Problem Set 1

September 12, 2021

All numbered exercises are from the textbook *Lectures on Real Analysis*, by F. Larusson.

1. Exercise 6.4.

2. Exercise 6.5.

3. Exercise 6.7.

4. Complete the proof of Theorem 6.4 in the textbook.

5. Review the proof of Theorem 5.26 and Example 6.8 in the textbook.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by the formula

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}\,. \end{cases}$$

(a) Use the $\epsilon - \delta$ definition of continuity to prove that f is *not* continuous at any point $c \neq 0$.

(b) Prove that f is differentiable at 0.

7. (a) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable, and find the formula for f'.

(b) Show that f' is discontinuous at 0. [Hint: Prove that there exists $\epsilon > 0$ and a sequence $(x_n)_{n=1}^{\infty}$ convergent to 0 such that $f'(x_n) \geq \epsilon$ for all $n \geq 1$.]

8. In this exercise you will prove the eqivalence of differentiability and the existence of local linearization of a function. We begin with a little definition:
Given $\delta > 0$, a function $\varphi : (-\delta, \delta) \to \mathbb{R}$ is said to be *little o of h* (denoted $\varphi \in o(h)$), when

$$\lim_{h \to 0} \frac{\varphi(h)}{h} \text{ exists and equals } 0\,.$$

(a) Prove that a function $f : I \to \mathbb{R}$ defined on a non-degenerate interval I is differentiable at a point $a \in I$ if and only if there exists a real number A and a function $\varphi \in o(h)$ such that

$$f(a+h) = f(a) + Ah + \varphi(h)\,.$$

(The above equality means that, in a small neighbourhood of a, f is nearly indistinguishable from the linear function $f(a) + Ah$, called its local *linearization*.)

(b) Assuming f is differentiable at $a \in I$, what is the exact value of the constant A above?