Practice Final Exam

December 4, 2021

All numbered exercises are from the textbook *Lectures on Real Analysis*, by F. Larusson.

0. Exercises from Problem Sets 1–9 and Practice Tests 1 and 2.

1. Exercise 9.13.

2. Exercise 9.19.

3. Exercise 10.2.

4. Exercise 10.10.

5. Exercise 10.11.

6. Exercise 10.14.

7. Give an example of a metric space (X,d) such that the function ϱ defined as

$$\varrho(x,y) = (d(x,y))^2 \qquad \text{for all } x,y \in X$$

is not a metric on X. Justify your answer.

8. For each of the following, give an explicit example (with justification) or prove one does not exist:

(a) A bounded complete metric space which is not compact.

(b) A continuous injection $f : (X,d) \to (Y,\varrho)$, where (X,d) is totally disconnected and (Y,ϱ) has precisely three connected components.

(c) A continuous bijection $f : (X,d) \to (Y,\varrho)$, where (X,d) is totally disconnected and (Y,ϱ) has precisely 2021 connected components.

(d) A continuous injection $f : (X,d) \to (Y,\varrho)$, where (X,d) is infinite totally disconnected and (Y,ϱ) has precisely 2021^{2021} connected components.

(e) A totally disconnected compact metric subspace of $(\mathbb{R}, |\cdot|)$.

(f) A totally disconnected complete metric subspace of $(\mathbb{R}, |\cdot|)$.

(g) A homeomorphism between the Cantor set C and the irrationals $\mathbb{R} \setminus \mathbb{Q}$ (as metric subspaces of $(\mathbb{R}, |\cdot|)$).

(h) An uncountable metric space with countably many connected components.

(i) A non-differentiable contraction $f : (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$.

(j) A sequence (f_n) of bounded continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ which contains no convergent subsequence.

(k) A uniformly convergent sequence (f_n) of non-integrable functions $f_n : [0,1] \to \mathbb{R}$ whose limit is differentiable.

(l) A Cauchy sequence (f_n) of integrable functions $f_n : [0,1] \to \mathbb{R}$ such that its limit with respect to the supremum metric is non-integrable.

(m) A pair of open sets $U, V \subset \mathbb{R}$ such that $U \cap \overline{V}$, $\overline{U} \cap V$, $\overline{U} \cap \overline{V}$, and $\overline{U \cap V}$ are all different.

(n) A family $\{K_i\}_{i \in I}$ of compact subsets of a metric space (X,d) such that $\bigcap_{i \in I} K_i$ is non-empty and non-compact.

(o) A connected metric space (X,d) such that the cardinality of X satisfies $2 \le |X| < \infty$.

(p) A continuous function $f : [0,1] \to \mathbb{R}$ satisfying $x \in \mathbb{Q} \iff f(x) \notin \mathbb{Q}$.