Problem Set 9

November 28, 2021

All numbered exercises are from the textbook *Lectures on Real Analysis*, by F. Larusson.

1. Let (E, ϱ) be a normed vector space, and let d_ϱ denote the metric induced by ϱ. Prove that the norm function

$$\varrho : (E, d_\varrho) \ni x \mapsto \varrho(x) \in (\mathbb{R}, |\cdot|)$$

is continuous.

2. Given normed vector spaces $(E_1, \varrho_1), \dots, (E_n, \varrho_n)$, one defines the *product norm* as

$$\varrho : E_1 \times \cdots \times E_n \ni (x_1, \dots, x_n) \mapsto \varrho_1(x_1) + \cdots + \varrho_n(x_n) \in [0, \infty)\,.$$

(a) Prove that the product norm is a norm on $E_1 \times \cdots \times E_n$.

(b) Prove that a sequence $(x_\nu)_\nu \subset E_1 \times \cdots \times E_n$, where $x_\nu = (x_\nu^1, \dots, x_\nu^n)$ for $\nu \in \mathbb{Z}_+$, is convergent to a point $a = (a^1, \dots, a^n)$ if and only if $(x_\nu^i)_\nu$ converges to a^i for all $i = 1, \dots, n$.

3. Let (E, ϱ) be a normed vector space. Prove that the functions

$$E \times E \ni (x, y) \mapsto x + y \in E \quad \text{and} \quad \mathbb{R} \times E \ni (\lambda, x) \mapsto \lambda \cdot x \in E$$

are continuous (where $E \times E$ and $\mathbb{R} \times E$ are equipped with product norms).

4. Let (X, d) be a metric space, let $E \subset X$ be connected, and let $Y \subset X$ be any set. Prove that, if $E \cap Y \neq \varnothing$ and $E \cap (X \setminus Y) \neq \varnothing$, then $E \cap \partial Y \neq \varnothing$.

5. A metric space (X, d) is called *totally disconnected* if all its connected components are singletons (i.e., all subsets of X with more than one element are disconnected).

(a) Prove that $\mathbb{Q}$ and the Cantor set C (with the Euclidean metric induced from $\mathbb{R}$) are totally disconnected.

(b) Let (X, d) be a connected metric space, and let (Y, ϱ) be totally disconnected. Prove that the only continuous functions from (X, d) to (Y, ϱ) are the constant functions.

6. We say that two sets A, B are *separated* when $A \cap \overline{B} = \overline{A} \cap B = \varnothing$. Let (X, d) be a metric space. Prove that, if C_1 and C_2 are connected components of X, then either C_1 and C_2 are separated or else $C_1 = C_2$.

7. (a) Let (X_1, d_1) and (X_2, d_2) be metric spaces, and let $\mathcal{F}$ and $\mathcal{G}$ denote the families of their connected components, respectively. Prove that, if $f : (X_1, d_1) \to (X_2, d_2)$ is a homeomorphism, then $f(C) \in \mathcal{G}$ for every $C \in \mathcal{F}$, and every element of $\mathcal{G}$ is obtained in this way.

(b) Prove that there is no homeomorphism between $\mathbb{R}$ and $\mathbb{R}^n$, for any $n \geq 2$. [Hint: For a proof by contradiction, suppose there is a homeomorphism $f : \mathbb{R} \to \mathbb{R}^n$ for some $n \geq 2$. Then f induces a homeomorphism $\tilde{f}$ between $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^n \setminus \{f(0)\}$.]

8. Let $\{E_i\}_{i \in I}$ be a family of connected subsets of a metric space (X, d). Suppose that there exists $i_0 \in I$ such that, for every $i \in I$, E_i and E_{i_0} are not separated. Prove that the union $\bigcup_{i \in I} E_i$ is connected.

9. Exercise 9.11.

10. Exercise 9.12.