}$, for all $i \geq 1$.

(c) Prove that $C$ contains no open intervals.

**20.** Let $X = \mathbb{R}^2$, $A = \mathbb{Q} \times (-\infty, -1]$, and $B = C \times \mathbb{R}$, where $C$ denotes the Cantor set in $\mathbb{R}$. Find metrics $\rho_1, \rho_2, \rho_3$, and $\rho_4$ in $X$ such that:

(a) $A$ is closed and $B$ is open w.r.t. $\rho_1$

(b) $A$ is closed and $B$ is not open w.r.t. $\rho_2$

(c) $A$ is not closed and $B$ is closed w.r.t. $\rho_3$

(d) $A$ is open and $B$ is closed w.r.t. $\rho_4$.

Justify.

**21.** Let $(X, d)$ be a metric space, and let $\{F_i\}_{i \in I}$ be any family of compact sets in $X$. Prove that $\bigcap_{i \in I} F_i$ is compact.