Problem Set 3

September 26, 2021

All numbered exercises are from the textbook *Lectures on Real Analysis*, by F. Larusson.

1. Exercise 7.1.

2. Recall that for any $a < b$ and any integrable function $f : [a, b] \to \mathbb{R}$, one defines $\int_b^a f$ as $-\int_a^b f$. Prove that, if $f : [\alpha, \beta] \to \mathbb{R}$ is integrable, then for any $a, b, c \in [\alpha, \beta]$ (i.e., regardless of their order on the real line) we have

$$\int_a^c f \;=\; \int_a^b f + \int_b^c f\,.$$

3. Exercise 7.2.

4. Exercise 7.4.

5. Exercise 7.5.

6. Exercise 7.6.

7. Suppose that f and g are integrable on $[a, b]$, and let $h : [a, b] \to \mathbb{R}$ be defined as

$$h(x) \;:=\; \max\{f(x), g(x)\}, \quad \text{for all } x \in [a, b]\,.$$

Prove that h is integrable.
[Hint: Show first that, for any $s, t \in \mathbb{R}$, one has $\max\{s, t\} = \frac{1}{2}(s + t + |s - t|)$.]

8. For a function $f : [a, b] \to \mathbb{R}$, define $f^+(x) := \max\{f(x), 0\}$ and $f^-(x) := \max\{-f(x), 0\}$, for all $x \in A$.

(a) Prove that f is integrable on $[a, b]$ iff f^+ and f^- are both integrable on $[a, b]$.

(b) Prove that, if f in integrable on $[a, b]$, then

$$\int_a^b f \;=\; \int_a^b f^+ - \int_a^b f^-\,.$$

9. Exercise 7.7.

10. Exercise 7.8.