Problem Set 8

November 21, 2021

All numbered exercises are from the textbook *Lectures on Real Analysis*, by F. Larusson.

1. Exercise 9.16.

2. Exercise 9.20.

3. Given metric spaces (X_1, d_1) and (X_2, d_2), one defines the *product metric* ρ on the set $X_1 \times X_2$ by the formula

$$\rho((x_1, x_2), (y_1, y_2)) := d_1(x_1, y_1) + d_2(x_2, y_2), \quad \text{for all } (x_1, x_2), (y_1, y_2) \in X_1 \times X_2.$$

(a) Prove that ρ is indeed a metric on $X_1 \times X_2$.

(b) Let (X_1, d_1) and (X_2, d_2) be metric spaces, and let ρ be the product metric on $X_1 \times X_2$. Let $p_j : (X_1 \times X_2, \rho) \to (X_j, d_j)$, $j = 1, 2$, be the coordinate projections; i.e., $p_1(x_1, x_2) = x_1$ and $p_2(x_1, x_2) = x_2$ for all $x_1 \in X_1, x_2 \in X_2$. Prove that p_1 and p_2 are continuous.

(c) Use the Weierstrass theorem (on compactness of continuous images of compact sets) proved in class to conclude that, if $A \subset X_1$ and $B \subset X_2$, then $A \times B$ is compact if and only if A and B are so.

4. Let (X, d) and (Y, ρ) be metric spaces. Prove that a function $f : (X, d) \to (Y, \rho)$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.

5. Let (X, d) be a compact metric space, and let $f : X \to X$ be a continuous function such that $f(x) \neq x$ for all $x \in X$. Prove that there exists $\epsilon > 0$ such that $d(x, f(x)) > \epsilon$ for all $x \in X$.

6. Let (X, d) and (Y, ρ) be metric spaces, let $A, B \subset X$ be such that $X = A \cup B$, and let $f : (A, d_A) \to (Y, \rho)$ and $g : (B, d_B) \to (Y, \rho)$ be continuous and such that $f|_{A\cap B} = g|_{A\cap B}$. Define a function $f \cup g$ as

$$(f \cup g)(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B\,. \end{cases}$$

Prove that, if A and B are simultaneously open or simultaneously closed in (X, d), then $f \cup g$ is continuous. Give an example showing the necessity of this assumption.

7. A function $f : (X, d) \to (Y, \rho)$ is called an *open function* when $f(U)$ is open in Y for every U open in X, and a *closed function* when $f(F)$ is closed in Y for every F closed in X. Prove that, if f is a bijection, then the following conditions are equivalent:

(i) f is a homeomorphism.

(ii) f is continuous and open.

(iii) f is continuous and closed.

8. Prove that, if (X, d) is a compact metric space and $f : (X, d) \to (Y, \rho)$ is a continuous bijection, then f is a homeomorphism.