Problem Set 5

October 10, 2021

All numbered exercises are from the textbook *Lectures on Real Analysis*, by F. Larusson.

1. Exercise 8.5.

2. Exercise 8.6.

3. Exercise 8.7.

4. Exercise 8.8.

5. Exercise 8.9.

6. Exercise 8.10.

7. (a) Prove that, if a series $\sum_n f_n$ of continuous functions $f_n : [a, b] \to \mathbb{R}$ converges uniformly to a function $f : [a, b] \to \mathbb{R}$, then f is integrable on $[a, b]$ and

$$\int_a^b f \;=\; \sum_n \int_a^b f_n \,.$$

(b) Prove that the function $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{n!}$ is integrable on the interval $[0, 1]$.

8. (a) For $n \in \mathbb{Z}_+$, let a function $f_n : [1, 2021] \to \mathbb{R}$ be defined by the formula

$$f_n(x) = \sin(\ln(nx)), \quad x \in [1, 2021]\,.$$

Prove that the sequence $(f_n)_{n\in\mathbb{Z}_+}$ contains a uniformly convergent subsequence.
[Hint: Use the Mean Value Theorem to show that the sequence $(f_n)_n$ is equicontinuous.]

(b) Construct an example of an equicontinuous sequence $(f_n)_n$ of functions on a closed interval $[a, b]$, such that $(f_n)_n$ does not contain a uniformly convergent subsequence.